TEST OF SYMMETRY OF A ONE-DIMENSIONAL DISTRIBUTION AGAINST POSITIVE BIASEDNESS

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Summary

Definitions of different strengths are given to the notion of 'a positively biased random variable'. This notion is related to that of 'a stochastically larger component of a two-dimensional random vector', which was introduced previously by the authors. Properties of common rank tests of symmetry about zero against our specification of alternatives are studied in detail. The positive biasedness is extended to 'positively more biased'. Test of symmetry of a two-dimensional random vector is also referred to.

1. Introduction

The purpose of this paper is to make clear the notion of a 'positively biased' one-dimensional random variable as an alternative to 'symmetry about zero'. This notion is useful to make more precise statements on the test of symmetry than discussed in previous publications like [1] and [2].

We introduce in Section 2 a series of definitions of different degrees of strictness and see their properties. We examine in Section 3 the notion from different points of view, which are based on the definitions of 'stochastically larger component of a random vector' introduced by the authors in [5]. In this aspect the present paper is a sequel to [5].

In Section 4 we study properties of common rank tests of symmetry about zero when the alternative hypothesis is given by a positively biased distribution, and improve the results in [2]. In Section 5 we define the notion of 'positively more biased,' and show a condition of monotonicity of the power function for one-parameter family of distributions. Our discussions on tests of symmetry can be applied straightly to those of a two-dimensional random vector against some special alternatives (Section 6). Most of the propositions in this paper can be easily proved and they are stated without proof.

2. Definitions of positive biasedness

Let X be a random variable on R^1 with the distribution function $F(x)$. X is symmetric about zero if $F(x)=1-F(-x-0)$ for any $x\geqq 0$, or equivalently if X and $-X$ have the same distribution. As alternatives of symmetry 'X is positively biased' in some different meanings which will be defined below and this fact is denoted by $X\succ 0$ $(\mathcal{B}_i)$ or by $F(\cdot)\succ 0$ $(\mathcal{B}_i)$.

DEFINITION 2.1.

$X\succ 0$ $(\mathcal{B}_0)$ iff (if and only if) $P(X>0)\geqq P(X<0)$, or equivalently $1-F(0)\geqq F(0-0)$.

$X\succ 0$ $(\mathcal{B}_1)$ iff $P(X>a_1)\geqq P(X<-a_1)$ for any $a_1>0$, or equivalently $F(x)+F(-x-0)\leqq 1$ for any $x\geqq 0$.

$X\succ 0$ $(\mathcal{B}_2)$ iff $P(a_2\geqq X>a_1)\geqq P(-a_1>X\geqq -a_2)$ for any $a_2>a_1>0$, or equivalently $F(x+y)-F(x)\geqq F(-x-0)-F(-x-y-0)$ for any $x, y\geqq 0$.

$X\succ 0$ $(\mathcal{B}_3)$ iff $\dfrac{P(a_3\geqq X>a_2)}{P(a_2\geqq X>a_1)}\geqq\dfrac{P(-a_2>X\geqq -a_3)}{P(-a_1>X\geqq -a_2)}$ for any $a_3>a_2>a_1>0$ such that the denominators are positive, or equivalently $(F(x+y)-F(y))/(F(-y-0)-F(-x-y-0))$ is nondecreasing in both $x>0$ and $y>0$.

$X\succ 0$ $(\mathcal{B}_4)$ iff $\dfrac{P(a_3\geqq X>a_2)}{P(a_2\geqq X>a_1)}\geqq\dfrac{P(-a_2>X\geqq -a_3)}{P(-a_1>X\geqq -a_2)}$ for any $a_3>a_2>a_1$ such that the denominators are positive, or equivalently $(F(x+y)-F(y))/(F(-y-0)-F(-x-y-0))$ is nondecreasing in both $x>0$ and y.

Notice that the suffix i of $\mathcal{B}_i$ corresponds almost to the number of parameters in the definitions. The notion of 'negative biasedness' can be defined similarly and denoted by $X\prec 0$ $(\mathcal{B}_i)$.

If X has a probability density function $f(x)$ with respect to Lebesgue measure, then the definitions are expressed as follows.

PROPOSITION 2.1.

$X\succ 0$ $(\mathcal{B}_2)$ iff $f(x)\geqq f(-x)$ for any $x>0$ (a.e.).

$X\succ 0$ $(\mathcal{B}_3)$ iff $f(x)/f(-x)$, $0<x<\infty$, is nondecreasing in x (a.e.) and both $f(-x)=0$ and $F(-x)\neq 0$ implies $f(x)=0$ (a.e.).

$X \succ 0$ ($\mathscr{B}_4$) iff $f(x)/f(-x)$, $-\infty < x < \infty$, is nondecreasing in x (a.e.) and $f(-x)=0$ and $F(-x) \neq 0$ jointly imply $f(x)=0$ (a.e.).

$$\begin{array}{llll} \mathscr{B}_4 \rightarrow \mathscr{B}_3 & & & \\ \quad \searrow \mathscr{B}_2 & \rightarrow \mathscr{B}_1 & \rightarrow \mathscr{B}_0 & \end{array}$$

Fig. 1 Implication scheme of $\mathscr{B}_i$'s.

PROPOSITION 2.2. The implication relationships shown by arrows in Fig. 1 and only these are valid.

PROOF. The validity of these relations are clear from the definitions. The impossibility of other relations are shown by the following example.

Example 2.1. Let $g(x)$ be a density function which is positive for $0 < x < a < \infty$ and X have the density

$$f(x) = \begin{cases} pg(x/c)/c\ , & \text{if } x > 0\ , \\ qg(-x)\ , & \text{if } x < 0\ , \end{cases}$$

where $p, q > 0$, $p+q=1$ and $c > 0$. Then we have

$X \succ 0$ ($\mathscr{B}_0$) iff $p \geqq q$,

$X \succ 0$ ($\mathscr{B}_1$) iff $p \geqq q$ and $c \geqq 1$,

$X \succ 0$ ($\mathscr{B}_2$) iff $p \geqq cq$, $c \geqq 1$ and $g(x)$ is nonincreasing ,

$X \succ 0$ ($\mathscr{B}_3$) if $-\log g(e^x)$ is convex and $c \geqq 1$,

$X \succ 0$ ($\mathscr{B}_4$) if the conditions for $\mathscr{B}_2$ and $\mathscr{B}_3$ are satisfied .

PROPOSITION 2.3.

The condition that $X \succ 0$ ($\mathscr{B}_3$) and ($\mathscr{B}_2$) implies that $X \succ 0$ ($\mathscr{B}_4$).

The condition that $X \succ 0$ ($\mathscr{B}_3$) and ($\mathscr{B}_0$) implies that $X \succ 0$ ($\mathscr{B}_1$).

When a random variable X is positively biased, this fact is due to a factor that the probability of positive X is larger than that of negative X and/or a factor that X under the condition that $X > 0$ is stochastically larger than $-X$ under the condition that $X < 0$. $X \succ 0$ ($\mathscr{B}_0$) and ($\mathscr{B}_2$) corresponds to the first factor and $X \succ 0$ ($\mathscr{B}_3$) to the second factor. Proposition 2.3 tells that $X \succ 0$ ($\mathscr{B}_4$) and ($\mathscr{B}_1$) are mixtures of these two factors. This fact appears also in the distributions of rank statistics as will be stated in Propositions 4.3–4.6.

PROPOSITION 2.4. The condition that $X \succ 0$ ($\mathscr{B}_1$) and $X \prec 0$ ($\mathscr{B}_1$) implies X to be symmetric about zero.

PROPOSITION 2.5. If $h(t)$ is an increasing odd function, then $X \succ 0$ $(\mathscr{B}_i)$ implies $h(X) \succ 0$ $(\mathscr{B}_i)$ for each $i=1, 2, 3$ and 4.

Remark. For weaker definition the condition for the function $h(t)$ can be weakened. For example, assume only that $h(t)>0$ if $t>0$ and $h(t)<0$ if $t<0$, and then $X \succ 0$ $(\mathscr{B}_0)$ implies $h(X) \succ 0$ $(\mathscr{B}_0)$.

PROPOSITION 2.6. We assume that X is symmetric. If $h(t)>0$ for $t>0$, then $h(X) \succ 0$ $(\mathscr{B}_0)$. If $h(t)$, $-\infty<t<\infty$, is monotone and $h(t) \geqq -h(-t) \geqq 0$, then $h(X) \succ 0$ $(\mathscr{B}_1)$.

Example 2.2. (Shift of a symmetric distribution.) If $F(x)$ is a distribution which is symmetric about zero, then

(1) $F(x-\theta) \succ 0$ $(\mathscr{B}_1)$ for any $\theta \geqq 0$.

(2) If $F(x)$ is also unimodal, then $F(x-\theta) \succ 0$ $(\mathscr{B}_2)$ for any $\theta \geqq 0$. Conversely $F(x-\theta) \succ 0$ $(\mathscr{B}_2)$ for any $\theta \geqq 0$ implies the distribution to be unimodal.

(3) If $F(x)$ has the density function $f(x)$, then $F(x-\theta) \succ 0$ $(\mathscr{B}_4)$ iff $f(x-\theta)/f(x)$ is nondecreasing.

3. Stochastic ordering and positive biasedness

In a previous paper [5] the authors introduced various definitions which express that 'a component X of a random vector (X, Y) is larger than the other component Y'. In this section we express a positively biased random variable X in terms of the definitions in [5]. Some new types of positive biasedness are presented here. However only those which are defined in Section 2 seem important.

Let 0 be a random variable which is degenerate on zero. Then $(X, 0)$ is a random vector which is degenerate on the line $y=0$.

PROPOSITION 3.1.

$$X \succ 0 \ (\mathscr{R}_{\mathrm{I}}) \quad \text{iff } X \succ 0 \ (\mathscr{B}_1)\,,$$

$$X \succ 0 \ (\mathscr{R}_{\mathrm{II}}) \quad \text{iff } X \succ 0 \ (\mathscr{B}_2)\,,$$

and $X \succ 0$ $(\mathscr{R}_1, \mathscr{R}_2$ or $\mathscr{R}_3)$ means just $F(0-0)=0$. More generally, for a random vector (X, Y),

$$X \succ Y \ (\mathscr{R}_0) \quad \text{iff } X-Y \succ 0 \ (\mathscr{B}_0)\,,$$

$$X \succ Y \ (\mathscr{R}_{\mathrm{I}}) \quad \text{iff } X-Y \succ 0 \ (\mathscr{B}_1)\,,$$

and

$$X \succ Y \ (\mathscr{R}_{\mathrm{II}}) \quad \text{iff } X-Y \succ 0 \ (\mathscr{B}_2)\,.$$

For a given random variable X, $(X, -X)$ is a random vector which is degenerate on the line $x+y=0$.

PROPOSITION 3.2.

$X \succ -X$ ($\mathcal{R}_1$ or $\mathcal{R}_I$) iff $X \succ 0$ ($\mathcal{B}_1$), and $X \succ -X$ ($\mathcal{R}_2$, $\mathcal{R}_3$ or $\mathcal{R}_{II}$) iff $X \succ 0$ ($\mathcal{B}_2$).

For a given random variable X, let X' be a random variable with the same distribution with X such that X and X' are mutually independent.

PROPOSITION 3.3.

(1) $X \succ -X'$ ($\mathcal{R}_1$) iff $X \succ 0$ ($\mathcal{B}_1$).

(2) $X \succ -X'$ ($\mathcal{R}_2'$) iff $\dfrac{F(x)}{F(x+y)} \leqq \dfrac{1-F(-x-0)}{1-F(-x-y-0)}$ for any x and $y>0$.

(3) $X \succ -X'$ ($\mathcal{R}_2''$) iff $\dfrac{1-F(x+y)}{1-F(x)} \geqq \dfrac{F(-x-y-0)}{F(-x-0)}$ for any x and $y>0$.

(4) $X \succ -X'$ ($\mathcal{R}_3$) iff $X \succ 0$ ($\mathcal{B}_4$).

For a random variable X with the distribution $F(x)$ we define an independent symmetric random variable X_0 with the distribution function $(F(x)+1-F(-x-0))/2$, that is an equal probability mixture of X and $-X'$.

PROPOSITION 3.4. The following three conditions are equivalent for $i=1$, 2 and 3:
(1) $X \succ -X'$ ($\mathcal{R}_i$),
(2) $X \succ X_0$ ($\mathcal{R}_i$),
(3) There exists a random variable Z which is symmetric about zero and such that $X \succ Z$ ($\mathcal{R}_i$).

For a random variable X we define independent truncated distributions $X_{tp}=\max(X, 0)$ and $X_{tn}=\max(-X', 0)$.

PROPOSITION 3.5. $X_{tp} \succ X_{tn}$ ($\mathcal{R}_1$ or $\mathcal{R}_3$) are equivalent to $X \succ -X'$ ($\mathcal{R}_1$ or $\mathcal{R}_3$) respectively.

For a random variable X we define $X|_{X>0}$ and $-X'|_{X'<0}$ to be X conditioned that $X>0$ and $-X'$ conditioned that $X'<0$ respectively. They are regarded as degenerate at 0 if $P(X>0)=0$ or $P(X'<0)=0$ respectively.

PROPOSITION 3.6.

$$X|_{X>0} \succ -X'|_{X'<0} \ (\mathcal{R}_1) \quad \text{iff} \quad \frac{F(x)-F(0)}{1-F(0)} \leqq \frac{F(0-0)-F(-x-0)}{F(0-0)},$$

and

$$X|_{X>0} \succ -X'|_{X'<0} \ (\mathscr{R}_3) \qquad \text{iff } X \succ 0 \ (\mathscr{B}_3) \,.$$

4. Statistics for test of symmetry

In this section we assume $F(x)$ to be continuous. Let $X_1, \cdots, X_n$ be a random sample from the distribution $F(x)$, and $Y_{(1)} \leqq \cdots \leqq Y_{(n)}$ be the ordered set of $Y_i = |X_i|$, $i=1, \cdots, n$. $Z=(Z_1, \cdots, Z_n)$ is the rank order, i.e. $Z_j = 1(-1)$ if $Y_{(j)}$ corresponds to a positive (negative) X_i. For a sample point $x=(x_1, \cdots, x_n)$ we denote by $z(x)=(z_1, \cdots, z_n)$ the corresponding value of z.

Let Π_n be the set of all n-vectors with -1 or 1 as components. Savage [3] defined partial orderings S and L in Π_n. We use also the third ordering SL which is weaker than L and S. Let $u=(u_1, \cdots, u_n)$ and $v=(v_1, \cdots, v_n)$ be elements of Π_n.

DEFINITION 4.1.

Partial ordering S: uSv iff $u_i \geqq v_i$, $i=1, \cdots, n$.

Partial ordering L: uLv iff either (1) there exists a pair of integers (i, j), $i \leqq j$, such that $u_j = v_i = 1$, $u_i = v_j = -1$, and $u_k = v_k$ for all $k \neq i, j$, or (2) there exists a sequence $w^{(1)}, \cdots, w^{(k)}$ of elements of Π_n such that $uLw^{(1)}Lw^{(2)} \cdots w^{(k)}Lv$ for L's in the sense of (1).

Partial ordering SL: $uSLv$ iff $\sum_{j=i}^{n} u_j \geqq \sum_{j=i}^{n} v_j$, $i=1, \cdots, n$.

PROPOSITION 4.1. *Each of uSv or uLv implies $uSLv$. Coversely, if $uSLv$, then there exists a vector $w \in \Pi_n$ such that $uSwLv$.*

PROOF. Let $V = 1/2 \sum_{j=1}^{n} (v_j + 1)$ and let $j_1, \cdots, j_V$ be the subscripts of the last V components of u such that $u_{j_1} = \cdots = u_{j_V} = 1$, and then define w by

$$w_j = \begin{cases} 1\,, & j = j_1, \cdots, j_V\,, \\ -1\,, & \text{otherwise}\,. \end{cases}$$

Clearly uSw and $wSLv$. We can show wLv by constructing a sequence $\{w^{(m)}\}$ as follows. Put $w^{(1)} = w$, and to construct $w^{(m+1)}$ from $w^{(m)}$ find the largest j such that $(w_j^{(m)}, v_j) = (-1, 1)$. Then there exists an i such that $i > j$ and $(w_i^{(m)}, v_i) = (1, -1)$ because $w^{(m)}SLv$ by induction. We define

$$w_i^{(m+1)} = -1\,, \quad w_j^{(m+1)} = 1 \quad \text{and} \quad w_k^{(m+1)} = w_k^{(m)}\,, \qquad k \neq i, j\,,$$

then $w^{(m)}Lw^{(m+1)}$ and $w^{(m+1)}SLv$. If there are no i's of the above property, then $w^{(m)}=v$.

PROPOSITION 4.2. If $X\succ 0$ ($\mathscr{B}_2$) then $P(Z=u)\geqq P(Z=v)$ for any u and v such that uSv.

PROOF. It is sufficient to prove the case where $u_k=1$, $v_k=-1$ and $u_j=v_j$, $j\neq k$. Then, writing

$$I(x,u,k)=\begin{cases}1, & \text{if } 0<u_1x_1<\cdots<u_{k-1}x_{k-1}<u_{k+1}x_{k+1}<\cdots<u_nx_n,\\ 0, & \text{otherwise},\end{cases}$$

we have

$$P(Z=u)=n!E[I(X,u,k)P(|X_{k-1}|<X_k<|X_{k+1}|\,|\,|X_{k-1}|,|X_{k+1}|)],$$

and

$$P(Z=v)=n!E[I(X,u,k)P(|X_{k-1}|<-X_k<|X_{k+1}|\,|\,|X_{k-1}|,|X_{k+1}|)].$$

From the assumption of the proposition it holds that $P(a<X_k<b)\geqq P(a<-X_k<b)$, which completes the proof.

PROPOSITION 4.3. If $X\succ 0$ ($\mathscr{B}_3$) then $P(Z=u)\geqq P(Z=v)$ for any u and v such that uLv.

PROOF. It is sufficient to prove the case where $u_{k+1}=v_k=1$, $u_k=v_{k+1}=-1$ and $u_j=v_j$, $j\neq k$, $k+1$. Then writing

$$J(x,u,k)=\begin{cases}1, & \text{if } 0<u_1x_1<\cdots<u_{k-1}x_{k-1}<u_{k+1}x_{k+1}<\cdots<u_nx_n,\\ 0, & \text{otherwise},\end{cases}$$

we have

$$P(Z=u)=n!E[J(X,u,k)P(|X_{k-1}|<-X_k<X_{k+1}<|X_{k+2}|\,|\,|X_{k-1}|,|X_{k+2}|)],$$

and

$$P(Z=v)=n!E[J(X,u,k)P(|X_{k-1}|<X_k<-X_{k+1}<|X_{k+2}|\,|\,|X_{k-1}|,|X_{k+2}|)].$$

As $X\succ 0$ ($\mathscr{B}_3$),

$$(F(b)-F(t))(F(-t)-F(-t-\Delta t))\geqq(F(-t)-F(-b))(F(t+\Delta t)-F(t)).$$

Integrating with respect to t, we get

$$P(a<-X_k<X_{k+1}<b)\geqq P(a<X_k<-X_{k+1}<b),\qquad \text{for any } 0<a<b,$$

which completes the proof.

PROPOSITION 4.4 (Corollary of Propositions 4.2 and 4.3). If $X\succ 0$

($\mathscr{B}_1$), then $P(Z=u)\geqq P(Z=v)$ for any u and v such that $uSLv$.

DEFINITION 4.2. A rank statistic $r(\cdot)$, or a function $r(x)=r^*(z)$ of $z=(z_1,\cdots,z_n)$ is called nondecreasing if $r^*(u)\geqq r^*(v)$ whenever $uSLv$.

PROPOSITION 4.5. The following two conditions on a rank statistic $r(x)=r^*(z)$ are equivalent:
(1) $r(x)\geqq r(y)$ if $x_i\geqq y_i$, $i=1,2,\cdots,n$, and
(2) $r(x)$ is nondecreasing.

PROOF. Let $u=z(x)$ and $v=z(y)$ be the rank orders of x and y. Suppose $x_i=y_i$, $i=1,\cdots,n-1$ and $x_n>y_n$. If either $y_n>0$ or $x_n<0$, then uLv, and if $x_n>0>y_n$ then it is seen that $uSLv$. Repeating the argument, we see that $x_i\geqq y_i$, $i=1,\cdots,n$ implies $uSLv$. So we have (2)$\Rightarrow$(1).

(1)$\Rightarrow$(2): Let $x=(u_1,2u_2,\cdots,nu_n)$ and $y=(v_1,2v_2,\cdots,nv_n)$, which are typical points such that $u=z(x)$ and $v=z(y)$. Suppose uLv in the strict sense (1) of Definition 4.1. Then $ku_k\geqq\sigma(k)u_{\sigma(k)}$ for $k=1,\cdots,n$, where $\sigma(i)=j$, $\sigma(j)=i$ and $\sigma(k)=k$ for $k\neq i,j$. If uSv, then $ku_k\geqq kv_k$ for $k=1,\cdots,n$. Thus using Proposition 4.1, $uSLv$ implies that $r^*(u)=r(x)\geqq r(y)=r^*(v)$.

PROPOSITION 4.6. If $X\succ 0$ ($\mathscr{B}_1$) and a statistic $r(\cdot)$ is nondecreasing then $P(r(X)\geqq c)\geqq P_0(r(X)\geqq c)$ for any c, where P_0 denotes the probability when X is symmetric. Therefore, the test of symmetry against $X\succ 0$ ($\mathscr{B}_1$) (excluding the symmetry case) with a rejection region $r(x)\geqq c$ is unbiased provided that $r(x)$ is nondecreasing.

PROOF. Let $F_0(x)=(F(x)+1-F(-x))/2$. Then $F(x)\leqq F_0(x)$ and $F_0^{-1}\cdot(F(-x))\leqq x$ for any x, where $F_0^{-1}(u)=\inf\{x;F(x)\geqq u\}$. Hence using Proposition 4.5, $P(r(X)\geqq c)\geqq P(r(F_0^{-1}(F(X))\geqq c)=P_0(r(X)\geqq c)$.

PROPOSITION 4.7. A rank statistic of the form $r(x)=\sum_{i=1}^n a_iz_i$, $a_1\leqq\cdots\leqq a_n$, is nondecreasing. The examples are the sign test statistic $\sum_{i=1}^n z_i$, Wilcoxon's one sample test statistic $\sum_{i=1}^n iz_i$ and Normal score test statistic $\sum_{i=1}^n a_{in}z_i$, where a_{in} is the expectation of the ith order statistic of a sample of size n from the distribution with the density $\sqrt{2/\pi}\cdot\exp(-x^2/2)$, $x\geqq 0$.

Example 4.1. Let the distribution function $F(x)$ have the following form:

$$F(x)=\begin{cases} q+pG(x)\,, & x\geqq 0\,,\\ q(1-G(-x))\,, & x<0\,,\end{cases}$$

where $G(x)$ is a continuous distribution function such that $G(0)=0$, $p, q \geqq 0$ and $p+q=1$. Then $F(x) \succ 0$ and $F(x) \prec 0$ ($\mathscr{B}_3$), while $F(x) \succ 0$ ($\mathscr{B}_2$) if $p \geqq q$ and $F(x) \prec 0$ if $p \leqq q$. For such a distribution

$$P(Z=(z_1, \cdots, z_n) \mid y_{(1)}, \cdots, y_{(n)}) = p^s q^{n-s}$$

where $s=\sum_{i=1}^{n} z_i$. Therefore the sign test of symmetry against the alternative $p>q$ in the above form of $F(x)$ is the uniformly most powerful test.

Incidentally we shall show that the equiprobability of the rank statistic Z, that is $P(Z=z=(z_1, \cdots, z_n))=2^{-n}$ for any $z \in \Pi_n$, characterizes symmetry of the population distribution for $n \geqq 3$.

When $n=2$ the probability is equal iff

$$X|_{X>0} \succ -X'|_{X'<0} \quad \text{and} \quad X|_{X>0} \prec -X'|_{X'<0} \ (\mathscr{R}_0) .$$

When $n=3$ the equiprobability means symmetry of the distribution, since if the rank probabilities are equal, then

$$\begin{aligned}
&\int (1-F(x)-F(-x))^2 dF(x) \\
&= \int_0^\infty (1-F(x))^2 dF(x) - 2\int_0^\infty (1-F(x))F(-x) dF(x) \\
&\quad + \int_0^\infty F^2(-x) dF(x) + \int_{-\infty}^0 (1-F(-x))^2 dF(x) \\
&\quad - 2\int_{-\infty}^0 (1-F(-x))F(x) dF(x) + \int_{-\infty}^0 F^2(x) dF(x) \\
&= \frac{1}{3}\{P(Z=(1,1,1)) - P(Z=(1,1,-1) \text{ or } (1,-1,1)) \\
&\qquad + P(Z=(1,-1,-1)) + P(Z=(-1,1,1)) - P(Z=(-1,1,-1) \\
&\qquad \text{or } (-1,-1,1)) + P(Z=(-1,-1,-1))\} \\
&= 0 .
\end{aligned}$$

When $n \geqq 4$ the equiprobability implies symmetry a fortiori.

5. Positively more biased

As an extension of positive biasedness $\mathscr{B}_1$ we compare two random variables X and Y with the distribution function $F(x)$ and $G(x)$ respectively. We write

$$F_0(t) = \{F(t)+1-F(-t-0)\}/2 ,$$

$$G_0(t) = \{G(t)+1-G(-t-0)\}/2 .$$

DEFINITION 5.1.

$X \succ Y$ ($\mathscr{B}$) ('X is positively more biased than Y') iff $F_0(s) \leqq G_0(t)$ implies $F(s) \leqq G(t)$.

Remark. For the distribution function $H(t)$ let $H^{-1}(u) = \inf\{t; H(t) \geqq u\}$. For some purposes an alternative definition, is more convenient: $X \succ Y$ ($\mathscr{B}$) iff $G^{-1}(u) \geqq G_0^{-1}(v)$ implies $F^{-1}(u) \geqq F_0^{-1}(v)$.

PROPOSITION 5.1. If both X and Y are symmetric about zero, then $X \succ Y$ ($\mathscr{B}$) and $Y \succ X$ ($\mathscr{B}$).

PROPOSITION 5.2. If $h(\cdot)$ is an increasing odd function, then $X \succ h(X)$ ($\mathscr{B}$) and $h(X) \succ X$ ($\mathscr{B}$) for any X.

PROPOSITION 5.3. If Z is symmetric about zero, then $X \succ Z$ ($\mathscr{B}$) implies $X \succ 0$ ($\mathscr{B}_1$). Conversely if $X \succ 0$ ($\mathscr{B}_1$), then $X \succ X_0$ ($\mathscr{B}$), where the notation X_0 means the same as in Section 3.

PROPOSITION 5.4. If $X \succ Y$ ($\mathscr{R}_1$) then $X \succ Y$ ($\mathscr{B}$).

Remark. This proposition and the following observations show that 'positively more biased' is a weaker notion than 'stochastically larger.'

PROOF. Let the distribution functions of X and Y be $F(x)$ and $G(y)$ respectively. We prove the contraposition of Definition 5.1. As $F(t) \leqq G(t)$, $F(s) > G(t)$ implies $s > t$. Then $F(-s-0) \leqq G(-t-0)$, so that $F(s)+1-F(-s-0) > G(t)+1-G(-t-0)$.

PROPOSITION 5.5. (1) Let S be a random variable such that $P(S \geqq 0) = 1$. Then $S \succ X$ ($\mathscr{B}$) for any X.
(2) Let T be a random variable such that $P(T > 0) = 1$. Then $X \succ T$ ($\mathscr{B}$) iff $P(X \geqq 0) = 1$.
The dual statements on nonpositive and negative random variables are valid. Then for a random variables S_0 which is degenerate on zero,
(3) $S_0 \succ X$ ($\mathscr{B}$) and $X \succ S_0$ ($\mathscr{B}$) for any X.

PROPOSITION 5.6. If X and Y have continuous distribution functions, $X \succ Y$ ($\mathscr{B}$) and $r(\cdot)$ is a nondecreasing statistic, then $P^X(r(X) \geqq c) \geqq P^Y(r(Y) \geqq c)$ for any c.

PROOF. Firstly we remark that because of the continuity of the distribution functions $X \succ Y$ ($\mathscr{B}$) implies the alternative definition in Remark of Definition 5.1: If $G^{-1}(u) \geqq G_0^{-1}(v)$ then $F^{-1}(u) \geqq F_0^{-1}(v)$. (This fact is essentially stated in Proposition 5.2 of [4].)

As a rank statistic $r(x)$ is invariant for the transformation of x by an increasing odd function

$$P^X(r(X)\geqq c)=P^X(r(F_0(X)-1/2)\geqq c)$$
$$=P^U(r(F_0(F^{-1}(U))-1/2)\geqq c)\ ,$$

where $r(F_0(X)-1/2)=r(F_0(X_1)-1/2,\cdots,F_0(X_n)-1/2)$ and $U=(U_1,\cdots,U_n)$ is a random sample from the uniform distribution over the interval (0, 1). Similarly

$$P^Y(r(Y)\geqq c)=P^U(r(G_0(G^{-1}(u))-1/2)\geqq c)\ .$$

Then the problem reduces to that of proving

$$(*)\qquad F_0(F^{-1}(u))\geqq G_0(G^{-1}(u))\qquad \text{for any } u\ ,$$

because of Proposition 4.5. If there exists a value of v, against $(*)$, such that $F_0(F^{-1}(u))<v<G_0(G^{-1}(u))$, then $G_0^{-1}(v)<G^{-1}(u)$ and $F^{-1}(u)<F_0^{-1}(v)$, which is a contradiction to the preceding remark.

PROPOSITION 5.7 (Corollary of 5.4 and 5.6). If a one-parameter family of continuous distributions $\{F_\theta(x)\}$ is increasing $(\mathscr{R}_1)$ and $F_\theta(x)$ is symmetric for a parameter value θ_0, then the power function of a nondecreasing rank test of symmetry is nondecreasing in θ.

Example 5.1. Let $F(x)$ be a continuous distribution function of a symmetric random variable and X_i, $i=1,2$, have the distribution function $F((x-\mu_i)/\sigma_i)$. Suppose that the distribution range is unlimited in both directions. Then $X_1\succ X_2$ $(\mathscr{B})$ iff $\mu_1/\sigma_1\geqq\mu_2/\sigma_2$.

6. Test of symmetry of a two-dimensional distribution

Let (X, Y) be a random vector with the distribution function $F(x, y)$. We test the hypothesis that (X, Y) is symmetric,

$$H_0:\ F(x,y)=F(y,x)\qquad \text{for any } x \text{ and } y\ .$$

Lehmann [2] discussed this problem without specifying the alternative. We may specify alternatives as follows.

$$H_1:\ X\succ Y\ (\mathscr{R}_0)\ ,$$
$$H_2:\ X\succ Y\ (\mathscr{R}_{\mathrm{I}})\ ,$$
$$H_3:\ X\succ Y\ (\mathscr{R}_{\mathrm{II}})\ .$$

Our discussions on test of symmetry of a one-dimensional distribution can be applied here straightly because of the following proposition, which is a modification of Proposition 3.1.

PROPOSITION 6.1. The problems to test H_0 against H_1, H_2 or H_3 are equivalent to test the symmetry of $X-Y$ against $(X-Y)\succ 0$ ($\mathscr{B}_0$, $\mathscr{B}_1$ or $\mathscr{B}_2$) respectively.

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