BAYESIAN ESTIMATES OF PARAMETERS IN SOME QUEUEING MODELS

M. V. MUDDAFUR

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1. Introduction and summary

A maximum likelihood procedure for estimating \( \lambda \) and \( \mu \), the input and service parameters, respectively, in a single channel Poisson input and exponential holding time, first come first served queue in equilibrium, has been derived by A. B. Clarke [1]. In general, to estimate \( \lambda \), one can observe the operation for a time of length \( T \); if \( n \) persons arrive, then one has \( \hat{\lambda} = n/T \). And if \( m \) persons are served and the total busy time of the service channel is \( \tau \), then \( \hat{\mu} = m/\tau \). If \( \rho \) is the traffic intensity and if \( \rho \geq 1 \) then the above estimates are the best one can find; otherwise if \( \rho < 1 \) it is possible to use the initial queue size to improve the above estimates for the queue in equilibrium.

Clarke's method can be outlined as follows. Let \( m \geq i \), \( i \) being the initial queue size, and suppose that the busy time of the channel is observed until a preassigned value is reached; let \( T \) be the time at which the \( m \)th departure occurs. Let the random variables \( X_j, Y_j, Z_j \), give the \( j \)th arrival time, and \( j \)th departure time, and the busy time of the channel up to the \( j \)th departure (all vanishing for \( j \leq 0 \)), respectively. Then we have recursively

\[
Y_j = \max (Y_{j-1}, X_{j-1}) + Z_j - Z_{j-1}.
\]

Knowing the initial number \( i \) and the \( X_j \) and \( Z_j \) (independent of \( X_j \)), and assuming the initial queue size \( i \) has a geometric distribution \((1 - \rho)^i \rho^i \) \((i=0, 1, 2, \ldots)\) which is of course the equilibrium solution of the queue size, he got the likelihood as

\[
L = K \left( 1 - \frac{\lambda}{\mu} \right) \mu^{m-i} \lambda^i e^{-\mu i - \lambda \tau}
\]

where \( K \) is a function which does not depend on \( \lambda \) or \( \mu \). Differentiating the likelihood and solving, Clarke gets maximum likelihood estimate of \( \rho \) the traffic intensity as the roots of the quadratic equation in \( \rho \),

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one being equal to unity and other as

\[ \hat{\rho}_i = \frac{n+i}{m-i} \frac{\tau}{T} \tag{3} \]

\[ \hat{\lambda} = \frac{(n+m)\hat{\rho}}{\hat{\rho}T+\tau} \tag{4} \]

and

\[ \hat{\mu} = \frac{n+m}{\hat{\rho}T+\tau} \tag{5} \]

whenever the approximation of \( \rho_i \) for \( \rho \) is valid simple approximations for \( \lambda \) and \( \mu \) are

\[ \hat{\lambda} \approx \frac{n+i}{T} \tag{6} \]

\[ \hat{\mu} \approx \frac{m-i}{\tau}. \tag{7} \]

Here in this paper we have found the Bayesian estimates of \( \lambda, \mu \) and \( \rho \) using the same likelihood given by Clarke and assuming some prior distributions for the parameters \( \lambda \) and \( \mu \).

2. Derivation of the results for \( M/M/1 \)

(i) Using natural-conjugate prior density

Now the likelihood is

\[ L = \left(1 - \frac{\lambda}{\mu}\right) e^{-\mu' - 2\tau} \mu x_{n-1}^\mu \lambda^{n+1} K. \]

Hence the kernel of the likelihood is (refer Raiffa and Schlaifer [2])

\[ = \left(1 - \frac{\lambda}{\mu}\right) e^{-\mu' - 2\tau} \mu x_{n-1}^\mu \lambda^{n+1}. \tag{8} \]

Then the natural conjugate prior density of \( \lambda \) and \( \mu \) may be taken as

\[ D'(\lambda, \mu|\tau', T') \propto \left(1 - \frac{\lambda}{\mu}\right) e^{-\mu' - 2\tau} \mu x_{n-1}^\mu \lambda^{n+1}. \tag{9} \]

Hence the posterior joint density of \( \lambda \) and \( \mu \) is

\[ D''(\lambda, \mu|\tau_1, T_1) \propto \left(1 - \frac{\lambda}{\mu}\right)^2 e^{\mu' - 2\tau} \mu x_{n-1}^\mu \lambda^{n+1}. \tag{10} \]

Where
\[ \tau_1 = \tau + \tau', \quad T_i = T + T', \quad m_i = m + m' - i, \quad n_i = n + n' + i. \]

The normalising constant \( k \) for \( D''(\lambda, \mu/\tau_1, T_i) \) is given by

\[
k \int_0^{\infty} \int_0^{\infty} D''(\lambda, \mu/\tau_1, T_i) d\lambda d\mu = 1
\]

(11)\]

\[
i.e., \quad \frac{1}{k} = \frac{\Gamma(m_i - 1) \Gamma(n_i + 1)}{\tau_1^{m_i + 1} T_i^{n_i + 3}} \left[ m_i(m_i - 1) T_i^2 + 2(m_i - 1)(n_i + 2) \tau_1 T_i + (n_i + 1)(n_i + 2) \tau_1^2 \right].
\]

**Bayesian estimates of \( \lambda, \mu \) and \( \rho \)**

Using the posterior density of \( \lambda, \mu \) we can find the expected values of \( \lambda, \mu \) and \( \rho \) which are ultimately Bayesian estimates of \( \lambda, \mu \) and \( \rho \). They are as follows.

\[
E^*(\lambda) = \frac{n_i + 1}{T_i} \left[ m_i(m_i - 1) T_i^2 + 2(m_i - 1)(n_i + 2) \tau_1 T_i + (n_i + 1)(n_i + 3) \tau_1^2 \right].
\]

(12)\]

\[
E^*(\mu) = \frac{m_i - 1}{\tau_1} \left[ m_i(m_i + 1) T_i^2 + 2m_i(n_i + 1) \tau_1 T_i + (n_i + 1)(n_i + 2) \tau_1^2 \right].
\]

(13)\]

\[
E^*(\lambda/\mu) = E^*(\rho) = \frac{n_i + 1}{m_i - 2} \frac{\tau_1}{T_i} \left[ (m_i - 1)(m_i - 2) T_i^2 + 2(m_i - 1)(n_i + 2) \tau_1 T_i + (n_i + 1)(n_i + 3) \tau_1^2 \right].
\]

(14)\]

**Taking the prior density of \( \lambda \) and \( \mu \) as of Gamma type**

Instead of considering the natural conjugate prior density for the parameters based on the kernel of the likelihood we may consider the prior densities of \( \lambda \) and \( \mu \) as gamma variates with parameters \( (T', n') \), \( (\tau', m') \) and assuming \( \lambda \) and \( \mu \) as independently distributed variates, the kernel of their joint density is

\[
e^{-m' - 1 \tau'} \lambda^{n' - 1} \mu^{m' - 1}.
\]

Then the posterior joint density of \( \lambda \) and \( \mu \) is

\[
D''(\lambda, \mu/\tau_1, T_i) \propto \left(1 - \frac{\lambda}{\mu}\right) e^{-m_1 - 1 \tau_1} \mu^{n_1} \lambda^{n_1}
\]

where
\[ \tau_n = \tau + \tau', \quad T_n = T + T', \]
\[ m_n = m + m' - i - 1, \quad n_n = n + n' + i - 1. \]

The normalising constant \( k \) is given by the integral
\[
k \int_{0}^{\infty} \int_{0}^{\infty} D'(\lambda, \mu; \tau_1, T_1) d\lambda d\mu = 1
\]
(17)

i.e.,
\[
\frac{1}{k} = \frac{I_m I_{m'}(n_1 + 1)}{\tau_{n+1}^{m+1} T_{n+1}^{m+1}} [m_1 T_1 - (n_1 + 1) \tau_1].
\]

**Bayesian estimates of \( \lambda, \mu \) and \( \rho \)**

Expected values of \( \lambda, \mu \) and \( \rho \) can be found by using the posterior density of \( \lambda \) and \( \mu \), and they are

\[
E^*(\lambda) = \frac{n_1 + 1}{T_1} \left[ \frac{m_1 T_1 - (n_1 + 2) \tau_1}{m_1 T_1 - (n_1 + 1) \tau_1} \right]
\]
(18)

\[
E^*(\mu) = \frac{m_1}{\tau_1} \left[ \frac{(m_1 + 1) T_1 - (n_1 + 1) \tau_1}{m_1 T_1 - (n_1 + 1) \tau_1} \right]
\]
(19)

\[
E^*(\rho) = \frac{n_1 + 1}{m_1 - 1} \frac{\tau_1}{T_1} \left[ \frac{(m_1 - 1) T_1 - (n_1 + 2) \tau_1}{m_1 T_1 - (n_1 + 1) \tau_1} \right]
\]
(20)

which are ultimately Bayesian estimates of \( \lambda, \mu \) and \( \rho \).

3. **Derivation of the results for \( M/M/\infty \)**

Following the same steps given by Clarke, and assuming the initial queue size \( i \) has a Poisson distribution \( e^{-\rho} \rho^i / i! \) \((i = 0, 1, 2, \cdots)\) which is the equilibrium solution of the queue size in this system \( M/M/\infty \) one can write the likelihood equation for this case as

\[
L = e^{-\tau - \tau \rho - i \tau} \mu^{-i} \lambda^i K.
\]
(21)

Before going for the Bayesian estimates one can have the maximum likelihood estimates of \( \lambda, \mu \) and \( \rho \) as follows.

\[
\hat{\lambda} = \frac{n + i}{1/\hat{\mu} + T}
\]
(22)

\[
\hat{\mu} = \frac{\tau - T(m - i) \pm \sqrt{[\tau - T(m - i)]^2 + 4\tau T(n + m)}}{2\tau T}
\]
(23)

and

\[
\hat{\rho} = \frac{2(n + i)\tau}{2\tau - T(m - i) \pm \sqrt{[\tau - T(m - i)]^2 + 4\tau T(n + m)}}.
\]
(24)
Bayesian estimates of $\lambda$, $\mu$ and $\rho$

Now the kernel of the likelihood is

$$= e^{-s-m-i\tau} \mu^{m-i} \lambda^{n+i}.$$

The conjugate prior density is

$$D'(\lambda, \mu/\tau', T') \propto e^{-s-m' + i\tau'} \mu^{m'} \lambda^{n'}.$$

Hence the posterior density is

$$D''(\lambda, \mu/\tau_1, T_1) \propto e^{-s-m'+i\tau_1} \mu^{m} \lambda^{n_1}$$

where

$$\tau_1 = \tau + \tau', \quad T_1 = T + T', \quad m_1 = m + m' - i, \quad n_1 = n + n' + i.$$

The normalising constant $k$ for $D''(\lambda, \mu/\tau_1, T_1)$ is given by the integral

$$k \int_0^\infty \int_0^\infty D''(\lambda, \mu/\tau_1, T_1) d\lambda d\mu = 1$$

(28)

i.e.,

$$\frac{1}{k} = \frac{\Gamma(n_1+1)}{\tau_1^{n_1+1} T_1^{n_1+1}} \sum_{r=0}^\infty \binom{n_1+r+1}{r} \left( -\frac{2\tau_1}{T_1} \right)^r \Gamma(m_1-r+1).$$

Hence the Bayesian estimates are given by the following expectations,

$$E^*(\lambda) = \frac{n_1+1}{T_1} \frac{\sum_{r=0}^\infty \binom{n_1+r+2}{r} \left( -\frac{2\tau_1}{T_1} \right)^r \Gamma(m_1-r+1)}{\sum_{r=0}^\infty \binom{n_1+r+1}{r} \left( -\frac{2\tau_1}{T_1} \right)^r \Gamma(m_1-r+1)}$$

(29)

$$E^*(\mu) = \frac{1}{\tau_1} \frac{\sum_{r=0}^\infty \binom{n_1+r+1}{r} \left( -\frac{2\tau_1}{T_1} \right)^r \Gamma(m_1-r+2)}{\sum_{r=0}^\infty \binom{n_1+r+1}{r} \left( -\frac{2\tau_1}{T_1} \right)^r \Gamma(m_1-r+1)}$$

(30)

$$E^*(\rho) = (n_1+1) \frac{\tau_1}{T_1} \frac{\sum_{r=0}^\infty \binom{n_1+r+2}{r} \left( -\frac{2\tau_1}{T_1} \right)^r \Gamma(m_1-r)}{\sum_{r=0}^\infty \binom{n_1+r+1}{r} \left( -\frac{2\tau_1}{T_1} \right)^r \Gamma(m_1-r+1)}.$$