ON THE DECOMPOSITION OF STABLE CHARACTERISTIC FUNCTIONS

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1. Decomposition of an infinitely divisible characteristic function (abbr. ch.f.)

$$(1)\quad \varphi(t)=\exp\left\{\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x)+\int_{-\infty}^0\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x)\right\}$$

into a product of infinitely divisible factors is nothing but a decomposition of the spectral functions $M(x)$ and $N(x)$ into sums of monotone non-decreasing functions. An infinitely divisible ch.f. (1) may, however, have a factor which is not infinitely divisible. All non-normal stable ch.f.'s are infinitely divisible and have non-infinitely divisible factors. The purpose of the present article is to give some of the factors, which are not necessarily infinitely divisible, of a stable ch.f. with the characteristic exponent α not greater than one.

2. The results of this section are based on the following well-known theorem due to Pólya [2].

THEOREM. *Let $\varphi(t)$ be a real valued continuous function defined for all real t and such that* (i) $\varphi(0)=1$, (ii) $\varphi(t)=\varphi(-t)$, (iii) $\lim_{t\to\infty}\varphi(t)=0$, *and* (iv) *$\varphi(t)$ is convex for $t>0$. Then $\varphi(t)$ is the ch.f. of an absolutely continuous distribution.*

We shall use the theorem in the following restricted but more convenient form.

COROLLARY 1. *Let $g(t)$ be a real valued twice differentiable even function such that $g(0)=0$, $\lim_{t\to\infty}g(t)=-\infty$, and $g''(t)+g'^2(t)\geqq 0$ for all $t>0$. Then $\varphi(t)=e^{g(t)}$ is a ch.f.*

From the corollary we can immediately derive the following

COROLLARY 2. *If $f(t)$ is a real valued continuous even function* *such that* $f(0)=0$, $f'(t)$ *and* $f''(t)$ *exist and bounded and* $f(t)=O(t)$ *as* $t\to\infty$. *Then for sufficiently large* $\lambda>0$,

$$\varphi(t)=\exp\{-\lambda|t|+f(t)\} \tag{2}$$

is a ch.f. and,

COROLLARY 3. *Let* p *be a positive integer and* $0<\alpha_1<\cdots<\alpha_p<1$ *be an arbitrary set of numbers. Let* $f_1(t),\cdots,f_p(t)$ *be a set of bounded real valued functions which have bounded first and second derivatives. Then for sufficiently large* $\lambda_1>0$ *and* $\lambda_p>0$,

$$\phi(t)=\exp\Big\{(-\lambda_1+f_1(\log|t|))|t|^{\alpha_1}+\sum_{k=2}^{p-1}f_k(\log|t|)|t|^{\alpha_k} \\ +(-\lambda_p+f_p(\log|t|))|t|^{\alpha_p}\Big\} . \tag{3}$$

is a ch.f.

Remark. Yu. V. Linnik [1] proved that if $0<\alpha_1<2$, $\alpha_1<\cdots<\alpha_p$ and if f_k's are constants with $-\lambda_p+f_p<0$, then for sufficiently large $\lambda_1>0$, (3) gives a ch.f. (Pôlya's theorem cannot be applied !). He also showed that if f_k's are polynomials, then it satisfies the relation of the form

$$\prod_{j=1}^{n}\varphi(a_jt)=\prod_{j=1}^{n}\varphi(b_jt) , \tag{4}$$

where a's and b's are real numbers such that

$$\sigma(x)=|a_1|^x+\cdots+|a_n|^x-|b_1|^x-\cdots-|b_n|^x\not\equiv 0 . \tag{5}$$

Though ch.f. $\varphi(t)$ given by the corollary 3 does not necessarily satisfy (4), we have the following

PROPOSITION. If f's are periodic functions with the common period $\rho>0$, then there exist sequences $\{a_j\}$ and $\{b_j\}$ of non-negative numbers less than 1 such that the series $\sum_1^\infty a_j^x$ and $\sum_1^\infty b_j^x$ are convergent for $x>0$, that $\varphi(t)$ satisfies the relation

$$\prod_{j=1}^{\infty}\varphi(a_jt)=\prod_{j=1}^{\infty}\varphi(b_jt) , \tag{6}$$

and that

$$\sigma(x)=\sum_{j=1}^{\infty}a_j^x-\sum_{j=1}^{\infty}b_j^x\not\equiv 0 . \tag{7}$$

If $e^{-\alpha_k\rho}$, $k=1,\cdots,p$ are all rational numbers, then (6) and (7) reduce to (4) and (5).

PROOF. Let $\xi=\xi(k)$ and $\eta=\eta(k)$ be positive integers such that

$$a=a(k)\equiv e^{-\alpha_k\rho}>b=b(k)\equiv 1-\xi a^{\eta}>0\ , \qquad k=1,\cdots,p\ .$$

Put $c'_{1,k}=0$, and

$$c'_{j,k}=\left[\left(b-\sum_{i=1}^{j-1}c'_{i,k}a^i\right)\Big/a^j\right]\ , \qquad j=2,\cdots\ .$$

where $[x]$ denotes the greatest integer not greater than x. Let $\{c_{j,k}\}$, $k=1,\cdots,p$ be sequences of integers defined by

$$c_{j,k}=\begin{cases}-1 & \text{if } j=0\\ c'_{j,k} & \text{if } j\geqq 1,\ j\neq\eta(k)\\ c'_{j,k}+\xi(k) & \text{if } j=\eta(k)\end{cases}$$

Then $c_{j,k}$, $1\leqq j$, $1\leqq k\leqq p$, are non-negative integers bounded by $\xi(k)+1/a(k)$ and we have $\sum_{j=1}^{\infty}c_{j,k}a^j(k)=1$. This means that $a(k)=e^{-\alpha_k\rho}$ is a zero of the series $g_k(x)=\sum_0^{\infty}c_{j,k}x^j$ which converges for $|x|<1$. Put $g(x)=\prod_1^p g_k(x)$ and represent it as

$$(8)\qquad g(x)=\sum_0^{\infty}l_jx^j-\sum_0^{\infty}m_jx^j\ ,$$

where l's and m's are non-negative integers, and put $l_{-1}=m_{-1}=1$ and

$$(9)\qquad a_{l_{n-1}+1}=\cdots=a_{l_{n-1}+l_n}=b_{m_{n-1}+1}=\cdots=b_{m_{n-1}+m_n}=e^{-n\rho}\ ,$$
$$n=0,1,2,\cdots\ .$$

Then we have for any $x>0$, $\sum_1^{\infty}a_j^x$ and $\sum_1^{\infty}b_j^x$ are convergent and

$$(10)\qquad \sigma(x)\equiv\sum_1^{\infty}a_j^x-\sum_1^{\infty}b_j^x=g(e^{-x\rho})\ .$$

Clearly $\sigma(\alpha_k)=0$, $k=1,\cdots,p$. It is not difficult to show that $\varphi(t)$ satisfies (6). When $e^{-\alpha_k\rho}$, is equal to a rational number r_k/r'_k, $k=1,\cdots,p$, we can use $g_k(x)=(r'_kx-r_k)$ in place of $\sum c_{j,k}x^j$, and (10) reduces to a finite sum. q.e.d.

Corollaries 2 and 3 enable us to give some decompositions of the stable ch.f.'s.

THEOREM 1. *Let $\xi(t)$ be a ch.f. without a zero. Suppose that $f(t)=\log|\xi(t)|$ has the bounded first and second derivatives for $t>0$. If $f(t)=O(t)$ as $t\to\infty$, then $\xi(t)$ is a factor of the Cauchy ch.f. $\varphi(t)=e^{-\lambda|t|}$, provided $\lambda>0$ is large. If $0<\alpha<1$, and if $f^{(k)}(t)=O(t^{-k+\alpha})$ as $t\to\infty$ $(k=0,1,2)$, then $\xi(t)$ is a factor of the symmetric stable ch.f. $\varphi(t)=e^{-\lambda|t|^{\alpha}}$ provided λ is large.*

PROOF. The stable ch.f. $\varphi(t)=e^{-\lambda|t|^{\alpha}}$ is formally decomposed as

$$\varphi(t)=\begin{cases} e^{-\lambda|t|+h(t)}\cdot\bar{\xi}(t)\cdot\xi(t)\ , & (\alpha=1) \\ e^{(-\lambda+h(\log|t|))|t|^{\alpha}}\cdot\bar{\xi}(t)\cdot\xi(t)\ , & (\alpha<1)\ , \end{cases} \tag{11}$$

where

$$h(t)=\begin{cases} -2f(t)\ , & (\alpha=1) \\ -2f(e^{t})e^{-\alpha t}\ , & (\alpha<1)\ . \end{cases} \tag{12}$$

Under the assumptions of the theorem, $h(t)$ has bounded derivatives and as $t\to\infty$ we have $h(t)=O(t)$ (when $\alpha=1$) and $h(t)=O(1)$ (when $\alpha<1$). The theorem is a consequence of the corollaries 2 and 3.

Example. For any real p such that $0<p<1$, $\xi(t)=p+(1-p)e^{-t^2/2}$ is the ch.f. of the distribution obtained by compounding the standard normal distribution and the distribution degenerate at zero. It is easy to see that it satisfies the conditions of the theorem. $\xi(t)$ is a factor of the symmetric stable ch.f. $e^{-\lambda|t|^{\alpha}}$, $0\leqq\alpha\leqq1$, if λ is large.

COROLLARY 4. *If $c(t)$ is the ch.f. of a distribution with the finite variance, then the ch.f. $\xi(t)=\exp\{\mu(c(t)-1)\}$, $\mu>0$ of the compound Poisson distribution is a factor of the Cauchy ch.f. $\varphi(t)=e^{-\lambda|t|}$, provided λ is sufficiently large. Especially the Poisson distribution is a factor of the Cauchy distribution.*

COROLLARY 5. *Every infinitely divisible ch.f. defined by* (1) *is a factor of a Cauchy ch.f., if as $|x|\to0$, $M(x)=O(x^{-1})$ and $N(x)=O(x^{-1})$, and if*

$$\int_0^{\infty}x^2\,dM(x)+\int_{-\infty}^{0}x^2\,dN(x)<\infty$$

COROLLARY 6. *Let $\xi(t)$ be the ch.f. of a distribution with the finite variance and suppose that $\inf_t|\xi(t)|>0$. Then it is a factor of a Cauchy ch.f.*

3. So far we have been concerned with the construction of symmetric ch.f.'s. The results of the preceeding section can be generalized using the following generalization of Pôlya's theorem.

THEOREM 2. *Let $\varphi(t)$ and $\psi(t)$ be real valued continuous functions which satisfy the conditions* (i) $\varphi(0)=1$, (ii) $1\geqq\varphi(t)=\varphi(-t)\geqq0$ *and* $\psi(t)=-\psi(-t)$ *for all real t,* (iii) *$\psi(t)$ is differentiable and absolutely integrable over* $(0,\infty)$, *and* (iv) *for all real c with* $|c|\leqq1$. $\varphi(t)+c\psi'(t)$ *and* $\varphi(t)+c\int_t^{\infty}\psi(\tau)\,d\tau$ *is convex on* $(0,\infty)$. *Then* $\varphi_0(t)=\varphi(t)+i\psi(t)$ *is the ch.f. of a* *distribution of which distribution function $F(x)$ has the form*

$$F(x)=p\cdot\epsilon(x)+qG(x) .$$

where $1\geqq p=1-q\geqq 0$, and $\epsilon(x)$ and $G(x)$ are the distribution functions, respectively of the distribution degenerate at $x=0$ and of an absolutely continuous distribution.

PROOF. Under the conditions of the theorem 2, $\varphi(t)$ is monotone non-increasing for $t>0$ and is bounded from below by 0, so that $p=\lim_{t\to\infty}\varphi(t)$ $(1\geqq p\geqq 0)$ exists. If $p=1$, then condition (iv) implies $\varphi(t)=\varphi_0(t)\equiv 1$. We therefore assume $p<1$. Put $\xi(t)=(\varphi(t)-p)/q$, $q=1-p$, or $\varphi(t)=p+q\xi(t)$. Then $\xi(t)$ satisfies the conditions of the Pólya theorem and hence is the ch.f. of an absolutely continuous distribution with the density function $p(x)$, say. Put, $r(x)=\frac{i}{2\pi q}\int_{-\infty}^{\infty}e^{-itx}\psi(t)\,dt$. Then we have for $x\neq 0$,

$$(13)\qquad p(x)+r(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\xi(t)+\frac{i}{q}\psi(t)\right)e^{-itx}\,dt$$
$$=\frac{1}{\pi}\left\{\int_0^{\infty}\xi(t)\cos tx\,dt+\int_0^{\infty}\frac{1}{q}\psi(t)\sin tx\,dt\right\}\qquad x\neq 0 ,$$

and

$$(14)\qquad \varphi_0(t)=p+q\int_{-\infty}^{\infty}(p(x)+r(x))\,e^{itx}\,dx .$$

Since $\int_{-\infty}^{\infty}(p(x)+r(x))\,dx=(\varphi_0(0)-p)/q=1$, we have only to show that $p(x)+r(x)\geqq 0$, for $x\neq 0$. We note first of all that $I_c(t)\equiv-\xi(t)-\frac{c}{q}\psi'(t)$ and $J_c(t)\equiv-\xi(t)-\frac{c}{q}\int_t^{\infty}\psi(\tau)\,d\tau$ are concave on $(-\infty,0)$ and on $(0,\infty)$ if $|c|\leqq 1$. Especially they have, for almost all t, the derivatives which are non-increasing on $(-\infty,0)$ and on $(0,\infty)$. When $0<|x|\leqq 1$, integrating the first term of (13) by parts, we obtain

$$(15)\qquad p(x)+r(x)=\frac{1}{\pi x}\int_0^{\infty}J_x'(t)\sin tx\,dt$$
$$=\frac{1}{\pi x}\sum_0^{\infty}\int_0^{\pi/x}\left\{J_x'\left(t+\frac{2n}{x}\pi\right)-J_x'\left(t+\frac{2n+1}{x}\pi\right)\right\}\sin tx\,dt\geqq 0$$

If $|x|\geqq 1$, integrating the second term of (13) by parts we have

$$(16)\qquad p(x)+r(x)=\frac{1}{\pi}\int_0^{\infty}I_{1/x}(t)\cos tx\,dt=\frac{1}{\pi x}\int_0^{\infty}I_{1/x}'(t)\sin tx\,dt$$

$$=\frac{1}{\pi x}\sum_0^\infty\int_0^{\pi/x}\left\{I'_{1/x}\left(t+\frac{2n}{x}\pi\right)-I'_{1/x}\left(t+\frac{2n+1}{x}\pi\right)\right\}\sin tx\,dt\geqq 0$$

q.e.d.

We can use the theorem to derive

THEOREM 3. *Let $\xi_0(t)$ and $\eta_0(t)$ be real valued and, respectively, even and odd functions which are differentiable three times. Suppose $\xi_0^{(k)}(t)$, $k=1, 2$ and 3, and $\eta_0^{(k)}(t)$, $k=0, 1, 2, 3$ and 4 are bounded and that $\xi_0(t)=O(t)$ as $t\to\infty$. Then for sufficiently large $\lambda_0>0$,*

$$\varphi_0(t)=\exp\{-\lambda_0|t|+\xi_0(t)+i\eta_0(t)\}$$

is a ch.f.

PROOF. Choose a positive integer k_0 so large that $|\eta_0(t)/k_0|\leqq\pi/3$ for all t. Let $k(\geqq k_0)$ be an integer and $\lambda>0$ be a large number, both being determined later. Put $\xi(t)=\xi_0(t)/k$, $\eta(t)=\eta_0(t)/k$, $\varphi(t)=e^{-\lambda|t|+\xi(t)}\cdot\cos\eta(t)$, $\psi(t)=e^{-\lambda|t|+\xi(t)}\cdot\sin\eta(t)$, $I_c(t)=\varphi(t)+c\psi'(t)$ and $J_c(t)=\varphi(t)+c\int_t^\infty\psi(\tau)\,d\tau$.

We have to prove that $I_c''(t)\geqq 0$ and $J_c''(t)\geqq 0$, for, $|c|\leqq 1$, $t>0$. Differentiating we have

$$I_c''(t)=\{a(t)\cos\eta(t)+b(t)\sin\eta(t)\}\,e^{-\lambda|t|+\xi(t)}\ ,$$

and

$$J_c''(t)=\{c(t)\cos\eta(t)+d(t)\sin\eta(t)\}\,e^{-\lambda|t|+\xi(t)}\ .$$

where

$$a(t)=\{(1+3c\eta')(-\lambda+\xi')^2+3c\eta''(-\lambda+\xi')+\xi''-\eta'^2+3c\eta'\xi''+c\eta'''-c\eta'^3\}\ ,$$

$$b(t)=\{c(-\lambda+\xi')^3-(2\eta'+3c\eta'^2-3c\xi'')(-\lambda+\xi')+c\xi'''-3c\eta'\eta''\}\ ,$$

$$c(t)=(-\lambda+\xi')^2+\xi''-\eta'^2-c\eta'\ ,\quad\text{and}\quad d(t)=-(2\eta'+c)(-\lambda+\xi')-\eta''\ .$$

Let $\lambda>0$ be so large that $a(t)\geqq 1$ and $c(t)\geqq 1$ hold for all t, $k\geqq k_0$ and $|c|\leqq 1$. Since $\cos\eta(t)\geqq\frac{\sqrt{3}}{2}>\frac{1}{2}\geqq|\sin\eta(t)|$, and since $b(t)$ and $d(t)$ are bounded, we can take k large enough to make

$$I_c''(t)\geqq(\cos\eta(t)-|b(t)|\sin\eta(t))\,e^{-\lambda|t|+\xi(t)}>0\ ,$$

and

$$J_c''(t)\geqq(\cos\eta(t)-|a(t)|\sin\eta(t))\,e^{-\lambda|t|+\xi(t)}>0\ .$$

By the theorem 2,

$$\varphi_k(t)=\varphi(t)+i\psi(t)=\exp\{-\lambda|t|+\xi(t)+i\eta(t)\}$$
$$=\exp\{-\lambda|t|+\xi_0(t)/k+i\eta_0(t)/k\}$$

is a ch.f.. Since k is a positive integer,

$$\varphi_0(t)=\exp\{-\lambda_0|t|+\xi_0(t)+i\eta_0(t)\}\ , \qquad \lambda_0\geqq\lambda k$$

is also a ch.f. q.e.d.

COROLLARY. *Suppose* $\eta_0(t)$ *satisfies the conditions of Theorem* 3. *Then for sufficiently large* $\lambda_0>0$,

$$\varphi(t)=e^{-\lambda_0|t|+i\eta_0(t)}$$

is a ch.f. for which $|\varphi(t)|=e^{-\lambda_0|t|}$.

REFERENCES

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