BAYESIAN ESTIMATES OF PARAMETERS IN SOME QUEUEING MODELS

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(Received Nov. 14, 1969; revised Nov. 2, 1970)

1. Introduction and summary

A maximum likelihood procedure for estimating λ and μ, the input and service parameters, respectively, in a single channel Poisson input and exponential holding time, first come first served queue in equilibrium, has been derived by A. B. Clarke [1]. In general, to estimate λ, one can observe the operation for a time of length T; if n persons arrive, then one has $\hat{\lambda}=n/T$. And if m persons are served and the total busy time of the service channel is τ, then $\hat{\mu}=m/\tau$. If ρ is the traffic intensity and if $\rho\geqq 1$ then the above estimates are the best one can find; otherwise if $\rho<1$ it is possible to use the initial queue size to improve the above estimates for the queue in equilibrium.

Clarke's method can be outlined as follows. Let $m>i$, i being the initial queue size, and suppose that the busy time of the channel is observed until a preassigned value is reached; let T be the time at which the mth departure occurs. Let the random variables X_j, Y_j, Z_j, give the jth arrival time, and jth departure time, and the busy time of the channel up to the jth departure (all vanishing for $j\leqq 0$), respectively. Then we have recursively

$$(1) \qquad Y_j=\max(Y_{j-1}, X_{j-i})+Z_j-Z_{j-1}\ .$$

Knowing the initial number i and the X_j and Z_j (independent of X_j), and assuming the initial queue size i has a geometric distribution $(1-\rho)\rho^i$ $(i=0, 1, 2, \cdots)$ which is of course the equilibrium solution of the queue size, he got the likelihood as

$$(2) \qquad L=K\left(1-\frac{\lambda}{\mu}\right)\mu^{m-i}\lambda^{n+i}e^{-\mu\tau-\lambda T}$$

where K is a function which does not depend on λ or μ. Differentiating the likelihood and solving, Clarke gets maximum likelihood estimate of ρ the traffic intensity as the roots of the quadratic equation in ρ, one being equal to unity and other as

$$\hat{\rho}_1=\frac{n+i}{m-i}\,\frac{\tau}{T} \tag{3}$$

$$\hat{\lambda}=\frac{(n+m)\hat{\rho}}{\hat{\rho}T+\tau} \tag{4}$$

and

$$\hat{\mu}=\frac{n+m}{\hat{\rho}T+\tau} \tag{5}$$

whenever the approximation of ρ_1 for ρ is valid simple approximations for λ and μ are

$$\hat{\lambda}\approx\frac{n+i}{T} \tag{6}$$

$$\hat{\mu}\approx\frac{m-i}{\tau}\,. \tag{7}$$

Here in this paper we have found the Bayesian estimates of λ, μ and ρ using the same likelihood given by Clarke and assuming some prior distributions for the parameters λ and μ.

2. Derivation of the results for $M/M/1$

(i) *Using natural-conjugate prior density*

Now the likelihood is

$$L=\left(1-\frac{\lambda}{\mu}\right)e^{-\mu\tau-\lambda T}\mu^{m-i}\lambda^{n+i}K\,.$$

Hence the kernel of the likelihood is (refer Raiffa and Schlaifer [2])

$$=\left(1-\frac{\lambda}{\mu}\right)e^{-\mu\tau-\lambda T}\mu^{m-i}\lambda^{n+i}\,. \tag{8}$$

Then the natural conjugate prior density of λ and μ may be taken as

$$D'(\lambda,\ \mu/\tau',\ T')\propto\left(1-\frac{\lambda}{\mu}\right)e^{-\mu\tau'-\lambda T'}\mu^{m'}\lambda^{n'}\,. \tag{9}$$

Hence the posterior joint density of λ and μ is

$$D''(\lambda,\ \mu/\tau_1,\ T_1)\propto\left(1-\frac{\lambda}{\mu}\right)^2 e^{\mu\tau_1-\lambda T_1}\mu^{m_1}\lambda^{n_1}\,. \tag{10}$$

Where

$$\tau_1=\tau+\tau'\,,\qquad T_1=T+T'\,,\qquad m_1=m+m'-i\,,\qquad n_1=n+n'+i\,.$$

The normalising constant k for $D''(\lambda, \mu/\tau_1, T_1)$ is given by

$$k\int_0^\infty\int_0^\infty D''(\lambda, \mu/\tau_1, T_1)d\lambda d\mu=1$$

(11)

$$\text{i.e.,}\quad \frac{1}{k}=\frac{\Gamma(m_1-1)\Gamma(n_1+1)}{\tau_1^{m_1+1}T_1^{n_1+3}}[m_1(m_1-1)T_1^2 -2(m_1-1)(n_1+1)\tau_1T_1+(n_1+1)(n_1+2)\tau_1^2]\,.$$

Bayesian estimates of λ, μ and ρ

Using the posterior density of λ, μ we can find the expected values of λ, μ and ρ which are ultimately Bayesian estimates of λ, μ and ρ. They are as follows.

(12)
$$E^*(\lambda)=\frac{n_1+1}{T_1}\cdot\left[\frac{m_1(m_1-1)T_1^2-2(m_1-1)(n_1+2)\tau_1T_1+(n_1+2)(n_1+3)\tau_1^2}{m_1(m_1-1)T_1^2-2(m_1-1)(n_1+1)\tau_1T_1+(n_1+1)(n_1+2)\tau_1^2}\right]$$

(13)
$$E^*(\mu)=\frac{m_1-1}{\tau_1}\cdot\left[\frac{m_1(m_1+1)T_1^2-2m_1(n_1+1)\tau_1T_1+(n_1+1)(n_1+2)\tau_1^2}{m_1(m_1+1)T_1^2-2(m_1-1)(n_1+1)\tau_1T_1+(n_1+1)(n_1+2)\tau_1^2}\right]$$

(14)
$$E^*(\lambda/\mu)=E^*(\rho)=\frac{n_1+1}{m_1-2}\,\frac{\tau_1}{T_1}\cdot\left[\frac{(m_1-1)(m_1-2)T_1^2-2(m-1)(n_1+2)\tau_1T_1+(n+2)(n_1+3)\tau_1^2}{m_1(m_1-1)T_1^2-2(m_1-1)(n_1+1)\tau_1T_1+(n_1+1)(n_1+2)\tau_1^2}\right].$$

(ii) *Taking the prior density of λ and μ as of Gamma type*

Instead of considering the natural conjugate prior density for the parameters based on the kernel of the likelihood we may consider the prior densities of λ and μ as gamma variates with parameters (T', n'), (τ', m') and assuming λ and μ as independently distributed variates, the kernel of their joint density is

(15)
$$=e^{-\mu\tau'-\lambda T'}\lambda^{n'-1}\mu^{m'-1}\,.$$

Then the posterior joint density of λ and μ is

(16)
$$D''(\lambda, \mu/\tau_1, T_1)\propto\left(1-\frac{\lambda}{\mu}\right)e^{-\mu\tau_1-\lambda T_1}\mu^{m_1}\lambda^{n_1}$$

where

$$\tau_1=\tau+\tau', \qquad T_1=T+T',$$
$$m_1=m+m'-i-1, \qquad n_1=n+n'+i-1.$$

The normalising constant k is given by the integral

(17)
$$k\int_0^\infty\int_0^\infty D''(\lambda, \mu/\tau_1, T_1)d\lambda d\mu=1$$
$$\text{i.e.,}\quad \frac{1}{k}=\frac{\Gamma m_1\Gamma(n_1+1)}{\tau_1^{m_1+1}T_1^{n_1+2}}[m_1T_1-(n_1+1)\tau_1].$$

Bayesian estimates of λ, μ and ρ

Expected values of λ, μ and ρ can be found by using the posterior density of λ and μ, and they are

(18)
$$E^*(\lambda)=\frac{n_1+1}{T_1}\left[\frac{m_1T_1-(n_1+2)\tau_1}{m_1T_1-(n_1+1)\tau_1}\right]$$

(19)
$$E^*(\mu)=\frac{m_1}{\tau_1}\left[\frac{(m_1+1)T_1-(n_1+1)\tau_1}{m_1T_1-(n_1+1)\tau_1}\right]$$

(20)
$$E^*(\rho)=\frac{n_1+1}{m_1-1}\,\frac{\tau_1}{T_1}\left[\frac{(m_1-1)T_1-(n_1+2)\tau_1}{m_1T_1-(n_1+1)\tau_1}\right]$$

which are ultimately Bayesian estimates of λ, μ and ρ.

3. Derivation of the results for $M/M/\infty$

Following the same steps given by Clarke, and assuming the initial queue size i has a Poisson distribution $e^{-\rho}\rho^i/i!$ $(i=0, 1, 2, \cdots)$ which is the equilibrium solution of the queue size in this system $M/M/\infty$ one can write the likelihood equation for this case as

(21)
$$L=e^{-\rho-\mu\tau-\lambda T}\mu^{m-i}\lambda^{n+i}K.$$

Before going for the Bayesian estimates one can have the maximum likelihood estimates of λ, μ and ρ as follows.

(22)
$$\hat{\lambda}=\frac{n+i}{1/\hat{\mu}+T}$$

(23)
$$\hat{\mu}=\frac{\tau-T(m-i)\pm\sqrt{[\tau-T(m-i)]^2+4\tau T(n+m)}}{2\tau T}$$

and

(24)
$$\hat{\rho}=\frac{2(n+i)\tau}{2\tau-T(m-i)\pm\sqrt{[\tau-T(m-i)]^2+4\tau T(n+m)}}.$$

Bayesian estimates of λ, μ and ρ

Now the kernel of the likelihood is

$$=e^{-\rho-\mu\tau-\lambda T}\mu^{m-i}\lambda^{n+i} . \tag{25}$$

The conjugate prior density is

$$D'(\lambda, \mu/\tau', T')\propto e^{-\rho-\mu\tau'-\lambda T'}\mu^{m'}\lambda^{n'} . \tag{26}$$

Hence the posterior density is

$$D''(\lambda, \mu/\tau_1, T_1)\propto e^{-2\rho-\mu\tau_1-\lambda T_1}\mu^{m_1}\lambda^{n_1} \tag{27}$$

where

$$\tau_1=\tau+\tau' , \qquad T_1=T+T' , \qquad m_1=m+m'-i , \qquad n_1=n+n'+i .$$

The normalising constant k for $D''(\lambda, \mu/\tau_1, T_1)$ is given by the integral

$$\begin{aligned} &k\int_0^\infty\int_0^\infty D''(\lambda, \mu/\tau_1, T_1)d\lambda d\mu=1 \\ \text{i.e.,}\quad &\frac{1}{k}=\frac{\Gamma(n_1+1)}{\tau_1^{m_1+1}T_1^{n_1+1}}\sum_{r=0}^{\infty}\binom{n_1+r+1}{r}\left(-\frac{2\tau_1}{T_1}\right)^r\Gamma(m_1-r+1) . \end{aligned} \tag{28}$$

Hence the Bayesian estimates are given by the following expectations,

$$E^*(\lambda)=\frac{n_1+1}{T_1}\frac{\sum_{r=0}^{\infty}\binom{n_1+r+2}{r}\left(-\frac{2\tau_1}{T_1}\right)^r\Gamma(m_1-r+1)}{\sum_{r=0}^{\infty}\binom{n_1+r+1}{r}\left(-\frac{2\tau_1}{T_1}\right)^r\Gamma(m_1-r+1)} \tag{29}$$

$$E^*(\mu)=\frac{1}{\tau_1}\frac{\sum_{r=0}^{\infty}\binom{n_1+r+1}{r}\left(-\frac{2\tau_1}{T_1}\right)^r\Gamma(m_1-r+2)}{\sum_{r=0}^{\infty}\binom{n_1+r+1}{r}\left(-\frac{2\tau_1}{T_1}\right)^r\Gamma(m_1-r+1)} \tag{30}$$

$$E^*(\rho)=(n_1+1)\frac{\tau_1}{T_1}\frac{\sum_{r=0}^{\infty}\binom{n_1+r+2}{r}\left(-\frac{2\tau_1}{T_1}\right)^r\Gamma(m_1-r)}{\sum_{r=0}^{\infty}\binom{n_1+r+1}{r}\left(-\frac{2\tau_1}{T_1}\right)^r\Gamma(m_1-r+1)} . \tag{31}$$

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