COMPLETENESS THEOREMS FOR CHARACTERIZING DISTRIBUTION-FREE STATISTICS

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1. Introduction and summary

In nonparametric testing problems, it is often useful to have a characterization of all distribution-free procedures. This is carried out by finding an appropriate group $\mathcal{S}$ of permutations or transformations of the data, such that under the null hypotheses the joint likelihood function of the data is invariant (Bell and Doksum [3], Bell and Haller [4], Smith [14]). Next one proves that the orbit of the data with respect to $\mathcal{S}$ is a complete sufficient statistic. Once this is done, the characterization follows from the well-known theorem on Neyman structure and similar tests. One usually considers invariant similar tests for practical applications. It can be shown that these procedures exhibit a stronger property than distribution-freeness. Again, one can state and prove a characterization theorem, and again, the proof depends on a completeness result.

In this paper, the basic completeness results needed for the above characterizations are given, along with the specific applications to tests of the randomness (multisample), independence, symmetry and k-factor design (Friedman model) hypotheses. The characterization of all distribution-free statistics (for a given hypotheses) will apply to multivariate situations. However, it will be necessary to restrict attention to univariate situations in order to find invariant procedures.

Similar characterization theorems can be proven for problems involving nonparametric hypotheses involving stochastic processes, including Poisson, spherically interchangeable, interchangeable and stationary independent increment processes (Bell, Woodroofe, Avadhani [6]).

2. Terminology

Vectors and matrices will always be denoted by boldface letters. Vectors will be thought of as column matrices:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix},$$

and $\boldsymbol{x}^T$ will denote the transpose of $\boldsymbol{x}$. If $\boldsymbol{x}$ is partitioned into sub-vectors, the following notation will be used:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{(1)} \\ \vdots \\ \boldsymbol{x}^{(q)} \end{bmatrix}.$$

If F is the distribution of a random vector $\boldsymbol{x}$, then F_i will denote the marginal distribution of $\boldsymbol{x}^{(i)}$. If $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ is a sample of size N from F, the N-fold power distribution will be written

$$F^{(N)}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N) = \prod_{j=1}^{N} F(\boldsymbol{x}_j).$$

The length of Euclidean norm of a vector will be written

$$\|\boldsymbol{x}\| = \left[\sum_{i=1}^{p} x_i^2\right]^{1/2}.$$

The symbol $\Omega_2(\mathcal{X})$ will denote the class of non-atomic probability measures on the space $\mathcal{X}$, and $\Omega_3(\mathcal{X}, \lambda)$ will denote the class of probability measures on $\mathcal{X}$ which are absolutely continuous with respect to the non-atomic measure λ. In this paper, $\mathcal{X}$ will generally be a subset of R^p, that is, p-dimensional Euclidean space.

If $\mathcal{X}$ is some interval (finite or infinite) contained in the real line, then, $\Omega_2^*(\mathcal{X})$ will denote the class of strictly increasing continuous distributions on $\mathcal{X}$ and $\Omega_3^*(\mathcal{X}, \mu)$ will denote the class of strictly increasing distributions which are absolutely continuous with respect to μ.

If a hypothesis H_0 is under consideration, then

$$\Omega(H_0) = \{F^{(N)} \mid F \in \Omega_2(\mathcal{X}^p) \text{ and } F \text{ satisfies } H_0\}.$$

If H_0 is tested against an alternative H_1, then $\Omega(H_0 \cup H_1)$ is the set of continuous joint distributions such that either H_0 or H_1 is satisfied. For example, if one considered the "bivariate symmetry" hypothesis

$$H_0 : F(X) = F(X_1, X_2) = F(X_2, X_1) \qquad \text{for all } X_1, X_2$$

against a general alternative, then

$$\Omega(H_0) = \{F^{(N)} \mid F \in \Omega_2(R^2),\ F(X_1, X_2) = F(X_2, X_1) \text{ for all } X_1, X_2\}$$

and

$$\Omega(H_0 \cup H_1) = \{F^{(N)} \mid F \in \Omega_2(R^2)\}.$$

If Ω is a class of probability measures, then $\Omega^{(n)}$ will denote the class of n-fold power measures of Ω. If $\boldsymbol{Z}=[X_{i_1,i_2\cdots i_k};\ i_j=1,\cdots,n_j,\ j=1,\cdots,k]$ is a generic sample point for a hypothesis testing problem and $\mathcal{S}$ is a group of transformations or permutations on the sample space, then $\mathcal{S}(\boldsymbol{Z})=\{\gamma\boldsymbol{Z}\mid\gamma\in\mathcal{S}\}$ is the orbit (or $\mathcal{S}$-orbit) of $\boldsymbol{Z}$. In the nonparametric problems considered in this paper, each hypothesis can be rephrased in terms of some group $\mathcal{S}$, and $\mathcal{S}(\boldsymbol{Z})$ will be the complete and sufficient statistic for $\Omega(H_0)$. In particular, $\mathcal{S}_k$ will represent the group of all permutations of k objects, and $\mathcal{O}_p$ will represent the group of orthogonal transformations on R^p.

3. Basic completeness theorems

Completeness is strictly a property of a class of distributions, although the term is commonly applied to a statistic. The exact definitions are given below.

DEFINITION 3.1. (i) A class of distributions Ω is complete if whenever $\int hdF=0$ for each $F\in\Omega$, one can conclude that $P_F[h\neq 0]=0$ for each $F\in\Omega$.

(ii) A statistic T is complete with respect to Ω if the family of distributions induced by T is complete in the sense of (i).

In univariate nonparametric problems, the statistic of interest is the order statistic $T(X_1,\cdots,X_N)=[X_{(1)},\cdots,X_{(N)}]$, where $X_{(1)}<X_{(2)}<\cdots<X_{(N)}$ are the ordered X's. If $h(X_1,\cdots,X_N)$ is a function of the order statistic, then h is *symmetric*. In other words

$$h(X_1,\cdots,X_N)=h(X_{i_1},\cdots,X_{i_N})\ ,$$

where $(i_1,\cdots,i_N)$ is a permutation of $(1,\cdots,N)$.

This symmetry property is independent of dimension and therefore can be used in multivariate work as well (Smith [16]). Accordingly, one makes the following definition.

DEFINITION 3.2. A class of distributions Ω is symmetrically complete if whenever h is a symmetric function of N arguments and $\int hdF^{(N)}=0$ for each $F\in\Omega$, then $P_{F^{(N)}}[h\neq 0]=0$ for each $F\in\Omega$.

The basic completeness theorems of this section were originally devised to show the completeness of the order statistic. However, they can be stated in sufficient generality to apply to multivariate situations, since they are essentially theorems on symmetric completeness.

THEOREM 3.1. *If μ is a non-atomic σ-finite measure on a measurable space $(\mathcal{X},\mathcal{B})$, then $\Omega_3(\mathcal{X},\mu)$ is symmetrically complete.*

PROOF. Fraser [8].

COROLLARY 3.1. *Let μ_i be a non-atomic σ-finite measure on a measurable space $(\mathcal{X}_i, \mathcal{B}_i)$, $(i=1, 2, \cdots, k)$. Let $\mathcal{S}=\times_{i=1}^{k} \mathcal{S}_{n_i}$. Then the $\mathcal{S}$-orbit is complete with respect to $\times_{i=1}^{k} \Omega_3^{(n_i)}(\mathcal{X}_i, \mu_i)$.*

PROOF. Fraser [8].

Theorem 3.1 and its corollary are important in their own right, if one wishes to consider tests similar with respect to the class of absolutely continuous distributions. However, their main value is in the proof of the following results, which apply to all non-atomic (i.e. continuous) distributions.

THEOREM 3.2. *For arbitrary $\mathcal{X}$, $\Omega_2(\mathcal{X})$ is symmetrically complete.*

PROOF. Bell, Blackwell and Breiman [2].

COROLLARY 3.2. *Let $\mathcal{S}=\times_{i=1}^{k} 1\mathcal{S}_{n_i}$. Then the orbit $\mathcal{S}(\boldsymbol{Z})$ is complete with respect to $\times_{i=1}^{k} \Omega_2^{(n_i)}(\mathcal{X}_i)$.*

PROOF. Let $\Omega=\times_{i=1}^{k} \Omega_2^{(n_i)}(\mathcal{X}_i)$ and let $h(x_{11}, \cdots, x_{1n_1}; x_{21}, \cdots, x_{2n_2}; \cdots, x_{k1}, \cdots, x_{kn_k})$ be symmetric in $(x_{11}, \cdots, x_{1n_1})$, in $(x_{21}, \cdots, x_{2,n_2})$, etc., such that $\int hdG=0$ for each $G \in \Omega$. Now $\Omega=\cup\left[\times_{i=1}^{k} \Omega_3^{(n_i)}(\mathcal{X}_i, \mu_i)\right]$, where the union is taken over all non-atomic measures on $(\mathcal{X}_i, \mathcal{B}_i)$. Choose $G^* \in \Omega$. Then $G^*=F_1^{*(n_1)}\times F_2^{*(n_2)}\times \cdots \times F_k^{*(n_k)}$ and $\int hdG^*=0$ for each $G^* \in \Omega= \times_{i=1}^{k} \Omega_3^{(n_i)}(\mathcal{X}_i, F_i)$.

But according to Corollary 3.1, the orbit is complete with respect to Ω^* so that $P_{G^*}[h\neq 0]=0$. But since G^* is arbitrary, the orbit is also complete with respect to Ω.

Theorem 3.2 and its corollary provide the basic completeness result used to characterize tests distribution-free with respect to the class of all continuous distributions satisfying certain (multivariate) hypotheses. The specific application of these results to common hypothesis testing problems will be delayed until Section 4. However, in the treatment of invariant and ranking procedures, one needs a completeness result of a slightly different sort.

LEMMA 3.1. *If $\mathcal{X}$ is an interval (finite or infinite), $\mathcal{B}$ is the class of Borel subsets of $\mathcal{X}$ and μ is a measure on $(\mathcal{X}, \mathcal{B})$, then $\Omega_3^*(\mathcal{X}, \mu)$ is symmetrically complete.*

PROOF. Let Ω_E be the class of probability distribution given by densities (with respect to μ) of the form

$$C(\theta_1,\cdots,\theta_N)\exp\left[-x^{2N}+\sum_{i=1}^{N}\theta_i x^i\right].$$

This class is symmetrically complete (Lehmann [12], p. 133). But Ω_E and $\Omega_3^*(\mathcal{X},\mu)$ have the same null class and $\Omega_E\subset\Omega_3^*(\mathcal{X},\mu)$, so that $\Omega_3^*(\mathcal{X},\mu)$ is also symmetrically complete.

LEMMA 3.2. *Let $\mathcal{X}_i$ be an interval, let $\mathcal{B}_i$ be the Borel subsets of $\mathcal{X}_i$ and let μ_i be a measure on $(\mathcal{X}_i,\mathcal{B}_i)$, $(i=1,2,\cdots,k)$. Let $\mathcal{S}=\times_{i=1}^{k}\mathcal{S}_{n_i}$. Then the orbit $\mathcal{S}(\boldsymbol{Z})$ is complete with respect to $\times_{i=1}^{k}\Omega_3^{*(n_i)}(\mathcal{X}_i,\mu_i)$.*

PROOF. Let the densities of Ω_E be of the form $\prod_{i=1}^{k}f_i d\mu_i$ where

$$f_i(x)=c(\theta_{i1},\cdots,\theta_{iN})\exp\left[-x^{2N}+\sum_{j=1}^{N_i}\theta_{ij}x^j\right].$$

The rest of the proof is the same as the proof of Lemma 3.1.

THEOREM 3.3. *If $\mathcal{X}$ is an interval (finite or infinite), then $\Omega_2^*(\mathcal{X})$ is symmetrically complete.*

PROOF. Note that $\Omega_2^*(\mathcal{X})=\bigcup_{\mu}\Omega_3^*(\mathcal{X},\mu)$. Therefore Lemma 3.1 and the argument of Corollary 3.2 yield the desired result.

COROLLARY 3.3. *Let $\mathcal{S}=\times_{i=1}^{k}\mathcal{S}_{n_i}$ and let $\mathcal{X}_i$ be an interval $(i=1,\cdots,k)$. Then the orbit $\mathcal{S}(\boldsymbol{Z})$ is complete with respect to $\times_{i=1}^{k}\Omega_2^{*(n_i)}(\mathcal{X}_i)$.*

PROOF. Analogous to Theorem 3.3.

4. Characterization of distribution-free tests

Four nonparametric hypotheses will be considered in this section: randomness, independence, symmetry and k-factor cross-classification. The data will be multidimensional, and no restriction will be made on the probability distributions except continuity. In this general setting it is desired to characterize all similar tests and critical regions, and all distribution-free statistics. The method is to rephrase the hypothesis in terms of a suitable permutation group (or transformation group in the case of some symmetry hypotheses). This group is the maximal group under which the likelihood function of the data (under H_0) remains invariant. The orbit of the data must be shown to be a complete sufficient statistic for the testing problem. Once this is proven, the theorem on Neyman structure (Lehmann [12], p. 134) yields the desired characterization. Since the sufficiency of the orbit statistic is quite easy to prove, the only difficulty is the proof of completeness. This is done by applying the theorems of the previous section.

Randomness (2-sample and c-sample)

In the randomness (c-sample) problem, one obtains samples from each of c multivariate populations and tests the hypothesis that the unknown population c.d.f.'s are the same. Therefore the hypothesis is

$$H_R : F_1 = F_2 = \cdots = F_c$$

and the generic data point is

$$\begin{aligned} \boldsymbol{Z} &= [\boldsymbol{X}_{11}, \boldsymbol{X}_{12}, \cdots, \boldsymbol{X}_{1n_1}; \boldsymbol{X}_{21}, \cdots, \boldsymbol{X}_{2n_2} : \boldsymbol{X}_{c1}, \cdots, \boldsymbol{X}_{cn_c}] \\ &= [\boldsymbol{Z}_1, \cdots, \boldsymbol{Z}_N] \end{aligned}$$

where $N = n_1 + n_2 + \cdots + n_c$. The null hypothesis class is $\Omega(H_R) = \Omega_2^{(N)}(R^p)$. The joint distribution function of the data is (under H_R)

$$F^{(N)}(\boldsymbol{Z}) = \prod_{j=1}^{N} F(\boldsymbol{Z}_j) ,$$

which is invariant under $\mathcal{S}_N$, the group of $N!$ permutations of the columns of $\boldsymbol{Z}$. The characterization is based on showing that $\mathcal{S}_N(\boldsymbol{Z})$ is complete. Then one obtains the following theorems:

THEOREM 4.1. *$\mathcal{S}_N(\boldsymbol{Z})$ is complete with respect to $\Omega(H_R)$.*

PROOF. Theorem 3.2 with $\mathcal{X} = R^p$.

THEOREM 4.2. *A test ϕ is similar of size α with respect to $\Omega(H_R)$ if, and only if,*

$$\sum_{\gamma \in \mathcal{S}_N} \phi(\gamma \boldsymbol{Z}) = N!\alpha \qquad \text{a.e. } [\Omega(H_R)] .$$

PROOF. Bell and Smith [5].

In the univariate case, $\mathcal{S}_N(\boldsymbol{Z})$ is equivalent to the order statistic. Therefore, the one-dimensional version of Theorem 4.2 reduces to the well-known theorem of Lehmann and Stein (Lehmann [12], p. 184). When $c=2$, of course, one is dealing with the classic two-sample problem, and when $c=N$, one has the randomness problem (against alternatives such as upward trend, serial correlation, etc.)

Independence

The most general version of the multivariate independence problem

(Anderson [1], Chapter 9) will be treated here. The data is a sample of size N from a continuous, unknown distribution F. The generic sample vector $\boldsymbol{X}$ (of dimension p) is partitioned into q subvectors of dimension p_i ($i=1, 2, \cdots, q$; $\sum p_i = p$), and the hypothesis is that the subvectors are independent:

$$H_I : F(\boldsymbol{x}) = F(\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(q)}) = \prod_{i=1}^{q} F_i(\boldsymbol{x}^{(i)}) .$$

Then the joint distribution of the sample is

$$\prod_{j=1}^{N} \prod_{i=1}^{q} F_i(\boldsymbol{X}^{(i)}) \qquad \text{and} \qquad \Omega(H_I) = \mathop{\times}_{i=1}^{q} \Omega_2(R^{p_i}) .$$

One notes that the joint distribution of

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{X}_1^{(1)} & \boldsymbol{X}_2^{(1)} & \cdots & \boldsymbol{X}_N^{(1)} \\ \boldsymbol{X}_1^{(2)} & \boldsymbol{X}_2^{(2)} & \cdots & \boldsymbol{X}_N^{(2)} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{X}_1^{(q)} & \boldsymbol{X}_2^{(q)} & \cdots & \boldsymbol{X}_N^{(q)} \end{bmatrix}$$

is invariant with respect to the group $\mathcal{G} = \mathop{\times}_{i=1}^{q} \mathcal{S}_N$, the group of order $(N!)^q$ which permutes the $\boldsymbol{X}_j^{(1)}$ among themselves, the $\boldsymbol{X}_j^{(2)}$ among themselves, etc. Therefore, the orbit $\mathcal{G}(\boldsymbol{Z})$ is the complete sufficient statistic for $\Omega(H_I)$.

THEOREM 4.3. *$\mathcal{G}(\boldsymbol{Z})$ is complete with respect to $\Omega(H_I)$.*

PROOF. Use Corollary 3.2 with $\mathcal{X}_i = R^{p_i}$ and $n_i = N$.

THEOREM 4.4. *A test ϕ is similar of size α with respect to $\Omega(H_I)$ if, and only if,*

$$\sum_{\gamma \in \mathcal{G}} \phi(\gamma \boldsymbol{Z}) = (N!)^q \alpha \qquad a.e. \ [\Omega(H_I)] .$$

PROOF. Bell and Smith [5].

When $p=q=2$, the problem is the familiar bivariate independence problem, considered by Bell and Doksum [3], along with many others. The orbit is the same as the ordered X's and the ordered Y's. In fact, whenever $p=q$, the orbit reduces to the order statistics of each component.

Symmetry

In multivariate work, some attention has been given to tests of spherical symmetry, and also to tests of interchangeability of the components of a random vector (Mauchly [13]; Smith [14]; Wilks [15]; Bell and Haller [4]; Bell and Smith [5]; Bell, Woodroofe and Avadhani [6]). These symmetry hypotheses can be subjected to a unified treatment by using the idea of invariance under a group of transformations of p-dimensional space. For example, spherical symmetry could be expressed as:

$$H_s : P_F[X \in A] = P_F[\boldsymbol{C}\boldsymbol{X} \in A] \qquad \text{for every orthogonal matrix } \boldsymbol{C}.$$

Similarly, interchangeability could be expressed as:

$$H_s : F(\boldsymbol{x}) = F(\gamma \boldsymbol{x}) \qquad \text{for all } \gamma \in \mathcal{S}_p$$

rather than the more familiar

$$H_s : F(x_1, x_2, \cdots, x_p) = F(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$$

for each permutation $(i_1, i_2, \cdots, i_p)$ of $(1, 2, \cdots, p)$ and for all $(x_1, \cdots, x_p)$. The group formulation allows one to treat other symmetry hypotheses, when symmetry can be expressed in terms of a group. Therefore, a generalized symmetry hypothesis will be considered here:

$$H_s : P_F[\boldsymbol{X} \in A] = P_F[\gamma \boldsymbol{X} \in A]$$

for all measurable A and for all $\gamma \in \mathcal{S}$ (where $\mathcal{S}$ is a compact transformation group). The likelihood function with respect to some measure μ on R^p of the generic data point. $\boldsymbol{Z} = [\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N]$ is

$$L(\boldsymbol{Z}) = \prod_{j=1}^{N} f(\boldsymbol{X}_j),$$

where $f(x) = f(\gamma x)$ under H_s. It is clear that if a transformation $\gamma_j \in \mathcal{S}$ is applied to $\boldsymbol{X}_j$, and then the $\gamma_j \boldsymbol{X}_j$ are permuted, $L(\boldsymbol{Z})$ does not change. Such transformations form a group, the so-called wreath product (Hall [11], Frucht [10]) of $\mathcal{S}$ and $\mathcal{S}_N$, written $\mathcal{W} = \mathcal{S} \wr \mathcal{S}_N$. The orbit $\mathcal{W}(\boldsymbol{Z})$ is the complete sufficient statistic for $\Omega(H_s) = \{F^{(N)} \mid P_F[\boldsymbol{X} \in A] = P_F[\gamma \boldsymbol{X} \in A], \gamma \in \mathcal{S}\}$. However, the proof of completeness cannot be given until some preliminary lemmas are proved.

One first needs a decomposition of R^p into the product of the group $\mathcal{S}$ and a space E, which is the space of the maximal invariant under $\mathcal{S}$. For example, if one considers the sphericity problem, $\mathcal{S} = \mathcal{O}_p$. A maximal invariant under $\mathcal{O}_p$ is the length of the vector $\boldsymbol{x}$, since rotations do not affect lengths. Since lengths are non-negative real numbers, $E = [0, \infty)$. For the interchangeability problem, $E = \{\boldsymbol{x} \mid x_1 < x_2 < \cdots < x_p\}$.

LEMMA 4.1. *The correspondence* $P_F \leftrightarrow P_F(\cdot \mid E)$ *is a one-to-one correspondence between the continuous symmetric distributions and the continuous distributions on* E.

PROOF. Write $X=(Y,\gamma)$, $Y\in E$, $\gamma\in\mathcal{S}$. The symmetry hypothesis states that X and γX are identically distributed. This implies that Y and γ are independent and that γ is distributed uniformly on $\mathcal{S}$. Now the condition $X\in E$ is the same as the condition $\gamma=e$, the identity element of $\mathcal{S}$. Therefore, given any probability measure Q on E, let P_F be given by $Q\times U$, where U is the uniform measure on $\mathcal{S}$. This shows the desired correspondence.

THEOREM 4.5. *The orbit $\mathcal{W}(Z)$ is complete with respect to $\Omega(H_s)$.*

PROOF. Express the generic data point as

$$Z=[Y_1,\gamma_1;Y_2,\gamma_2;\cdots;Y_N,\gamma_N]\,,\qquad (Y_j\in E,\ \gamma_j\in\mathcal{S},\ j=1,\cdots,N)$$

and let h be invariant under $\mathcal{W}$ and such that

$$\int hdF^{(N)}=0\qquad\text{for each } F^{(N)}\in\Omega(H_s)\,.$$

Then h must be independent of the γ_j and symmetric in the Y_j so that

$$\int_{R^{pN}}hdF^{(N)}=\int_{E^N}hdG^{(N)}$$

where G is a distribution on E. However, $\Omega_2(E)$ is symmetrically complete (Theorem 3.2) so that $h(Y_1,\cdots,Y_N)=0$ a.e. $[\Omega_2^{(N)}(E)]$. But using the correspondence of Lemma 4.1 and the invariance of h under $\mathcal{W}$, it follows that $h(Z)=0$ a.e. $[\Omega(H_s)]$, so that $\mathcal{W}(Z)$ is complete.

THEOREM 4.6. *A test ϕ is similar of size α with respect to $\Omega(H_s)$ if, and only if,*

$$\sum_{\pi\in S_N}\int\cdots\int\phi(\pi Z)\prod_{j=1}^{N}dU(\gamma_j)=N!\alpha$$

where $U(\cdot)$ denotes the uniform measure on $\mathcal{S}$, the existence of which is guaranteed by the compactness assumption on $\mathcal{S}$. [Some specific symmetry hypotheses are given in Table I, along with the corresponding $\mathcal{S}$'s].

k-factor layouts

The k-factor designs considered here are extensions of the Friedman [9] model. It is assumed that one has collected data of the form

$$Z=\{X_{i_1i_2\cdots i_kj}\,|\,1\leqq i_v\leqq c_v,\ v=1,\cdots,k;\ j=1,\cdots,n(i_1,\cdots,i_k)\}\,.$$

That is, the data is vector valued, with possibly several observations per cell. One wishes to test the hypothesis of "no first factor effect." The crucial assumption of the Friedman model is that there is no interaction under the hypothesis. When $k=2$, $n(i_1, i_2)=1$, then Friedman's original two-factor classification model results.

The hypothesis becomes $H_F : F_{i_1 i_2 \cdots i_k} = \bar{F}_{i_2 \cdots i_k}$, and under H_F, the joint distribution of the data is invariant if, for fixed $i_2, i_3, \cdots, i_k$, the "i_1-subscripts" are permuted. If one introduces the notation

$$N_{(i_2,\ldots,i_k)} = \sum_{i_1=1}^{c_1} n(i_1, \cdots, i_k)$$

and considers the permutation group $\mathcal{F} = \underset{(i_2 \cdots i_k)}{\times} \mathcal{S}_{N_{(i_2 \cdots i_k)}}$ then the orbit $\mathcal{F}(\boldsymbol{Z})$ is the complete sufficient statistic for

$$\Omega(H_F) = \underset{(i_2 \cdots i_k)}{\times} \Omega_2^{(N(i_2 \cdots i_k))}(R^p) .$$

THEOREM 4.7. *$\mathcal{F}(\boldsymbol{Z})$ is complete with respect to $\Omega(H_F)$.*

PROOF. Corollary 3.2 with $\mathcal{X}_{i_2 \cdots i_k} = R^p$.

THEOREM 4.8. *A test ϕ is similar of size α with respect to $\Omega(H_F)$ if, and only if,*

$$\sum_{\gamma \in \mathcal{F}} \phi(\gamma \boldsymbol{Z}) = \alpha \prod_{(i_2 \cdots i_k)} N(i_2, \cdots, i_k)! \qquad \textit{for a.e. } \boldsymbol{Z} .$$

PROOF. Analogous to Theorem 4.2.

The results of this section are summarized in Table I. For each of the common nonparametric problems, the table lists the corresponding transformation or permutation group, the cardinality of the orbit, and the relevant completeness theorem.

5. Invariant and strongly distribution-free tests

Nonparametric problems are usually simplified in practice by using the *invariance principle.* The use of invariance has practical importance in that it leads to the use of ranks, and in addition it simplifies the mathematical problems inherent in nonparametric work, enabling one to construct optimal tests.

The invariance method is based on the use of a group $\mathcal{G}$ of transformations of the sample space. This group has two properties:
(i) If $F \in \Omega(H_0)$, then $F_g = F[g^{-1}(\cdot)] \in \Omega(H_0)$;
(ii) If $F \in \Omega(H_1)$ then $F_g \in \Omega(H_1)$.
As an example, in the univariate two-sample problem, $\mathcal{G}$ is the group of strictly increasing continuous transformations of the real line onto itself. Having found a suitable $\mathcal{G}$, one then restricts attention to *invariant tests,* i.e., such that $\phi(\boldsymbol{Z}) = \phi(g\boldsymbol{Z})$. It can be shown that every

Table I. Characterization of distribution-free statistics

Problem H_0	Data	Group	Size of orbit	Completeness thm.
2-sample H_R	$\boldsymbol{X}_1, \cdots, \boldsymbol{X}_m$ $\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_n$	$\mathcal{S}_N \quad N=m+n$	$N!$	Thm. 3.2 $\mathcal{X}=R^p$
c-sample H_R	$\boldsymbol{X}_{11}, \cdots, \boldsymbol{X}_{1n_1}$ $\boldsymbol{X}_{21}, \cdots, \boldsymbol{X}_{2n_2}$ $\cdots\cdots\cdots\cdots$ $\boldsymbol{X}_{c1}, \cdots, \boldsymbol{X}_{cn_c}$	$\mathcal{S}_N \quad N=\sum n_i$	$N!$	Thm. 3.2 $\mathcal{X}=R^p$
Randomness H_R	$\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N$	$\mathcal{S}_N$	$N!$	Thm. 3.2 $\mathcal{X}=R^p$
Independence H_I	$\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N$ $\boldsymbol{X}_j^T=[\boldsymbol{X}_j^{(1)T}, \cdots, \boldsymbol{X}^{(2)T}]$ $\dim \boldsymbol{X}^{(i)}=p_i$	$\overset{q}{\underset{i=1}{\times}} \mathcal{S}_N$	$(N!)^q$	Cor. 3.2 $\mathcal{X}_i=R^{p_i},\ n=N$
Sphericity H_s $(\mathcal{S}=\mathcal{O}_p)$ (univariate symmetry if $p=1$)	$\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N$	$\mathcal{O}_p \wr \mathcal{S}_N$ ($\mathcal{O}_1=$"sign change" group)	uncountable if $p>1$ $N!2^N$ if $p=1$	Lemma 4.1, Thm. 3.2 $\mathcal{X}=[0, \infty)=E$
Interchangeability (of components) H_s $(\mathcal{S}=\mathcal{S}_p)$	$\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N$ $\boldsymbol{X}_j=[X_{ij}, \cdots, X_{pj}]$	$\mathcal{S}_p \wr \mathcal{S}_N$	$N!(p!)^N$	Lemma 4.1, Thm. 3.2 $\mathcal{X}=E=\{\boldsymbol{x} \mid x_1<x_2<\cdots<x_p\}$
Interchangeability (of sets of components) H_s $(\mathcal{S}=\mathcal{S}_q)$	$\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N$ $\boldsymbol{X}_j^T=[\boldsymbol{X}^{(i)T}, \cdots, \boldsymbol{X}^{(q)T}]$ $\dim(\boldsymbol{X}^{(i)})=r=p/q$	$\mathcal{S}_q \wr \mathcal{S}_N$	$N!(q!)^N$	Lemma 4.1, Thm. 3.2 $\mathcal{X}=E=\{\boldsymbol{x} \mid x_1^{(1)}<\cdots<x_1^{(q)}\}$
k-factor H_F	$\boldsymbol{X}_{i_1,i_2,\ldots,i_k,j}$ $i_1=1, \cdots, c_1$ $i_2=1, \cdots, c_2$ $\cdots\cdots\cdots\cdots$ $i_k=1, \cdots, c_k$ $j=1, \cdots, n(i_1, \cdots, i_k)$	$\underset{(i_2,\cdots,i_k)}{\times} \mathcal{S}_{N(i_2,\ldots,i_k)}$ $N_{(i_2,\ldots,i_k)}=\sum_{i_1=1}^{c_1} n(i_1, \cdots, i_k)$	$\prod_{(i_2,\cdots,i_k)} N(i_2, \cdots, i_k)!$	Cor. 3.2 $\mathcal{X}_{i_2,\ldots,i_k}=R^p$

invariant procedure is a function of a so-called maximal invariant statistic. In the univariate two-sample problem, a maximal invariant is the ranks of the pooled sample.

As a rule, invariant procedures do not exist for multivariate problems, either because a suitable group has not been found or because the invariance method eliminates too much information (Smith [14]; Bell, Woodroofe and Avadhani [6]). In this section, therefore, the dimension of the data will be restricted.

The study of the structural properties of invariant procedures depends very much on the alternatives under consideration, and the idea of completeness is an important tool. First, however, it is necessary to introduce two concepts closely related to invariance.

DEFINITION 5.1. (i) A statistic T is almost invariant with respect to a class of probability measures Ω and a group $\mathcal{G}$ if

$$P_F[T(x)\neq T(gx)]=0 \qquad \text{for every } g\in\mathcal{G} \text{ and for every } F\in\Omega .$$

The exceptional set on which $T(x)\neq T(gx)$ may depend on g, but not on F.

(ii) A set B is almost invariant if I_B (the indicator function of B) is an almost invariant statistic.

DEFINITION 5.2. A statistic T is strongly distribution-free with respect to a class Ω of distributions and a group $\mathcal{G}$ if for all real t and all $F\in\Omega$

$$P_F[T(x)\leqq t]=P_G[T(x)\leqq t]$$

whenever $G(x)=F_g(x)=F[g^{-1}(x)]$ for some $g\in\mathcal{G}$. The power function of a strongly distribution-free test is constant over each equivalence class of $\Omega=\Omega(H_0\cup H_1)$ under $\mathcal{G}$.

The study of the structure of invariant statistics does not always lead to a full characterization. However, if the procedure is nonsequential (i.e., does not depend on the "chronologicalorder" of the data), characterizations are possible. This is because each nonsequential statistic is a function of a complete sufficient statistic for $\Omega(H_0\cup H_1)$. The formal definition is given below.

DEFINITION 5.3. Let the hypothesis H_0 be tested against a general alternative H_1 and let $\mathcal{S}'$ be the maximal group of permutations (or transformations) on $\boldsymbol{Z}$ such that the joint distribution of $\boldsymbol{Z}$ under H_1 remains invariant. If $T(\boldsymbol{Z})$ is invariant with respect to $\mathcal{S}'$, then T is nonsequential (for the testing problem H_0 vs. H_1).

For example, if one tested H_R against the two-sample alternative (i.e., $X_1,\cdots,X_m$ form a random sample from F and $X_{m+1},\cdots,X_{m+n}$ form a random sample from G, with $G \neq F$), then any statistic which is invariant under permutations of $X_1, \cdots, X_m$ among themselves and also under permutations of $X_{m+1}, \cdots, X_{m+n}$ among themselves is non-sequential. For this example, $\mathcal{S}' = \mathcal{S}_m \times \mathcal{S}_n$. If one considered a c-sample alternative then $\mathcal{S}' = \mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \cdots \times \mathcal{S}_{n_c}$. Another way to define a non-sequential statistic is to say that T is a function of the sufficient statistic for $\Omega(H_0 \cup H_1)$.

Almost invariant and strongly distribution-free statistics are related by the following lemma.

LEMMA 5.1. (i) *If T is an almost invariant statistic with respect to $\Omega(H_0 \cup H_1)$ and $\mathcal{G}$, then T is strongly distribution-free.*

(ii) *If T is non-sequential, then T is almost invariant if, and only if, it is strongly distribution-free.*

PROOF. Smith [14] provides a proof for the independence problem. However, the proof is easily applied to any of the other nonparametric problems of Table I. Note that completeness of $\Omega(H_0 \cup H_1)$ (which follows from Corollary 3.2) is required to prove (ii).

The relationship between strongly distribution-free and invariant (i.e., rank) statistics is based on a theorem of Berk and Bickel [7]. They consider the sample space of some random variable $\boldsymbol{Z}$ and a family of distributions Ω generated by a group $\mathcal{G}$. That is, $\Omega = \{F_g \mid g \in \mathcal{G}\}$ for any $F \in \Omega$. They also assume that I is a maximal invariant statistic and that S is some other statistic such that the correspondence $\boldsymbol{Z} \leftrightarrow (I, S)$ is one-one, bimeasurable and such that $\mathcal{G}$ induces a group $\mathcal{G}_s$ acting on S. That is, if $\boldsymbol{Z} \leftrightarrow (I, S)$, then $g\boldsymbol{Z} \leftrightarrow (I, g_s S)$. Their theorem can be stated as follows.

THEOREM 5.1. *If S is sufficient and boundedly complete, then any test which is almost invariant (with respect to Ω and $\mathcal{G}$) is Ω-equivalent to an invariant test.*

In the nonparametric problems considered here, the role of S is played by the orbit and I is usually the rank statistic. However, $\Omega(H_0)$ is not of the form $\{F_g \mid g \in \mathcal{G}\}$, so that one must prove Theorem 5.1 for a special subclass of $\Omega(H_0)$, then extend the results to the entire null hypothesis class. Eventually, one can prove a theorem of the form:

THEOREM 5.2. (i) *If T is almost invariant with respect to $\Omega(H_0 \cup H_1)$ and $\mathcal{G}$, then T is equivalent to rank statistic.*

(ii) *Let T be non-sequential. Then T is strongly distribution-free if, and only if, T is equivalent to a rank statistic.*

In the remaining portion of this section, the method outlined above will be applied to each of the nonparametric hypotheses. For each hypothesis, the group, the maximal invariant and the necessary completeness theorem are given. The results of this section are summarized in Table II.

Randomness vs. c-sample alternatives

The null hypothesis H_R of Section 4 is tested against the general c-sample alternative

$$H_{cs}: F_i \neq F_j \qquad \text{for some } i,\ j\,.$$

The maximal group of permutations of the data which leaves likelihood function invariant under H_1 is $\mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \cdots \times \mathcal{S}_{n_c}$, the group of permutations which only permutes observations within each sample. Therefore, a statistic T is non-sequential for this problem if it is invariant under $\mathcal{S}_{n_1} \times \cdots \times \mathcal{S}_{n_c}$. The method of invariance leads to good results only if the data is univariate, i.e., $p=1$. In this case, $\mathcal{G}$ is the group of strictly increasing continuous transformations of the real line onto itself. The maximal invariant is the set of ranks of the combined sample, that is,

$$R_{ij} = \sum_{k=1}^{c} \sum_{v=1}^{n_k} \epsilon\,(X_{ij} - X_{kv}) \qquad (i=1,\cdots,c;\ j=1,\cdots,n_i)\,.$$

To prove the structure theorem (Theorem 5.2) for this problem, one considers $\Omega^*(H_R) = \{F_0^{(N)} \mid F \in \Omega_2^*(R)\}$, where $\Omega_2^*(R)$ is the class of strictly increasing continuous distributions on the real line. The first step is to apply the Berk-Bickel theorem (Theorem 5.1) to $\Omega^*(H_R)$. Here, S is the $\mathcal{S}_N$-orbit and I is the rank statistic. The bimeasurability condition is obviously satisfied and S is a sufficient statistic (in fact, S is the order statistic of the combined sample). Therefore one needs only to prove the following lemma.

LEMMA 5.2. *The orbit $\mathcal{S}_N(\boldsymbol{Z})$ is complete with respect to $\Omega^*(H_R)$.*

PROOF. This follows immediately from Theorem 3.3 with $\mathcal{X}=R$.

Lemma 5.2 implies that the Berk-Bickel Theorem holds in the univariate c-sample problem, if attention is restricted to strictly increasing distributions. However, the more general result is Theorem 5.2 which is proven here for the special case of the c-sample problem.

THEOREM 5.2a. (i) *If T is almost invariant with respect to $\Omega(H_R \cup H_{cs})$, then T is equivalent $[\Omega(H_R)]$ to a rank statistic.*

(ii) *Let T be a non-sequential statistic. Then T is strongly distribution-free if, and only if, T is equivalent to a rank statistic.*

PROOF. Clearly, if T is almost invariant with respect to $\Omega(H_R \cup H_{cs})$,

Table II. Invariant and strongly distribution-free tests

Problem	Group defining non-sequential statistic	Transformations	Dimensional restrictions	$\Omega^*(H_0)$	Completeness of $\Omega^*(H_0)$
2-sample	$\mathcal{S}_m \times \mathcal{S}_N$	$X_j \to g(X_j)$ $Y_j \to g(Y_j)$ $g \in \mathcal{G}_2^*(R)$	$p=1$	$\Omega_2^{*(N)}(R)$	Thm. 3.3 $\mathcal{X}=R$
c-sample	$\mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \cdots \times \mathcal{S}_{n_c}$	$X_{ij} \to g(X_{ij})$ $g \in \mathcal{G}_2^*(R)$	$p=1$	$\Omega_2^{*(N)}(R)$	Thm. 3.3 $\mathcal{X}=R$
Randomness	$\{e\}$	$X_j \to g(X_j)$ $g \in \mathcal{G}_2^*(R)$	$p=1$	$\Omega_2^{*(N)}(R)$	Thm. 3.3 $\mathcal{X}=R$
Total independence	$\mathcal{S}_N$	$X_{ij} \to g_i(X_{ij})$ $i=1,\cdots,p,\ j=1,\cdots,N$ $g_i \in \mathcal{G}_2^*(R)$	$q=p$ $p_i=1$	$\underset{i=1}{\overset{p}{\times}} \Omega_2^{*(N)}(R)$	Cor. 3.3 $\mathcal{X}_i=R$
Sphericity	$\mathcal{S}_N$	$(\lVert X_j \rVert, \theta_j) \to (g(\lVert X_j \rVert), \theta_j)$ $j=1,\cdots,N$ $g \in \mathcal{G}_2^*([0,\infty))$	None	$\{F^{(N)} \mid F$ spherical, $F_{\lVert X \rVert} \in \Omega_2^*([0,\infty))\}$	Thm. 3.3 $\mathcal{X}=[0,\infty)$
Interchangeability	$\mathcal{S}_N$	unknown	—	—	—
k-factor	$\underset{(i_1,\cdots,i_k)}{\times} \mathcal{S}_{n(i_1,\cdots,i_k)}$	$X_{i_1,\cdots,i_k,j} \to g(X_{i_1,\cdots,i_k,j})$ $g \in \mathcal{G}_2^*(\mathrm{R})$	$p=1$	$\underset{(i_2,\cdots,i_p)}{\times} \Omega_2^{*(N(i_2,\cdots,i_p))}(\mathrm{R})$	Cor. 3.3 $\mathcal{X}=R$

then it is almost invariant with respect to $\Omega^*(H_R)$. The Berk-Bickel Theorem states that T is equivalent $[\Omega^*(H_R)]$ to a rank statistic. But the null classes of $\Omega(H_R)$ and $\Omega^*(H_R)$ are equal, so the result may be extended to equivalence $[\Omega(H_R)]$. If T is non-sequential and strongly distribution-free, then Lemma 5.1 says T is almost invariant, and (i) then applies. Conversely, if T is non-sequential and equivalent to a rank statistic, it is almost invariant. Applying Lemma 5.1 again, one concludes that T is strongly distribution-free.

Total independence

The general treatment of independence does not lend itself to the invariance method. However, the special case of total independence

$$H_{TI} : F(\boldsymbol{x}) = \prod_{i=1}^{p} F_i(x_i)$$

can be treated effectively by this method. For this problem, a non-sequential statistic is one invariant with respect to $\mathcal{S}_N$. The transformation group is the group of strictly increasing continuous, $1-1$ transformations applied to each co-ordinate of the sample vector $\boldsymbol{X}$. The maximal invariant is the sets of ranks R_{ij} of the co-ordinates, where

$$R_{ij} = \sum_{k=1}^{N} \epsilon\,(X_{ij} - X_{ik}) \ .$$

To explore the structure of rank statistics, one again uses the Berk-Bickel Theorem. For this problem, S is the orbit $\mathcal{G}(\boldsymbol{Z})$, or, in other words, the order statistics of each co-ordinate. (Recall that $\mathcal{G} = \overset{p}{\underset{i=1}{\times}} \mathcal{S}_N$). As in the c-sample case, one first proves Theorem 5.1 for

$$\Omega^*(H_{TI}) = \left\{ F^{(N)} \mid F(\boldsymbol{x}) = \prod_{i=1}^{p} F_i(x_i),\ F_i \in \Omega_2^*(R) \right\} \ .$$

The necessary preliminary result is the following lemma.

LEMMA 5.3. *The orbit $\mathcal{G}(\boldsymbol{Z})$ is complete with respect to $\Omega^*(H_{TI})$.*

PROOF. Corollary 3.3 with $n_i = N$, $\mathcal{X}_i = R$, $i = 1, \cdots, p$.

Once this lemma is obtained, then Theorem 5.2 can be proven for the problem of testing H_{TI} against the general alternative H_R. The proof is an exact parallel to the proof for the randomness hypothesis, and so will not be repeated.

Sphericity

The general alternative to the p-variate sphericity hypothesis is H_R, i.e., the data consists of a random sample from a non-spherical distribution. Accordingly, a non-sequential statistic is one which is invariant with respect to $\mathcal{S}_N$. For the sphericity problem the appropriate transformation group is

$$\mathcal{G}_R=\{g\mid g(\|\boldsymbol{X}\|,\boldsymbol{\theta})=(\bar{g}(\|\boldsymbol{X}\|),\boldsymbol{\theta})\ ,$$

where $\bar{g}$ is a strictly increasing continuous transformation of $[0, \infty)$ onto itself$\}$. Here the vector $\boldsymbol{X}$ has been expressed in spherical co-ordinates:

$$\begin{aligned}
X_1\ &=\|\boldsymbol{X}\|\sin\theta_1\\
X_2\ &=\|\boldsymbol{X}\|\cos\theta_1\sin\theta_2\\
&\vdots\\
X_{p-1}&=\|\boldsymbol{X}\|\cos\theta_1\cos\theta_2\cdots\cos\theta_{p-2}\sin\theta_{p-1}\\
X_p\ &=\|\boldsymbol{X}\|\cos\theta_1\cos\theta_2\cdots\cos\theta_{p-2}\cos\theta_{p-1}\ ,
\end{aligned}$$

where

$$\|\boldsymbol{X}\|=(X_1^2+\cdots+X_p^2)^{1/2}\ ,\qquad -\pi/2\leqq\theta_i\leqq\pi/2\qquad(i=1,\cdots,p-2)\ ,$$

and $-\pi\leqq\theta_{p-1}\leqq\pi$. In words, $\mathcal{G}_R$ is the group which makes monotone distortions of the length of X while leaving its direction fixed. The maximal invariant under $\mathcal{G}_R$ is the set of ranks of the norms and the set of direction angle vectors (Smith [14]). That is,

$$I=(R(\|\boldsymbol{X}_1\|),\boldsymbol{\theta}_1;R(\|\boldsymbol{X}_2\|),\boldsymbol{\theta}_2;\cdots;R(\|\boldsymbol{X}_N\|)\boldsymbol{\theta}_N)$$

where

$$R(\|\boldsymbol{X}_j\|)=\sum_{k=1}^{N}\epsilon\,(\|\boldsymbol{X}_j\|-\|\boldsymbol{X}_k\|)\ .$$

To apply the Berk-Bickel Theorem, one notes that the orbit $(\mathcal{O}_p \wr \mathcal{S}_N)(\boldsymbol{Z})$ satisfies the bimeasurability conditions. One first considers the class

$$\Omega^*(H_s)=\{F^{(N)}\mid F\text{ is spherical and }F_{\|\boldsymbol{X}\|}\in\Omega_2^*[0,\infty)\}$$

(where $F_{\|\boldsymbol{X}\|}$ is the $\|\boldsymbol{X}\|$-marginal). One must prove the completeness of $(\mathcal{O}_p \wr \mathcal{S}_N)(\boldsymbol{Z})$ with respect to $\Omega^*(H_s)$.

LEMMA 5.4. *The orbit $(\mathcal{O}_p \wr \mathcal{S}_N)(\boldsymbol{Z})$ is complete with respect to $\Omega^*(H_s)$.*

PROOF. The orbit $(\mathcal{O}_p \wr \mathcal{S}_N)(\boldsymbol{Z})$ is equivalent to the ordered norms, since the ordered norms are the maximal invariant under $\mathcal{O}_p \wr \mathcal{S}_N$. To prove the completeness of the ordered norm statistic, one applies Theorem 3.3 with $\mathcal{X}=[0,\infty)$.

Now the structure theorem (Theorem 5.2) follows using the same arguments as in the randomness case.

As yet, the other symmetry problems have not yielded to the invariance technique, because it has not been possible to find an appropriate group $\mathcal{G}$. In fact, for the bivariate interchangeability problem, it has been conjectured that no such group exists (Bell and Haller [4]).

Univariate k-way layouts

When the observations are univariate, the invariance method can be used to treat k-factor designs. If there are several observations per cell, then these observations may be permuted without changing the value of the likelihood function under H_1. Therefore, a non-sequential statistic is one which is invariant under $\underset{(i_1\cdots i_k)}{\times} \mathcal{S}_{n(i_1,\cdots,i_k)}$, since this group permutes observations within a given cell. The appropriate transformation group for this problem is the group of strictly increasing, continuous transformations of the real line onto itself. To construct the maximal invariant, one takes each combination of "block" effects (i.e., factor 2, factor 3, $\cdots$, factor k effects) and ranks the observations. In other words, one calculates

$$R(X_{i_1,i_2,\cdots,i_k,j})=\sum_{m=1}^{n_1}\sum_{v=1}^{n(i_1\cdots i_k)} \epsilon\,(X_{i_1\cdots i_k,j}-X_{m_{i_2\cdots i_k,v}})$$
$$\text{for each } (i_2,\cdots,i_k)\,.$$

The Berk-Bickel Theorem is first applied to

$$\Omega^*(H_F)=\underset{(i_2\cdots i_k)}{\times}\ \Omega_2^{*(N(i_2\cdots i_k))}(R)$$

where as before, $N(i_2\cdots i_k)=\sum_{i_1=1}^{n_1} n(i_1\cdots i_k)$. Thus, all that is needed is to prove the completeness of the orbit with respect to $\Omega^*(H_F)$.

LEMMA 5.5. *The orbit* $\left[\underset{(i_2\cdots i_k)}{\times} \mathcal{S}_{N(i_2\cdots i_k)}\right](Z)$ *is complete with respect to* $\Omega^*(H_F)$.

PROOF. Corollary 3.3 with each $\mathcal{X}_i=R$. The rest follows as in the randomness case.

The results of this section are summarized in Table II. For each testing problem considered, the transformation group, dimensional restrictions and "non-sequential" group is given, along with the completeness result needed to prove the completeness of the orbit with respect to $\Omega^*(H_0)$. The notation $\mathcal{G}_2^*(\mathcal{X})$ denotes the group of strictly increasing transformations of $\mathcal{X}$ onto itself, where $\mathcal{X}$ is an interval.

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