SOME SEQUENTIAL PROCEDURES FOR RANKING MULTIVARIATE NORMAL POPULATIONS[1]

M. S. SRIVASTAVA AND V. S. TANEJA

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Summary

Consider k p-variate normal populations π_i with means μ_i and common covariance matrix Σ, i.e., $\pi_i : N(\mu_i, \Sigma)$. The problem is to design a sequential procedure to rank these populations with respect to some distance function. We consider two distance functions $\mu_i'\mu_i$ and $\mu_i'\Sigma^{-1}\mu_i$. Procedures on the lines of Chow and Robbins [3], Paulson [5] and Hoel and Majumdar [4] are obtained.

1. Introduction

Suppose an experimenter is concerned with comparing k categories, such as k new machines or k new drugs, etc. We assume that k categories are of experimental nature (that is, no standard or control is used). The experimenter is practically certain that the k categories differ among themselves, and his objective is to select the " best category " on the basis of information supplied by taking measurements from each category. Denote the k categories or populations by $\pi_1, \pi_2, \cdots, \pi_k$. It is assumed that π_i $(i=1, 2, \cdots, k)$ is a p-variate normal population with mean μ_i and covariance matrix Σ, that is, $\pi_i : N(\mu_i, \Sigma)$. Each category is characterized with respect to some distance function. We consider two distance functions, the Euclidean,

$$\delta^2 = \mu_i'\mu_i \ , \tag{1.1}$$

and the Mahalanobis distance function,

$$\tau = \mu_i'\Sigma^{-1}\mu_i \ . \tag{1.2}$$

The best category is defined as the category with the largest value of the distance function.

In Section 2, sequential procedures similar to Chow and Robbins [3]

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are obtained. It follows from Srivastava and Bhargava [8] that the suggested procedure is not only asymptotically efficient but $EN-C$ is uniformly bounded where C is the fixed sample size required to reach a decision when Σ is known.

In Section 3, Paulson's [5] selection procedure is extended to the multivariate case. For the non-truncated procedure, it is proved that the proposed procedure terminates with probability one.

Let X_{ij} denote the jth observation vector on category π_i, ($i=1, 2, \cdots, k$, $j=1, 2, \cdots$). We assume that X_{ij} is a sequence of independent p-vectors, with $E(X_{ij})=\mu_i$ and $\text{Cov}(X_{ij})=\Sigma$.

2. Asymptotic procedures

Suppose that the ordered set of δ $(=(\mu'\mu)^{1/2})$ values of $\pi_1, \pi_2, \cdots, \pi_k$ are denoted by $\delta_{[1]}\leqq\delta_{[2]}\leqq\cdots\leqq\delta_{[k]}$. The δ-values are assumed to be unknown and the best category is the one which corresponds to $\delta_{[k]}$. In this section we apply Chow and Robbins [3] sequential theory to design a class ζ of sequential procedures for selecting the best category so that

$$\lim_{\Delta\to 0} P\{\text{selecting } \pi_{[k]} \mid \delta_{[k]}-\delta_{[k-1]}\geqq\Delta\}\geqq 1-\alpha , \tag{2.1}$$

where Δ and α are preassigned constants determined in advance of the experiment on the basis of practical considerations. For univariate results see [6] and [7].

2.1. Σ *known case*

Let $\bar{X}_{in}=n^{-1}\sum_{j=1}^{n} X_{ij}$, $i=1, 2, \cdots, k$. If Σ were known, one could take a sample of size

$$n\geqq a^2\lambda_1/\Delta^2\equiv C \tag{2.2}$$

from each category and select the population with the largest $\bar{X}_{in}'\bar{X}_{in}$ as the best population, where

$$\lambda_1=\max_{c\,:\,c'c=1} c'\Sigma c \tag{2.3}$$

that is, λ_1 is the maximum characteristic root of Σ and

$$\Phi(-a/\sqrt{2})=\alpha(k-1)^{-1} , \qquad \Phi(x)=\int_{-\infty}^{x} [e^{-u^2/2}(2\pi)^{-1/2}]du . \tag{2.4}$$

It can be shown (see the unknown Σ case below) that (2.1) is met.

2.2. Σ *unknown case*

We now consider the case when Σ is unknown. Let

(2.5) $$S_n=(nk)^{-1}\sum_{i=1}^{k}\sum_{s=1}^{n}(X_{is}-\bar{X}_{in})(X_{is}-\bar{X}_{in})' .$$

Define

(2.6) $$\lambda_{1n}=\max_{b\,:\,b'b=1} b'S_nb .$$

Note that $\lim_{n\to\infty}\lambda_{1n}=\lambda_1$ a.s. Let $a_1, a_2, \cdots$ be any sequence of positive constants such that $\lim_{n\to\infty} a_n=a$, where a is defined by (2.4). Then the sequence $\{a_n\}$ determines a member of the class ζ of sequential procedures as follows:

(1) Start by taking $n_0>p$ observations from each category and then sample one observation at a time from each category and stop according to the stopping rule defined by

(2.7) $$N=\text{smallest } n\geqq n_0 \text{ such that } \lambda_{1n}\leqq n\Delta^2/a_n^2 .$$

(2) When the sampling is stopped at $N=n$, select the population with the largest $\bar{X}_{in}'\bar{X}_{in}$ as the best category. It may be noted that the random sample size N depends on Δ and $N(\Delta)\to\infty$ a.s. as $\Delta\to 0$. It follows from Srivastava and Bhargava [8] that

(2.8) $$EN\leqq C+0(1) ,$$

where C is defined by (2.2). Thus the sequential procedure is not only asymptotically efficient but $EN-C$ is uniformly bounded.

PROOF OF (2.1). Let θ_1 denote the parameter configuration $\delta_{[k]}\geqq\delta_{[k-1]}+\Delta$, and let θ_1^* denote the parameter configuration $\delta_k\geqq\delta_j+\Delta$ for $j=1, 2, \cdots, k-1$. Then from the symmetry of the sequential procedure it is sufficient to prove that the probability of a correct selection is $\geqq 1-\alpha$ when $[k]=k$ and the parameter point belongs to θ_1^*. Hence

$$\begin{aligned}
&P(\pi_k \text{ is eliminated}\,|\,\theta_1^*)\\
&\quad=P(\text{Incorrect selection}\,|\,\theta_1^*)\\
&\quad=P\{\text{For at least one value of } i\ (i=1, 2, \cdots, k-1),\\
&\qquad\qquad \bar{X}_{kN}'\bar{X}_{kN}-\bar{X}_{iN}'\bar{X}_{iN}<0\,|\,\theta_1^*\}\\
&\quad\leqq\sum_{j=1}^{k-1} P\{\bar{X}_{kN}'\bar{X}_{kN}-\bar{X}_{jN}'\bar{X}_{jN}<0\,|\,\theta_1^*\}\\
&\quad=\sum_{j=1}^{k-1} P\{(\bar{X}_{jN}-\bar{X}_{kN})'(\bar{X}_{jN}+\bar{X}_{kN})\geqq 0\,|\,\theta_1^*\}\\
&\quad=\sum_{j=1}^{k-1} P\{(\bar{X}_{jN}-\bar{X}_{kN}-(\mu_j-\mu_k))'(\bar{X}_{jN}+\bar{X}_{kN})\\
&\qquad\qquad \geqq(\mu_k-\mu_j)'(\bar{X}_{jN}+\bar{X}_{kN})\,|\,\theta_1^*\}
\end{aligned}$$

$$=\sum_{j=1}^{k-1} P\left\{(N/2)^{1/2}\frac{(\bar{X}_{jN}-\bar{X}_{kN}-(\mu_j-\mu_k))'(\bar{X}_{jN}+\bar{X}_{kN})}{[(\bar{X}_{jN}+\bar{X}_{kN})'\Sigma(\bar{X}_{jN}+\bar{X}_{kN})]^{1/2}}\right.$$

$$\left.\geqq(N/2)^{1/2}\frac{(\mu_k-\mu_j)'(\bar{X}_{jN}-\bar{X}_{kN})}{[(\bar{X}_{jN}+\bar{X}_{kN})'\Sigma(\bar{X}_{jN}+\bar{X}_{kN})]^{1/2}}\,\middle|\,\theta_1^*\right\}$$

$$=\sum_{j=1}^{k-1} EP\left\{(N/2)^{1/2}\frac{(\bar{X}_{jN}-\bar{X}_{kN}-(\mu_j-\mu_k))'(\bar{X}_{jN}+\bar{X}_{kN})}{[(\bar{X}_{jN}+\bar{X}_{kN})'\Sigma(\bar{X}_{jN}+\bar{X}_{kN})]^{1/2}}\right.$$

$$\left.\geqq(N/2)^{1/2}\frac{(\mu_k-\mu_j)'(\bar{X}_{jN}+\bar{X}_{kN})}{[(\bar{X}_{jN}+\bar{X}_{kN})'\Sigma(\bar{X}_{jN}+\bar{X}_{kN})]^{1/2}}\,\middle|\,\theta_1^*,\ \bar{X}_{jN}+\bar{X}_{kN}\right\}$$

$$=\sum_{j=1}^{k-1}\left\{1-E\Phi\left[(N/2)^{1/2}\frac{(\mu_k-\mu_j)'(\bar{X}_{jN}+\bar{X}_{kN})}{[(\bar{X}_{jN}+\bar{X}_{kN})'\Sigma(\bar{X}_{jN}+\bar{X}_{kN})]^{1/2}}\right]\right\}.$$

Since $\Phi(\cdot)$ is a continuous function, $N\Delta^2/\lambda_1\to a^2$ a.s. (see [3] and [6]), and $\bar{X}_{jN}+\bar{X}_{kN}\to\bar{\mu}_j+\bar{\mu}_k$ a.s., it follows that

$$\Phi\left[(N\Delta^2/2)^{1/2}\frac{(\mu_k-\mu_j)'(\bar{X}_{jN}+\bar{X}_{kN})}{\Delta[(\bar{X}_{jN}+\bar{X}_{kN})'\Sigma(\bar{X}_{jN}+\bar{X}_{kN})]^{1/2}}\right]$$

$$\to\Phi\left[(a^2/2)^{1/2}\frac{\lambda_1^{1/2}(\mu_k-\mu_j)'(\mu_j+\mu_k)}{\Delta[(\mu_j+\mu_k)'\Sigma(\mu_j+\mu_k)]^{1/2}}\right]\text{ a.s.}\,.$$

Since $\Phi(\cdot)\leqq 1$, it follows from Lebesgue bounded convergence theorem that

$$E\Phi\left[(N\Delta^2/2)^{1/2}\frac{(\mu_k-\mu_j)'(\bar{X}_{jN}+\bar{X}_{kN})}{\Delta[(\bar{X}_{jN}+\bar{X}_{kN})'\Sigma(\bar{X}_{jN}+\bar{X}_{kN})]^{1/2}}\right]$$

$$\to\Phi\left[(a^2/2)^{1/2}\frac{\lambda_1^{1/2}(\mu_k-\mu_j)'(\mu_k+\mu_j)}{\Delta[(\mu_k+\mu_j)'\Sigma(\mu_k+\mu_j)]^{1/2}}\right]$$

$$\geqq\Phi\left[(a^2/2)^{1/2}\frac{(\mu_k-\mu_j)'(\mu_k+\mu_j)}{\Delta[\mu_k'\mu_k+\mu_j'\mu_j+2(\mu_k'\mu_k)^{1/2}(\mu_j'\mu_j)^{1/2}]^{1/2}}\right]$$

$$=\Phi\left[(a^2/2)^{1/2}\frac{\mu_k'\mu_k-\mu_j'\mu_j}{\Delta[(\mu_k'\mu_k)^{1/2}+(\mu_j'\mu_j)^{1/2}]}\right]$$

$$=\Phi\left[(a^2/2)^{1/2}\frac{(\mu_k'\mu_k)^{1/2}-(\mu_j'\mu_j)^{1/2}}{\Delta}\right]$$

$$\geqq\Phi\left[(a^2/2)^{1/2}\right]$$

since $(\mu_k'\mu_k)^{1/2}=\delta_k$ and $(\delta_k-\delta_j)\geqq\Delta$. Hence,

$$\lim_{\Delta\to 0} P[\pi_k \text{ is eliminated}\,|\,\theta_1^*]\leqq(k-1)[1-\Phi(a/\sqrt{2})]=\alpha\,,$$

which proves (2.1). It may be noted that the assumption of normality is not required.

3. Extension of Paulson's results

Denote the ranked τ's $(=\mu'\Sigma^{-1}\mu)$ by $\tau_{[1]}\leqq\tau_{[2]}\leqq\cdots\leqq\tau_{[k]}$. The τ-values are assumed to be unknown and the best category is the one which corresponds to the largest τ value. The problem is to design a procedure for selecting the best category so that

$$P\{\text{selecting } \pi_{[k]} \mid \tau_{[k]}-\tau_{[k-1]}\geqq d\}\geqq 1-\alpha . \tag{3.1}$$

Here d and α are constants which are specified by the experimenter on the basis of practical considerations, and $\pi_{[j]}$ is the category with parameter $\tau_{[j]}$.

When Σ is known this problem for fixed sample size has been considered by Alam and Rizvi [1]. However, the tables for n are not yet available to carry out the procedure. In this section we are concerned with the sequential aspect of this problem. Truncated and non-truncated sequential procedures similar to Paulson [5] and, Hoel and Mujumdar [4] are obtained.

The case when Σ is not known and a preliminary sample is taken to estimate Σ, the procedure similar to Paulson [5] can also be obtained.

In the sequential procedures developed in this section the inferior categories are eliminated before the final stage of the experiment, which tend to decrease the number of observations required to reach a decision.

We also prove that in the non-truncated case the proposed sequential procedure terminates with probability one.

The following lemma is needed in the sequel:

LEMMA 1. *Let $Z_1, Z_2, \cdots$ be a sequence of independently distributed random variables having the same distribution as $Z=x-y$ where x and y are independent non-central chi-square random variables with p degrees of freedom and non-centrality parameter θ and ϕ respectively. Let $\theta<\phi$, $0\leqq\lambda<\phi-\theta$ and $b>0$. Then*

$$P\left\{\sup_m \sum_{i=1}^m (Z_i+\lambda)>b\right\}\leqq e^{-t^*b} ,$$

where $t^*>0$ is the solution of

$$\max\{t : (1-4t^2)^{-p/2}e^{t[\lambda-4\lambda t^2+2t(\phi+\theta)-(\phi-\theta)]/(1-4t^2)}\leqq 1\} , \qquad 0<t<1/2 . \tag{3.2}$$

For proof refer to ([2], p. 164).

In Table I, the solution of (2.1) is given for selected values of p, $\phi-\theta$ and $\phi+\theta$. It may be remarked that an exact solution of (3.2) is not possible unless an upper bound of $\phi+\theta$ is given. This does not seem to seriously limit the use of the sequential procedures of Section 4 since

one may have a lower bound for $\phi-\theta$ and an upper bound for $\phi^2-\theta^2$.
In Table II, solution of (3.2) is given for the special case $\lambda=0$.

3.1. *Non-truncated sequential procedure*

Let

$$U_{is}=X_{is}'\Sigma^{-1}X_{is}\ ,\qquad i=1,2,\cdots,k,\ s=1,2,\cdots\ ,\tag{3.3}$$
$$z_{in}=\sum_{s=1}^{n}U_{is}\ ,$$

$$C_d=(1/t^*)\log((k-1)/\alpha)\tag{3.4}$$

where t^* is the solution of (3.2), with $\lambda=0$. We start with one observation on each category π_i and compute z_{i1}, $i=1,2,\cdots,k$. Then we eliminate from further consideration any category π_j for which

$$Z_{j1}\leqq\max_s(Z_{s1})-C_d\ .\tag{3.5}$$

If all but one category are eliminated, we stop the experiment and select the remaining category as the best one. Otherwise we go on to the second stage of the experiment and take one observation on each category not eliminated. Proceeding this way, at the mth stage of the experiment $(m=2,3,\cdots)$ we take one measurement on each category not eliminated after the $(m-1)$ stage, and then eliminate any category π_j for which

$$Z_{jm}=\max_s(Z_{sm})-C_d\ ,\tag{3.6}$$

where the max is taken over all categories left after the $(m-1)$ stage. We shall terminate the procedure at the stage when only one population has not been eliminated and select it as the best. We now prove that this procedure guarantees (3.1).

Let θ_2 denote the parameter configuration $\tau_{[k]}\geqq\tau_{[k-1]}+d$, and let θ_2^* denote the parameter configuration $\tau_k\geqq\tau_j+d$ for $j=1,2,\cdots,k-1$. It is obvious from the symmetry of the sequential procedure that it is sufficient to prove that the probability of a correct selection is $\geqq1-\alpha$ when $[k]=k$ and the parameter point belongs to θ_2^*. Hence

$$\begin{aligned}
&P\{\text{Incorrect selection}\,|\,\theta_2^*\}\\
&\quad=P\{\pi_k\text{ is eliminated}\,|\,\theta_2^*\}\\
&\quad\leqq P\{\text{For at least one value of } i\ (i=1,2,\cdots,k-1),\\
&\qquad\qquad\qquad Z_{km}\leqq Z_{im}-C_d\text{ for some } m<\infty\,|\,\theta_2^*\}\\
&\quad\leqq\sum_{i=1}^{k-1}P\{(Z_{im}-Z_{km})\geqq C_d\text{ for some } m<\infty\,|\,\theta_2^*\}
\end{aligned}$$

$$\leqq (k-1)e^{-\log((k-1)/\alpha)} \text{ from Lemma 1 with } \lambda=0 \text{ and (3.2)}$$
$$=\alpha .$$

We will now show that the procedure terminates with probability one[2]. Let M denote the integer, $1 \leqq M \leqq k$, for which $\max_{1 \leqq i \leqq k} Z_{iN}$ occurs. Then

$$\begin{aligned} P\{N=\infty\} &= P\{\text{For at least one value of } i \neq M \ (i=1, 2, \cdots, k) \\ &\qquad\qquad Z_{MN}-C_d < Z_{iN} \text{ for all } N \mid \theta_2^*\} \\ &\leqq \sum_{i=1}^{k-1} P\{Z_{kN}-C_d < Z_{iN} \text{ for } N \mid \theta_2^*\} \\ &= (k-1)P\{N((Z_{kN}/N))-(Z_{iN}/N)) < C_d \text{ for all } N \mid \theta_2^*\} . \end{aligned}$$

Since for the parameter point θ_2^*, $(Z_{kN}/N)-(Z_{iN}/N) \rightarrow \phi-\theta > 0$, a.s., it follows that $P\{N=\infty\}=0$.

3.2. *Truncated sequential procedure*

It has been proved in Section 3.1, that the non-truncated sequential procedure terminates with probability one. However, it has not been possible to obtain any upper bound of the expected sample size as in Section 2. Thus, a truncated sequential procedure similar to Paulson [5] may be desirable.

We introduce a parameter η, $0<\eta<d$, and start by specifying a class S_η of truncated sequential procedures and then show that for each η, $0<\eta<d$, the corresponding procedure satisfies (3.1). Let

$$C_\eta=(1/t_\eta) \log ((k-1)/\alpha) , \tag{3.7}$$

where t_η is the solution of (3.2) with $\lambda=\eta$. Let W_η be the largest integer less than C_η/η. We start sampling as in non-truncated case with (3.5) and (3.6) replaced by (3.8) and (3.9) defined below,

$$Z_{j1} \leqq \max_s (Z_{s1})-C_\eta+\eta \tag{3.8}$$

and

$$Z_{jm} \leqq \max_s (Z_{sm})-C_\eta+m\eta . \tag{3.9}$$

If more than one category remain after W_η stage, the experiment is terminated at $(W_\eta+1)$ stage and from the remaining categories, the one for which Z value is maximum is selected as the best category. We now show that for each η, the corresponding procedure satisfies (3.1).

[2] The statement of this kind has been made in the literature, see, e.g., [4], but this seems to be the first simple, neat proof.

$$
\begin{aligned}
&P[\text{Incorrect selection} \mid \theta_2^*] \\
&\quad = P[\pi_k \text{ is eliminated on or before } (W_\eta+1) \text{ stage} \mid \theta_2^*] \\
&\quad \leqq P[\text{For at least one value of } i \ (i=1, 2, \cdots, k-1) \\
&\qquad \text{there exist an integer } n_i \leqq (W_\eta+1) \text{ so that} \\
&\qquad Z_{kn_i} \leqq Z_{in_i} - C_\eta + n_i\eta \mid \theta_2^*] \\
&\quad \leqq \sum_{i=1}^{k-1} P[Z_{kn} \leqq Z_{in} - C_\eta + n\eta \text{ for some } n \leqq (W_\Delta+1) \mid \theta_2^*] \\
&\quad \leqq \sum_{i=1}^{k-1} P[Z_{kn} \leqq Z_{in} - C_\eta + n\eta \text{ for some } n < \infty \mid \theta_2^*] \\
&\quad = \sum_{i=1}^{k-1} P\left[\sum_{s=1}^{n} (U_{is} - U_{ks} + \eta) > C_\eta \text{ for some } n < \infty \mid \theta_2^*\right] \\
&\quad \leqq (k-1)e^{-\log((k-1)/\alpha)} \text{ from Lemma 1 with } \lambda=\eta \text{ and (3.7)} \\
&\quad = \alpha\,.
\end{aligned}
$$

At present the "optimum" choice of η is unknown. As recommended by Paulson [5], one may take $\eta = d/4$.

Table I. Values of t

$\phi-\theta$	λ	$\phi+\theta$	t values		
			$p=2$	3	4
0.5	0.1	4.5	0.0308	0.0267	0.0235
	0.1	6.5	0.0235	0.0211	0.0190
	0.2	4.5	0.0231	0.0200	0.0177
	0.2	6.5	0.0177	0.0158	0.0143
	0.3	4.5	0.0154	0.0133	0.0118
	0.3	6.5	0.0118	0.0105	0.0095
	0.4	4.5	0.0077	0.0067	0.0059
	0.4	6.5	0.0059	0.0053	0.0048
1.0	0.2	5.0	0.0574	0.0502	0.0446
	0.2	7.0	0.0446	0.0401	0.0364
	0.5	5.0	0.0359	0.0314	0.0279
	0.5	7.0	0.0279	0.0251	0.0228
	0.7	5.0	0.0215	0.0188	0.0167
	0.7	7.0	0.0167	0.0150	0.0137
	0.9	5.0	0.0072	0.0063	0.0056
	0.9	7.0	0.0056	0.0050	0.0045
2.5	0.6	6.5	0.1143	0.1019	0.0920
	0.6	8.5	0.0917	0.0836	0.0769
	1.2	6.5	0.0784	0.0699	0.0630
	1.2	8.5	0.0629	0.0573	0.0526
	1.8	6.5	0.0419	0.0374	0.0337
	1.8	8.5	0.0337	0.0307	0.0282
	2.4	6.5	0.0059	0.0053	0.0048
	2.4	8.5	0.0048	0.0044	0.0040

Table I. (Continued)

$\phi-\theta$	λ	$\phi+\theta$	t values		
			$p=2$	3	4
5.0	1.0	9.0	0.1911	0.1745	0.1604
	1.0	11.0	0.1590	0.1473	0.1372
	2.2	9.0	0.1355	0.1231	0.1129
	2.2	11.0	0.1124	0.1038	0.0965
	3.5	9.0	0.0716	0.0651	0.0597
	3.5	11.0	0.0597	0.0552	0.0513
	4.8	9.0	0.0092	0.0084	0.0077
	4.8	11.0	0.0077	0.0072	0.0067
8.0	1.6	12.0	0.2472	0.2298	0.2144
	1.6	14.0	0.2114	0.1985	0.1870
	3.5	12.0	0.1783	0.1644	0.1526
	3.5	14.0	0.1515	0.1416	0.1329
	6.5	12.0	0.0566	0.0524	0.0489
	6.5	14.0	0.0488	0.0457	0.0430
	7.8	12.0	0.0072	0.0067	0.0063
	7.8	14.0	0.0063	0.0059	0.0056
12.0	3.0	16.0	0.2821	0.2657	0.2507
	3.0	18.0	0.2465	0.2338	0.2223
	6.0	16.0	0.1932	0.1803	0.1691
	6.0	18.0	0.1678	0.1583	0.1498
	9.0	16.0	0.0920	0.0862	0.0811
	9.0	18.0	0.0810	0.0766	0.0726
	11.5	16.0	0.0141	0.0134	0.0127
	11.5	18.0	0.0127	0.0121	0.0115

Table II. Values of t for $\lambda=0$

$\phi-\theta$	$\phi+\theta$	t values		
		$p=2$	3	4
0.5	4.5	0.0381	0.0325	0.0287
	6.5	0.0287	0.0259	0.0231
1.0	5.0	0.0709	0.0625	0.0550
	7.0	0.0550	0.0493	0.0447
2.5	6.5	0.1478	0.1328	0.1197
	8.5	0.1197	0.1094	0.1000
5.0	9.0	0.2321	0.2134	0.1966
	11.0	0.1947	0.1806	0.1684
8.0	12.0	0.2931	0.2753	0.2594
	14.0	0.2537	0.2406	0.2275
12.0	16.0	0.3437	0.3287	0.3137
	18.0	0.3062	0.2931	0.2819

University of Tronto and University of Connecticut
Western Illinois University and University of Tronto

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