ASYMPTOTICALLY EFFICIENT ESTIMATION BY LOCAL LOCATION-PARAMETER APPROXIMATIONS*

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1. Introduction

Let $F(x, \theta)$ be a family of distribution functions indexed by a parameter θ belonging to an open subset Ω of the real line. If this family has density $f(x, \theta)$, the information integral is defined as

$$I(\theta)=\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\log f(x, \theta)\right]^2 f(x, \theta)dx .$$

Supposing $X_1, \cdots, X_n$ to be a random sample from $F(x, \theta)$, a sequence of estimators $T_n=T_n(X_1, \cdots, X_n)$ of θ is said to be *asymptotically efficient* (AE) if

$$\mathcal{L}\{\sqrt{n}(T_n-\theta)\}\to N(0, I(\theta)^{-1}) \tag{1.1}$$

when θ is the true parameter value. Here and below we use a standard notation for convergence in law.

It is well known that there exist AE estimators for any suitably regular location or scale parameter family. Such estimators can be obtained by use of appropriate linear combinations of order statistics, and are discussed in Section 3 of [1]. Here we utilize this fact to give AE estimators for any suitably regular stochastically increasing family $F(x, \theta)$. The idea is as follows: if $F(x, \theta)$ is stochastically increasing, Fraser [2] observed that there exists for any $\theta_1 \in \Omega$ a transformation $Y=S(X|\theta_1)$ such that the distribution of Y is approximately a location parameter family $L(y-\theta|\theta_1)$ for θ near θ_1. Suppose that $\hat{\theta}_n$ is a consistent estimator of θ and that for each $\theta_1 \in \Omega$ $T_n(y_1, \cdots, y_n|\theta_1)$ is an AE sequence for the approximating family $L(y-\theta|\theta_1)$. Then we will investigate the sequence of estimators defined by

$$T_n(S(X_1|\hat{\theta}_n), \cdots, S(X_n|\hat{\theta}_n)|\hat{\theta}_n) . \tag{1.2}$$

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It turns out that under suitable regularity conditions these intuitively reasonable estimators are AE for θ. Section 2 concerns the local approximation used and the AE linear combinations of order statistics for location parameters. Section 3 proves that the estimators (1.2) are asymptotically efficient.

Section 4 presents an example of the computation of our estimators in a particular case and remarks on how such computations are made in practice. The family $F(x, \theta)$ discussed there satisfies the stringent regularily conditions imposed in Section 3 and is such that the maximum likelihood estimators are not explicitly computable. In such situations our estimators, which are given in closed form, are an alternative to the use of maximum likelihood when asymptotic efficiency is desired and inefficient $\hat{\theta}_n$ are available.

Throughout this paper F_1 and F_2 will denote the partial derivatives of $F(x, \theta)$ with respect to its variables, with similar notation for higher derivatives. $Z_n \rightarrow 0$ (P) denotes convergence of Z_n to 0 in probability.

2. Preliminaries

Let $L(y-\theta)$ be a location parameter family of distributions having density $l(y-\theta)$. We will collect some basic results on estimation of θ, most of which may be found in Section 3 of Chernoff, Gastwirth and Johns [1]. Define

$$Q(y) = -l'(y)/l(y)$$

so that

$$\frac{\partial}{\partial\theta} \log l(y-\theta) = Q(y-\theta) .$$

If l'' exists and $y\ l'(y) \rightarrow 0$ as $y \rightarrow \pm\infty$, the information integral

$$I = \int_{-\infty}^{\infty} [Q(y)]^2 l(y) dy$$

for the family $L(y-\theta)$ is equal to

$$\int_{-\infty}^{\infty} Q'(y) l(y) dy .$$

Define also

$$I_2 = \int_{-\infty}^{\infty} y Q'(y) l(y) dy ,$$

and for $0 < u < 1$,

$$J(u) = I^{-1} Q'(y) \tag{2.1}$$

where $y=L^{-1}(u)$. Then under suitable conditions on J and L the sequence of estimators

$$T_n=n^{-1}\sum_{i=1}^{n} J(i/(n+1))X_{in}-I_2I^{-1} \tag{2.2}$$

is AE for θ. Here $X_{1n},\cdots,X_{nn}$ are the order statistics from a random sample of size n from $L(x-\theta)$. Conditions sufficient for (1.1) to hold with T_n as in (2.2) can be found in [1], [4] and [5]. We will assume that the T_n corresponding to the approximating location parameter families introduced below are AE for those families. In any application of our results it will therefore be necessary to check this, usually by reference to [1], [4] and [5].

A family $F(x,\theta)$ of continuous distributions is said to be *stochastically increasing* in θ near θ_1 if for all x

$$F_2(x,\theta_1)<0\ . \tag{2.3}$$

Define a transformation by the indefinite integral

$$S(x\,|\,\theta_1)=-\int^x \frac{F_1(t,\theta_1)}{F_2(t,\theta_1)}\,dt\ .$$

If (2.3) holds $S(\cdot\,|\,\theta_1)$ is a strictly increasing transformation. Fraser [2] observed that if X has distribution function $F(x,\theta)$ then $Y=S(X\,|\,\theta_1)$ has distribution function $H(y,\theta\,|\,\theta_1)$ satisfying

$$H_2(y,\theta_1\,|\,\theta_1)=-H_1(y,\theta_1\,|\,\theta_1) \qquad \text{for all } y\ . \tag{2.4}$$

If we define a location parameter family $L(y-\theta\,|\,\theta_1)$ by setting

$$L(y\,|\,\theta_1)=H(y+\theta_1,\theta_1\,|\,\theta_1)$$

we obtain the following lemma by Taylor's series and (2.4).

LEMMA 1 (*Fraser*). *Let the partial derivatives H_{11}, H_{12} and H_{22} be bounded uniformly in y and uniformly in θ in a neighborhood of θ_1. Then the location parameter family $L(y-\theta\,|\,\theta_1)$ is a local approximation to $H(y,\theta\,|\,\theta_1)$ at $\theta=\theta_1$ in the sense that*

$$H(y,\theta\,|\,\theta_1)=L(y-\theta\,|\,\theta_1)+O((\theta-\theta_1)^2) \tag{2.5}$$

uniformly in y.

Fraser used this result for other purposes than ours. We will use it to implement the method of estimation described in the Introduction. Let $F(x,\theta)$ be stochastically increasing in θ, that is, (2.3) holds for all $\theta_1\in\Omega$. If $X_1,\cdots,X_n$ are a random sample from $F(x,\theta)$, then for any θ_1 the random variables $S(X_1\,|\,\theta_1),\cdots,S(X_n\,|\,\theta_1)$ are a random sample from a distribution which is approximately $L(y-\theta|\theta_1)$ for θ near θ_1. Let $J(u, \theta_1)$ be the coefficient function (2.1) for the family $L(y-\theta|\theta_1)$ and $T_n(y_1, \cdots, y_n|\theta_1)$ the corresponding estimators (2.2).

Note that any indefinite integral of $-F_1(t, \theta_1)/F_2(t, \theta_1)$ has the properties specified in Lemma 1. We may therefore modify $S(\cdot|\theta_1)$ by the addition of any constant, which may depend on θ_1. This freedom of choice will now be invoked to cancel the correction for bias $I_2 I^{-1}$ appearing in (2.2).

For any $\theta_1 \in \Omega$ the function Q for the approximating family $L(\cdot|\theta_1)$ is

$$Q(y) = -H_{11}(y+\theta_1, \theta_1|\theta_1)/H_1(y+\theta_1, \theta_1|\theta_1) .$$

Differentiate $Q(y)$ and evaluate the integral I_2 for the family $L(y-\theta|\theta_1)$, calling the result $I_2(\theta_1)$. Temporarily writing $H(y, \theta)$ for $H(y, \theta|\theta_1)$ we see that if $H_{111}(y, \theta_1)$ exists and $yH_{11}(y, \theta_1) \to 0$ as $y \to \pm\infty$ an elementary integration by parts gives

$$I_2(\theta_1) = \int_{-\infty}^{\infty} y\left[\frac{H_{11}(y+\theta_1, \theta_1)}{H_1(y+\theta_1, \theta_1)}\right]^2 H_1(y+\theta_1, \theta_1)dy .$$

The relation (2.4) implies that $H_{11}(y, \theta_1) = -H_{12}(y, \theta_1)$ for all y, so that if $t = y + \theta_1$ we have

$$\begin{aligned} I_2(\theta_1) &= \int_{-\infty}^{\infty} (t-\theta_1)\left[\frac{H_{12}(t, \theta_1)}{H_1(t, \theta_1)}\right]^2 H_1(t, \theta_1)dt \\ &= -\theta_1 I(\theta_1) + \int_{-\infty}^{\infty} t\left[\frac{H_{12}(t, \theta_1)}{H_1(t, \theta_1)}\right]^2 H_1(t, \theta_1)dt . \end{aligned}$$

Reference to (2.2) and (1.1) shows that the mean of the asymptotic distribution of $n^{-1}\sum J(i/(n+1))S(X_{in}|\theta_1)$ when θ_1 is true is

$$\begin{aligned} \mu(\theta_1) &= \theta_1 + I_2(\theta_1)/I(\theta_1) \\ &= I(\theta_1)^{-1}\int_{-\infty}^{\infty} t\left[\frac{H_{12}(t, \theta_1)}{H_1(t, \theta_1)}\right]^2 H_1(t, \theta_1)dt \\ &= I(\theta_1)^{-1}\int_{-\infty}^{\infty} S(x|\theta_1)\left[\frac{\partial}{\partial\theta}\log f(x, \theta)|_{\theta_1}\right]^2 f(x, \theta_1)dx \end{aligned}$$

where the last expression follows after some calculation from the substitution $t = S(x|\theta_1)$. It is now clear that for each $\theta_1 \in \Omega$ we may choose $S(\cdot|\theta_1)$ so that $\mu(\theta_1) = \theta_1$. Finally, if $G(u, \theta|\theta_1)$ is an inverse of $H(\cdot, \theta|\theta_1)$, it is shown in [1] that

$$\mu(\theta_1) = \int_0^1 G(u, \theta_1|\theta_1)J(u, \theta_1)du .$$

We summarize these facts in a lemma.

LEMMA 2. *Let $J(u, \theta_1)$ be the coefficient function* (2.1) *for the family $L(y-\theta|\theta_1)$. Suppose $H(y, \theta|\theta_1)$ has density $h(y, \theta)$ such that $h_{11}(y, \theta_1)$ exists and $yh_1(y, \theta_1)\to 0$ as $y\to\pm\infty$. Then there exists a choice of $S(\cdot|\theta_1)$ having the properties stated in Lemma* 1 *and such that*

$$\int_0^1 G(u, \theta_1|\theta_1)J(u, \theta_1)du=\theta_1 .$$

3. Asymptotic efficiency

We can now formulate a theorem on the asymptotic efficiency of the estimators (1.2) when $S(\cdot|\theta)$ is chosen as in Lemma 2 and the T_n are the estimators of (2.2). The regularity conditions to be required are as follows:

(A) The distribution function H of $Y=S(X|\theta_1)$ satisfies the hypotheses of Lemmas 1 and 2 for any $\theta_1 \in \Omega$.

(B) For any $\theta_0 \in \Omega$ the estimators

$$n^{-1}\sum_{i=1}^{n} J(i/(n+1), \theta_0)Y_{in}$$

are AE for the family $L(y-\theta|\theta_0)$ if the Y_{in} are order statistics of a random sample from this family.

(C) For every $\eta>0$ there is a $B>0$ such that when θ_0 is true

$$P[\sqrt{n}\,|\hat{\theta}_n-\theta_0|>B]<\eta \qquad \text{for all } n .$$

Let $V(\theta_0)$ denote a (small) neighborhood in Ω of θ_0 and $h(u)$ a non-negative function in $L_1[0, 1]$. We require that for any $\theta_0 \in \Omega$ there exist $V(\theta_0)$ and $h(u)$ such that

(D1) There is an $\eta>0$ such that for $|\lambda|\leqq\eta$ and $\theta \in V(\theta_0)$

$$|G(u, \theta_0|\theta)[J_1(u+\lambda, \theta)-J_1(u, \theta)]|\leqq|\lambda|h(u)$$

(D2) There is an $\eta>0$ such that for $|\lambda|\leqq\eta$ and $\theta \in V(\theta_0)$

$$\left|\left[\frac{\partial}{\partial\theta}G(u, \theta_0|\theta)\right][J(u+\lambda, \theta)-J(u, \theta)]\right|\leqq|\lambda|h(u)$$

(D3) Let $A(u, \theta)=G(u, \theta_0|\theta)J(u, \theta)$. Then $A_2(u, \theta_0) \in L_2[0, 1]$ and for $\theta \in V(\theta_0)$

$$|A_2(u, \theta)-A_2(u, \theta_0)|\leqq|\theta-\theta_0|h(u)$$

(D4) For $\theta_1, \theta_2 \in V(\theta_0)$

$$|G_{22}(u, \theta_2|\theta_1)J(u, \theta_1)|\leqq h(u) .$$

THEOREM. *Let $F(x, \theta)$ be a stochastically increasing family of distri-*

bution functions. Suppose $S(\cdot|\theta)$ is chosen as in Lemma 2 *and that* (A), (B), (C) *and* (D1)-(D4) *hold. Then when $\theta_0 \in \Omega$ is true,*

$$\mathcal{L}\left\{\sqrt{n}\left(n^{-1}\sum_{i=1}^{n} J(i/(n+1), \hat{\theta}_n)S(X_{in}|\hat{\theta}_n)-\theta_0\right)\right\}\rightarrow N(0, I(\theta_0)^{-1}) .$$

PROOF. Since $S(X_{in}|\theta_0)$ are the order statistics of a random sample from $L(y-\theta_0|\theta_0)$, it follows from (B) and Lemma 2 that

$$\mathcal{L}\left\{\sqrt{n}\left(n^{-1}\sum_{i=1}^{n} J(i/(n+1), \theta_0)S(X_{in}|\theta_0)-\theta_0\right)\right\}\rightarrow N(0, I(\theta_0)^{-1}) .$$

Here $I(\theta_0)$ is the information in $L(\cdot|\theta_0)$ and is easily computed to be equal to the information in the original family $F(x, \theta)$ at $\theta=\theta_0$. So we need only show that $D_n\rightarrow 0(P)$, where

$$D_n=\sqrt{n}\left(n^{-1}\sum_{i=1}^{n} J(i/(n+1), \hat{\theta}_n)S(X_{in}|\hat{\theta}_n) - n^{-1}\sum_{i=1}^{n} J(i/(n+1), \theta_0)S(X_{in}|\theta_0)\right) .$$

If we let $U_n(u)$ be the empiric distribution function from a random sample of size n from the uniform distribution on [0, 1], we see that we can write

$$D_n=\sqrt{n}\left(\int_0^1 G(u, \hat{\theta}_n)J\left(\frac{n}{n+1}U_n(u), \hat{\theta}_n\right)dU_n(u) - \int_0^1 G(u, \theta_0)J\left(\frac{n}{n+1}U_n(u), \theta_0\right)dU_n(u)\right)$$

where we have used the abbreviation $G(u, \theta)=G(u, \theta_0|\theta)$. The plan of the proof is as follows. Define

$$\bar{D}_n=\sqrt{n}\left(\int_0^1 G(u, \hat{\theta}_n)J(u, \hat{\theta}_n)dU_n(u)-\int_0^1 G(u, \theta_0)J(u, \theta_0)dU_n(u)\right)$$

$$D_n^*=\sqrt{n}\left(\int_0^1 G(u, \hat{\theta}_n)J(u, \hat{\theta}_n)du-\int_0^1 G(u, \theta_0)J(u, \theta_0)du\right) .$$

We shall show successively that $D_n-\bar{D}_n\rightarrow 0(p)$, $\bar{D}_n-D_n^*\rightarrow 0(P)$ and $D_n^*\rightarrow 0(P)$.

Set for convenience $U_n^*(u)=nU_n(u)/(n+1)$ and

$$A(v, u, \theta)=G(u, \theta)[J(v, \theta)-J(u, \theta)] .$$

Then

$$D_n-\bar{D}_n=\sqrt{n}\int_0^1 [A(U_n^*(u), u, \hat{\theta}_n)-A(U_n^*(u), u, \theta_0)]dU_n(u)$$

$$=\sqrt{n}\,(\hat{\theta}_n-\theta_0)\int_0^1 \frac{\partial}{\partial\theta}A(U_n^*(u),\,u,\,\theta_n^*)dU_n(u)$$

where θ_n^* lies between $\hat{\theta}_n$ and θ_0 and hence $\theta_n^*\rightarrow\theta_0(P)$. Now (D1), (D2) and the fact that $\sup_u |U_n^*(u)-u|\rightarrow 0$ almost surely imply that for any $\eta>0$ there is an N_0 such that for $n>N_0$

$$\left|\frac{\partial}{\partial\theta}A(U_n^*(u),\,u,\,\theta_n^*)\right|\leqq h(u)\,|U_n^*(u)-u|$$

holds with probability $>1-\eta$. Thus for large n

$$|D_n-\bar{D}_n|\leqq\sqrt{n}\,|\hat{\theta}_n-\theta_0|\cdot\sup_u|U_n^*(u)-u|\cdot\int_0^1 h(u)dU_n(u)$$

with probability $>1-\eta$. The first term on the right is bounded in probability by (C). The second approaches 0 almost surely. The third approaches $\int h(u)du<\infty$ almost surely by the law of large numbers. It follows that $D_n-\bar{D}_n\rightarrow 0(P)$.

Set now $W_n(u)=\sqrt{n}(U_n(u)-u)$ for $0\leqq u\leqq 1$ and define $A(u,\theta)$ as in (D3). Then

$$\bar{D}_n-D_n^*=\int_0^1[A(u,\,\hat{\theta}_n)-A(u,\,\theta_0)]dW_n(u)$$

$$=(\hat{\theta}_n-\theta_0)\int_0^1 A_2(u,\,\theta_n^*)dW_n(u)$$

where again θ_n^* falls between $\hat{\theta}_n$ and θ_0. Write this expression as $I_{1n}+I_{2n}$, where

$$I_{1n}=(\hat{\theta}_n-\theta_0)\int_0^1 A_2(u,\,\theta_0)dW_n(u)$$

$$I_{2n}=(\hat{\theta}_n-\theta_0)\int_0^1[A_2(u,\,\theta_n^*)-A_2(u,\,\theta_0)]dW_n(u)\,.$$

Since

$$\int_0^1 A_2(u,\,\theta_0)dW_n(u)=\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n A_2(U_i,\,\theta_0)-\int_0^1 A_2(u,\,\theta_0)du\right)$$

where $U_1,\cdots,U_n$ are independent uniform random variables, this integral is asymptotically normal by (D3) and the central limit theorem. This and $\hat{\theta}_n-\theta_0\rightarrow 0(P)$ imply that $I_{1n}\rightarrow 0(P)$. (D3) and $\theta_n^*\rightarrow\theta_0(P)$ imply that with probability arbitrarily near 1 we have for sufficiently large n that

$$|I_{2n}|\leqq\sqrt{n}\,|\hat{\theta}_n-\theta_0|\cdot|\theta_n^*-\theta_0|\cdot\int_0^1 h(u)dV_n(u) \tag{3.1}$$

where $V_n(u)$ is the total variation of $U_n(t)-t$ from 0 to u. Calculation shows that $E[V_n(u)]=2u$ for all n so that $E\left[\int h(u)dV_n(u)\right]=2\int h(u)du<\infty$. The right side of (3.1) is therefore of the form $\alpha_n\beta_n$, where $\alpha_n\rightarrow 0(P)$ and $\beta_n\geqq 0$, $E\beta_n=K<\infty$ for all n. Clearly $\alpha_n\beta_n\rightarrow 0(P)$ and hence $I_{2n}\rightarrow 0(P)$.

It remains only to show that $D_n^*\rightarrow 0(P)$. From Lemma 2 we see that

$$D_n^*=\sqrt{n}\left(\int_0^1 G(u,\theta_0|\hat{\theta}_n)J(u,\hat{\theta}_n)du-\theta_0\right). \tag{3.2}$$

One can check that for all u and θ

$$G_2(u,\theta|\theta)=1$$

$$\int_0^1 J(u,\theta)du=1\ .$$

Therefore by Taylor's theorem

$$G(u,\theta_0|\hat{\theta}_n)=G(u,\hat{\theta}_n|\hat{\theta}_n)+(\theta_0-\hat{\theta}_n)+(\theta_0-\hat{\theta}_n)^2G_{22}(u,\theta_n^*|\hat{\theta}_n)$$

where θ_n^* lies between $\hat{\theta}_n$ and θ_0. Hence

$$\int_0^1 G(u,\theta_0|\hat{\theta}_n)J(u,\hat{\theta}_n)du=\theta_0+(\theta_0-\hat{\theta}_n)^2\int_0^1 G_{22}(u,\theta_n^*|\hat{\theta}_n)J(u,\hat{\theta}_n)du$$

and this with (3.2) implies that

$$D_n^*=\sqrt{n}(\hat{\theta}_n-\theta_0)^2\int_0^1 G_{22}(u,\theta_n^*|\hat{\theta}_n)J(u,\hat{\theta}_n)du\ .$$

This is easily shown to converge to 0 in probability by use of (D4) and (C). This completes the proof of the theorem.

4. An example

We will now give an example of a family of distributions satisfying the hypotheses of the theorem. This will also afford an opportunity to point out a way of calculating J which does not require explicit determination of the approximating location parameter family.

Consider the density

$$f(x,\theta)=\begin{cases}\frac{1}{2}(1+\theta x) & -1<x<1\\ 0 & \text{elsewhere}\end{cases}$$

for $\theta\in\Omega=(-1,1)$. The distribution function and its partial derivative with respect to θ are

$$F(x, \theta)=\frac{\theta}{4}(x^2-1)+\frac{1}{2}(x+1)$$

$$F_2(x, \theta)=(x^2-1)/4 .$$

Since $F_2(x, \theta)<0$ for all x and θ this is a stochastically increasing family. Computation shows that an indefinite integral of $-f(t, \theta)/F_2(t, \theta)$ is

$$\log \frac{(1+x)^{1-\theta}}{(1-x)^{1+\theta}} .$$

Applying the correction for bias to satisfy the requirements of Lemma 2 gives

$$S(x \mid \theta)=\theta+\log \frac{(1+x)^{1-\theta}}{(1-x)^{1+\theta}}$$

which maps $(-1, 1)$ onto $(-\infty, \infty)$. That (A) is satisfied can be seen by expressing derivatives of H in terms of derivatives of F. Since $E_\theta X=\theta/3$ we can satisfy (C) by using $\hat{\theta}_n=3\bar{X}_n$. For this estimator

$$\mathcal{L}\{\sqrt{n}(\hat{\theta}_n-\theta)\}\rightarrow N(0, 3-\theta^2)$$

when θ is true. Computation shows that

$$I(\theta)=\theta^{-2}\left[(2\theta)^{-1}\log\frac{1+\theta}{1-\theta}-1\right]$$

when $\theta\neq 0$ and $I(0)=1/3$. The asymptotic efficiency of $\hat{\theta}_n$ therefore varies from 1 when $\theta=0$ to 0 as $\theta\rightarrow\pm 1$. The method of this paper uses the inefficient estimators $\hat{\theta}_n$ to obtain AE estimators.

Routine computation shows that for $\theta\neq 0$

$$F^{-1}(u, \theta)=\theta^{-1}[\varphi(u, \theta)-1]$$

$$G(u, \theta \mid \theta_1)=S(F^{-1}(u, \theta) \mid \theta_1)$$

$$=\theta_1+\log\frac{[1+\theta^{-1}(\varphi(u, \theta)-1)]^{1-\theta_1}}{[1-\theta^{-1}(\varphi(u, \theta)-1)]^{1+\theta_1}}$$

where

$$\varphi(u, \theta)=[(1-\theta)^2+4\theta u]^{1/2} .$$

These quantities and $I(\theta)$ are continuous at $\theta=0$. We will not give separately the results for $\theta=0$.

It can be shown from the definitions of H and L that if the required derivatives exist, the coefficient function may be expressed in terms of the original distribution function $F(x, \theta)$ as

$$J(u, \theta)=I(\theta)^{-1}(F_2F_{11}F_{21}-F_1F_2F_{211})/F_1^3$$

where all functions on the right are evaluated at (x, θ) for $x=F^{-1}(u, \theta)$. This is usually the easiest way to compute J. In the present case we obtain

$$\begin{aligned}J(u, \theta)&=[2I(\theta)]^{-1}(1-x^2)(1+\theta x)^{-3}|_{x=F^{-1}(u,\theta)}\\&=[2\theta^2 I(\theta)]^{-1}[\theta^2-(\varphi(u, \theta)-1)^2][\varphi(u, \theta)]^{-3}.\end{aligned}$$

In particular $J(0)=J(1)=0$. We can now check condition (B). It is not hard to verify that $J_1(u, \theta)$ is continuous and of bounded variation in $0\leqq u\leqq 1$ and that

$$\int_0^1 |G(u, \theta|\theta)|\,du<\infty$$

for any $\theta\in\Omega$. These simple conditions are sufficient for (B) by a theorem of the author [3]. Assumptions (D1)-(D4) may be checked by using the existence and boundedness of higher derivatives to bound the differences occurring in those assumptions. For example, (D1) holds if J_{11} is bounded for $0\leqq u\leqq 1$ and θ near θ_0 and if $G(u, \theta_0|\theta)$ is bounded by an L_1 function for θ near θ_0. These facts may be checked by computation from the expressions given for J and G.

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