ASYMPTOTICALLY MOST POWERFUL RANK TESTS FOR REGRESSION PARAMETERS IN MANOVA*

M. S. SRIVASTAVA

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Summary

Srivastava [5] proposed a class of rank score tests for testing the hypothesis that $\beta_1=\cdots=\beta_p=0$ in the linear regression model $y_i=\beta_1x_{1i}+\beta_2x_{2i}+\cdots+\beta_px_{pi}+\varepsilon_i$ under weaker conditions than Hájek [2]. In this paper, under the same weak conditions, a class of rank score tests is proposed for testing $\boldsymbol{\beta}_1=\cdots=\boldsymbol{\beta}_q=0$ in the multivariate linear regression model $\boldsymbol{y}_i=\boldsymbol{\beta}_1x_{1i}+\cdots+\boldsymbol{\beta}_px_{pi}+\boldsymbol{\varepsilon}_i$, $q\leqq p$, where $\boldsymbol{\beta}_i$'s are k-vectors. The limiting distribution of the test statistic is shown to be central χ^2_{qk} under H and non-central χ^2_{qk} under a sequence of alternatives tending to the hypothesis at a suitable rate.

1. Introduction

Let $\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n$, $n\geqq p$, be n independent random k-vector with

$$\boldsymbol{y}_i=\beta\boldsymbol{x}^{(i)}+\boldsymbol{\varepsilon}_i\ , \tag{1.1}$$

β an unknown $k\times p$ matrix, $\boldsymbol{x}^{(i)}$ a known p-vector and $\boldsymbol{\varepsilon}_i$ a random k-vector obeying an unknown distribution function F such that

$$P_\beta(\boldsymbol{y}_i\leqq\boldsymbol{y})=P_\beta(\boldsymbol{\varepsilon}_i\leqq\boldsymbol{y}-\beta\boldsymbol{x}^{(i)})=F(\boldsymbol{y}-\beta\boldsymbol{x}^{(i)})\ ,$$

where P_β denotes that the probability is being computed for the parameter value β. We note that the specification (1.1) can be rewritten in the matrix notation as

$$Y_n=\beta X_n+\varepsilon_n\ , \tag{1.1a}$$

where $Y_n=(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_n)$ is a $k\times n$ matrix of observations, and $X_n=(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(n)})$, a $p\times n$, $p\leqq n$, matrix of known constants (design matrix); we assume that X_p is of full rank, i.e., $\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(p)}$ are linearly independent vectors. $\varepsilon_n=(\boldsymbol{\varepsilon}_1,\cdots,\boldsymbol{\varepsilon}_n)$ where the random vector ε_i obeys an un-

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known distribution function F with density function $f(x)$; let f_i denote the marginal density function of ε_i, $i=1, 2, \cdots, k$, and f_{ij} the joint density function of $(\varepsilon_i, \varepsilon_j)$, $i \neq j$. The form of F is not known but we shall assume that $F \in \mathcal{F}$ where

$$\text{(C1)} \quad \mathcal{F} = \Big\{ \text{absolutely continuous } F: $$
$$\text{(i)} \quad \int_{-\infty}^{\infty} f_i'(x)dx = 0, \quad i=1, 2, \cdots, k,$$
$$\text{(ii)} \quad \int_{-\infty}^{\infty} \Big[f_i'(x)/f_i(x)\Big]^2 f_i(x)dx < \infty, \quad i=1, 2, \cdots, k,$$
$$\text{(iii)} \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big[f_i'(x)f_j'(y)/f_i(x)f_j(y)\Big] f_{ij}(x, y)\,dxdy < \infty, \quad i \neq j \Big\} .$$

The condition (i) can be dropped if we modify the function $\phi(u)$ defined in Section 3 so as to have $\int_0^1 \phi(u)du = 0$. We will refer in the sequel to the above conditions on the class $\mathcal{F}$ of distribution functions as condition (C1). Let

$$\beta = (\beta_1, \beta_2)\,, \qquad \beta_1 = (\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_q)\,, \qquad \beta_2 = (\boldsymbol{\beta}_{q+1}, \cdots, \boldsymbol{\beta}_p) \tag{1.2}$$
$$X_n = ((x_{ij})) = (\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}) = \begin{pmatrix} \boldsymbol{x}_1^{(1)}, \cdots, \boldsymbol{x}_1^{(n)} \\ \boldsymbol{x}_2^{(1)}, \cdots, \boldsymbol{x}_2^{(n)} \end{pmatrix} = \begin{pmatrix} X_n^{(1)} \\ X_n^{(2)} \end{pmatrix}$$

where $\boldsymbol{x}_1^{(i)}$'s are q-vectors and $\boldsymbol{x}_2^{(i)}$'s are $(p-q)$-vectors. We wish to test the hypothesis $H: \beta_1 = 0$. For the univariate case $(k=1)$ and for $p=2$, $q=1$, Hájek [2] proposed a class of rank score tests for testing the hypothesis $H: \beta_1 = 0$ under the following conditions:

$$\text{(i)} \quad x_{2j} = 1 \qquad \text{for all } j=1, 2, \cdots, n$$
$$\text{(ii)} \quad \lim_{n\to\infty} \Big\{ \max_{1\le j\le n} (x_{1j} - \bar{x}_{1n})^2 \Big/ \sum_{j=1}^{n} (x_{1j} - \bar{x}_{1n})^2 \Big\} = 0 \tag{1.3}$$
$$\text{(iii)} \quad \lim_{n\to\infty} \Big\{ n^{-1} \sum_{j=1}^{n} (x_{1j} - \bar{x}_{1n})^2 \Big\} < \infty$$

where

$$\bar{x}_{1n} = n^{-1} \sum_{j=1}^{n} x_{1j}\,. \tag{1.4}$$

That the requirements in (1.3) are too restrictive can be seen from the following example which shows that an important class of problems do not meet the above requirements in (1.3).

Example. Consider the problem of polynomial regression. For convenience of computation, we will consider the case

$$y_t=\alpha+\beta t+\varepsilon_t ,$$

$t=1, 2,\cdots, n$; where ε_t are independently distributed with distribution function F. Identifying it with the specification in (1.1a), we have $p=2$, and

$$X_n=\begin{pmatrix} 1 & 2\cdots n \\ 1 & 1\cdots 1 \end{pmatrix} .$$

Hence

$$X_nX_n'=\begin{pmatrix} n(n+1)(2n+1)/6 & n(n+1)/2 \\ n(n+1)/2 & n \end{pmatrix} ,$$

$$\bar{x}_{1n}=(n+1)/2 , \qquad \sum_{j=1}^{n}(x_{1j}-\bar{x}_{1n})^2=n(n+1)(n-1)/12 ,$$

and the condition (iii) of (1.3) is not satisfied. In this paper, we propose a class of rank score tests for the multivariate ($k\geqq 1$) regression model for any p and $q\leqq p$, under a weaker condition (C2): that the maximum (in magnitude) of the elements in $T_n^{-1}X_n\to 0$ as $n\to\infty$, where T_n is the *unique* $p\times p$ *upper* triangular matrix, such that

$$X_nX_n'=T_nT_n'=\begin{bmatrix} T_n^{(1)} & T_n^{(12)} \\ 0 & T_n^{(2)} \end{bmatrix}\begin{bmatrix} T_n^{(1)\prime} & 0 \\ T_n^{(12)\prime} & T_n^{(2)\prime} \end{bmatrix} . \tag{1.5}$$

$T_n^{(1)}$ and $T_n^{(2)}$ are triangular matrices of order $q\times q$ and $(p-q)\times(p-q)$ respectively. It can easily be shown that the condition (C2) is satisfied for the above example.

We now show that the condition (C2) is in fact weaker than the condition (1.3), i.e., the condition (C2) holds whenever the condition (1.3) is satisfied. We proceed as follows:

From (1.5), it follows that

$$X_n=T_nL_n \tag{1.6}$$

where

$$L_n=(l_n^{(1)},\cdots, l_n^{(n)})=((l_{ij}(n)))=T_n^{-1}X_n \tag{1.7}$$

is a $p\times n$, $p\leqq n$, semi-orthogonal matrix, $L_nL_n'=I_p$. Consequently the condition (C2) is equivalent to

$$\lim_{n\to\infty}\max_{1\leqq i\leqq n} l_n^{(i)\prime}l_n^{(i)}=0 . \tag{C2}$$

From (1.7), it follows that

$$(1.8)\qquad \begin{aligned} \boldsymbol{l}_n^{(i)\prime}\boldsymbol{l}_n^{(i)} &= \boldsymbol{x}^{(i)\prime}(T_nT_n')^{-1}\boldsymbol{x}^{(i)} \\ &= \text{MCR of } [(T_nT_n')^{-1}\boldsymbol{x}^{(i)}\boldsymbol{x}^{(i)\prime}] \\ &\leqq [\text{MCR of } (T_nT_n')^{-1}][\text{MCR of } \boldsymbol{x}^{(i)}\boldsymbol{x}^{(i)\prime}] \\ &\leqq \operatorname{tr}(T_nT_n')^{-1}(\boldsymbol{x}^{(i)\prime}\boldsymbol{x}^{(i)}) , \end{aligned}$$

where 'MCR' denotes the maximum characteristic root. Considering the case $p=2$, we have from (i) of (1.3),

$$\operatorname{tr}(T_nT_n')^{-1}=(1+n^{-1}\sum x_{1j}^2)/\sum (x_{1j}-\bar{x}_{1n})^2$$

and

$$\boldsymbol{x}^{(i)\prime}\boldsymbol{x}^{(i)}=1+x_{1i}^2 .$$

Hence

$$(1.9)\qquad \boldsymbol{l}_n^{(i)\prime}\boldsymbol{l}_n^{(i)}\leqq\left(1+n^{-1}\sum_{j=1}^n x_{1j}^2\right)\left[\left\{\sum_{j=1}^n (x_{1j}-\bar{x}_{1n})^2\right\}^{-1}+x_{1i}^2\Big/\sum_{j=1}^n (x_{1j}-\bar{x}_{1n})^2\right] .$$

It follows from (1.3) that

$$(1.10)\qquad \begin{gathered} \lim_{n\to\infty}\left[\sum_{j=1}^n x_{1jn}^2\right]<\infty ; \qquad \lim_{n\to\infty}\sum_{j=1}^n (x_{1j}-\bar{x}_{1n})^2=cO(n) , \quad c>0 ; \\ \lim_{n\to\infty}\max_{1\leqq i\leqq n} x_{1i}^2\Big/\sum_{j=1}^n (x_{1j}-\bar{x}_{1n})^2=0 , \end{gathered}$$

where $O(n)$ means that $O(n)/n\to 1$ as $n\to\infty$. Consequently under conditions (1.3) the right side of (1.9) tends to zero as $n\to\infty$.

2. Test based on least square estimator

We recall in this section about the test based on the least square estimator. It is known that the semi-orthogonal matrix L_n can be completed by an arbitrary $n-p\times n$ matrix M_n such that $\Gamma_n'=(L_n', M_n')$ is an orthogonal matrix; $\Gamma_n\Gamma_n'=\Gamma_n'\Gamma_n=L_n'L_n+M_n'M_n=I_n$. The theory of least squares suggests to minimize with respect to β

$$(2.1)\qquad (Y_n-\beta X_n)(Y_n-\beta X_n)'=(Y_nL_n'-\beta T_n)(L_nY_n'-T_n'\beta')+Y_nM_n'M_nY_n' .$$

Hence, the least square estimate of β is

$$(2.2)\qquad \hat{\beta}'(n)=T_n^{-1\prime}L_nY_n'=(X_nX_n')^{-1}X_nY_n' .$$

When $\beta_1=0$, the least square estimator of β_2 is

$$(2.3)\qquad \hat{\beta}_2'(n)=T_n^{(2)-1\prime}L_n^{(2)}Y_n' ,$$

where

$$X_n=\begin{bmatrix} X_n^{(1)} \\ X_n^{(2)} \end{bmatrix}=\begin{bmatrix} T_n^{(1)} & T_n^{(12)} \\ 0 & T_n^{(2)} \end{bmatrix}\begin{bmatrix} L_n^{(1)} \\ L_n^{(2)} \end{bmatrix}, \tag{2.4}$$

and $L_n^{(1)}$ and $L_n^{(2)}$ are semi-orthogonal matrices orthogonal to each other; $L_n^{(1)}L_n^{(1)\prime}=I_q$, $L_n^{(2)}L_n^{(2)\prime}=I_{p-q}$, $L_n^{(1)}L_n^{(2)\prime}=0$, $L_n^{(2)}L_n^{(1)\prime}=0$. If we assume that the random vector $\boldsymbol{\varepsilon}^{(i)}$ obeying an unknown distribution function $F\in\mathcal{F}$ has

$$\text{(C3)}\qquad \begin{aligned} &E(\boldsymbol{\varepsilon}^{(i)})=\mathbf{0}\ , \\ &\text{Cov.}\,(\varepsilon^{(i)})=I\ , \qquad \text{without loss of generality}\ , \end{aligned}$$

then a test statistic based on the least square estimator for the hypothesis H depends upon the characteristic roots of the matrix

$$\begin{aligned} A_n&=[\hat{\beta}(n)(X_nX_n')\hat{\beta}'(n)-\hat{\beta}_2(n)(X_n^{(2)}X_n^{(2)\prime})\hat{\beta}_2'(n)] \\ &=[Y_nL_n'L_nY_n'-Y_nL_n^{(2)\prime}L_n^{(2)}Y_n] \\ &=Y_nL_n^{(1)\prime}L_n^{(1)}Y_n\ , \end{aligned} \tag{2.5}$$

and hence several test criterion are available. One such criterion could be taken as

$$C_n=\operatorname{tr} A_n\ . \tag{2.6}$$

In order to find the the distribution of C_n under H, and under a sequence of alternatives K_n (defined in 2.8) tending to the hypothesis at a suitable rate, we need the following lemma and corollary:

LEMMA 2.1. *Let $u_1, u_2, \cdots$ be independent identically distributed random variables with mean* 0 *and variance* 1; *let $a_1, a_2, \cdots$ be a sequence of numbers such that as $n\to\infty$*

$$a_1^2+a_2^2+\cdots+a_n^2\to 1 \quad \textit{and} \quad \max_{1\le i\le n}|a_i|\to 0\ .$$

Then

$$\lim_{n\to\infty}\left(\sum_{i=1}^n a_iu_i\right)=N(0,1)\ ,$$

where $N(0,1)$ denotes the standard normal random variable with mean 0 *and variance* 1.

For proof, refer to Gnedenko, B. V. and Kolmogorov, A. N.* ([1954], p. 103).

COROLLARY 2.1. *Let*

$$Y_n=(\boldsymbol{y}_1^*,\cdots,\boldsymbol{y}_k^*)'=((y_{ij}))\ . \tag{2.7}$$

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Under conditions (C2) *and* (C3)

$$\lim_{n\to\infty} (L_n^{(1)} y_i^{*\prime} | P_0) = N(0, I_q)$$

where P_0 *denotes that the probability is being computed under* $H: \beta_1 = 0$.

For proof; refer to Srivastava [4].

From Corollary 2.1, it follows that the test statistic C_n has a central χ^2_{kq} (chi-square with qk degrees of freedom) and has a non-central χ^2_{kq} under the sequence of alternatives

$$\begin{aligned} K_n : \beta_{1n} &= n^{-1/2}(n^{-\alpha_1} b_1, \cdots, n^{-\alpha_q} b_q) \\ &= n^{-1/2}(b_1^{(n)}, \cdots, b_q^{(n)}) , \qquad \text{say} \\ &= n^{-1/2} b_1^{(n)} , \end{aligned} \tag{2.8}$$

where the elements of b_i are real constants and $\max_{1 \leq j \leq n} |x_{ij}| = O(n^{\alpha_i})$, $\alpha_i \geqq 0$. Hájek [2] considered the case when $\alpha_i = 0$. The non-centrality parameter is

$$\begin{aligned} \delta^2 &= \lim_{n\to\infty} \operatorname{tr} \beta_{1n}(T_n^{(1)} T_n^{(1)\prime}) \beta_{1n}' \\ &= \lim_{n\to\infty} n^{-1} \operatorname{tr} b_1^{(n)}(T_n^{(1)} T_n^{(1)\prime}) b_1^{(n)\prime} . \end{aligned} \tag{2.9}$$

3. Proposed tests

Let

$$\psi_i(u) = -[g_i'(G_i^{-1}(u))/g_i(G_i^{-1}(u))] , \qquad 0<u<1 , \tag{3.1}$$

where G_i^{-1} is the inverse of G_i, and G_i is the marginal distribution function of $G \in \mathcal{F}$ corresponding to the ith character; G is known. The (3.1)-function that corresponds to F is

$$\phi_i(u) = -[f_i'(F_i^{-1}(u)/f_i(F_i^{-1}(u)))] , \qquad 0<u<1 . \tag{3.2}$$

Observe that unlike $\psi(u)$ of (3.1), $\phi(u)$ is not known since it is defined through the unknown F. We will consider only those F for which $\psi(u)$ and $\phi(u)$ are non-decreasing functions of u. We will refer to this condition in the sequel as condition (C4). From (C1) it follows that

$$0 = \int_0^1 \psi_i(u) du = \int_0^1 \phi_i(u) du \tag{3.3}$$

and

$$\int_0^1 \psi_i^2(u) du < \infty , \qquad \int_0^1 \phi_i^2(u) du < \infty . \tag{3.4}$$

As in (2.4), let $Y_n'=(\boldsymbol{y}_1^*,\cdots,\boldsymbol{y}_k^*)=((y_{ij}))'$. Let R_{ij} be the rank of y_{ij} in the ordered sample $O_{i1}<\cdots<O_{in}$, i.e., $y_{ij}=O_{R_{ij}}$, $1\leqq j\leqq n$. Consider a $k\times n$ matrix of statistics defined by

$$Z_n=(Z_1^{(n)},\cdots,Z_k^{(n)})'=((\phi_n(R_{ij}/n+1))) \tag{3.5}$$

where

$$\phi_n(u)=\phi(j/n+1)\ , \qquad (j-1)/n<u\leqq j/n\ . \tag{3.6}$$

From Hájek [1], and condition (C4) we have

$$\lim_{n\to\infty}\int_0^1[\phi_n(u)-\phi(u)]^2du=0\ . \tag{3.7}$$

Define

$$\bar{C}_n(\phi)=\operatorname{tr}\Sigma^{-1}Z_nL_n^{(1)\prime}L_n^{(1)}Z_n'\ , \tag{3.8}$$

$$\Sigma=((\sigma_{ij}))\ , \qquad \sigma_{ii}=\int_0^1\phi_i^2(t)dt\ , \tag{3.9}$$

$$\sigma_{ij}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{g_i'(x)g_j'(y)}{g_i(x)g_j(y)}\right]g_{ij}(x,y)dxdy\ .$$

We propose $\bar{C}_n(\phi)$ as one of the class of rank score test statistics for testing $H:\beta_1=0$. While several other tests criterion could be proposed we will consider only $\bar{C}_n(\phi)$ in this paper.

4. Limiting distribution of $\bar{C}_n$ under the hypothesis

We note that under H, the distribution of the random matrix Z_n defined by (3.5) is not independent of F and depends on the function ϕ and hence on G, through which ϕ is defined. The following theorem gives the limiting null distribution of $\bar{C}_n$., which however, does not depend upon F.

THEOREM 4.1. *Under conditions* (C1), (C2) *and* (C4)

$$\lim_{n\to\infty}P_0\{\bar{C}_n\leqq y\}=P\{\chi_{kq}^2\leqq y\} \tag{4.1}$$

where P_0 denotes that the probability is being computed under $H:\beta_1=0$.

PROOF. The idea of the proof is as in Hájek [2] of replacing the random vector $Z_i^{(n)}$ by another random vector $W_i^{(n)}$ whose components are independent and identically distributed and then apply Corollary 2.1 to $L_n^{(1)}W_i^{(n)}$. The result will then follow from the multivariate central limit theorem. We achieve this goal in two stages. First we introduce a $k\times n$ random matrix V_n defined by

(4.2) $$V_n=(V_1^{(n)},\cdots,V_k^{(n)})'=(\psi_n(F(y_{ij})))\ .$$

We will now show that as $n\to\infty$

(4.3) $$L_n^{(1)}Z_i^{(n)}-L_n^{(1)}V_i^{(n)}\to 0\quad\text{in } P_0\text{-probability},\qquad i=1,2,\cdots,k\ ,$$

i.e., as $n\to\infty$, for $i=1,2,\cdots,k$, and $j=1,2,\cdots,q$

(4.4) $$\sum_{r=1}^{n} l_{jr}^{(1)}(n)(Z_{ij}^{(n)}-V_{ij}^{(n)})\to 0\quad\text{in } P_0\text{-probability}\ ,$$

i.e., as $n\to\infty$, for $i=1,2,\cdots,k$, and $j=1,2,\cdots,q$,

(4.5) $$\sum_{r=1}^{n} l_{jr}^{(1)}(n)[\psi_n(R_{ir}/n+1)-\psi_n(U_{ir})]\to 0\quad\text{in } P_0\text{-probability}$$

where $U_{ir}=F(y_{ir})$ are independent random variables uniformly distributed over [0, 1]. Under condition (C4), (4.5) follows from Lemma 3.1 of Hájek [1].

Now, we introduce another $k\times n$ random matrix W_n defined by

(4.6) $$W_n=(W_1^{(n)},\cdots,W_k^{(n)})'=((\psi(U_{ij})))\ ,$$

where $U_{ij}=F(y_{ij})$. We will now show that as $n\to\infty$

(4.7) $$L_n^{(1)}V_i^{(n)}-L_n^{(1)}W_i^{(n)}\to 0\quad\text{in } P_0\text{-probability},\qquad i=1,2,\cdots,k\ ,$$

i.e., as $n\to\infty$, for $i=1,2,\cdots,k$, and $j=1,2,\cdots,q$,

(4.8) $$\sum_{r=1}^{n} l_{jr}^{(1)}(n)(V_{ir}^{(n)}-W_{ir}^{(n)})\to 0\quad\text{in } P_0\text{-probability}\ .$$

This follows from (3.7) and Chebycheff's inequality. Combining (4.5) and (4.8), we obtain that

(4.9) $$L_n^{(1)}Z_n-L_n^{(1)}W_n\to 0\quad\text{in } P_0\text{-probability}\ .$$

Note that $\psi(U_{ij})$, $j=1,2,\cdots,n$, are independent and identically distributed random variables. Since

(4.10) $$E_0\psi(U_{ij})=0$$

and

(4.11) $$\operatorname{Var}\psi(U_{ij})=\sigma_{ii}=\int_0^1\psi_i^2(u)du\ ,$$

we get $W_{i1},W_{i2},\cdots,W_{in}$ as independent identically distributed random variables with mean 0 and variance γ_i^2. Also,

(4.12) $$\operatorname{Cov}(W_{ir},W_{js})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_i(G_i(y_i))\psi_j(G_j(y_j))g_{ij}(y_i,y_j)dy_idy_j\ .$$

Hence from Corollary (2.1) and multivariate central limit theorem we get

$$\lim_{n\to\infty} P\{\bar{C}_n \leqq y\} = P\{\chi^2_{kq} \leqq y\} . \tag{4.13}$$

5. Limiting distribution of $\bar{C}_n$ under near alternatives

In order to determine the efficiency of the class $\bar{C}_n(\psi)$ of test statistics, it is necessary to find its distribution under a sequence of alternatives tending to the hypothesis at a suitable rate. In this section, we discuss the distribution of $\bar{C}_n$ for alternatives K_n (defined by 2.8) tending to H at the rate of $n^{-(1/2+\alpha_i)}$, $i=1, 2, \cdots, q$, and for this, we shall follow the method based on LeCam's contiguity lemma (see Hájek [2] and LeCam [3]).

First we recall the definition of contiguity.

DEFINITION 5.1. Let P_0 and P_1 be two probability measures on a measurable space (X, A). If for any $A_n \in A$, $P_0(A_n)\to 0$ entails $P_1(A_n)\to 0$, we say that the probability measure P_1 is contiguous to the probability measure P_0.

Now we give the set-up under which the contiguity principle is applicable.

Let $P_{ni}=\prod_{j=1}^{n} P_{ij}$ be the distribution of $(Y_{i1}, \cdots, Y_{in})$ under a sequence K_n of alternatives defined by (2.8), and let

$$r_{ij}=p_{ij}(Y_{ij})/p_{0i}(Y_{ij}) , \qquad \text{for } p_{0i}(Y)>0 , \tag{5.1}$$

where p_{ij}, $j=1, 2, \cdots, n$, are densities corresponding to P_{ij} and p_{0i} corresponds to the distribution P_{0i}, under the hypothesis.

Define

$$Q_{ni}=2\sum_{j=1}^{n} (r_{ij}^{1/2}-1) . \tag{5.2}$$

Let

$$\beta_2=(\boldsymbol{\beta}_{q+1}, \cdots, \boldsymbol{\beta}_p)=(\boldsymbol{\beta}^*_{21}, \cdots, \boldsymbol{\beta}^*_{2k})' ,$$

$$b_1^{(n)}=(\boldsymbol{b}_1^{(n)}, \cdots, \boldsymbol{b}_q^{(n)})=(\boldsymbol{b}_{11}^{(n)}*, \cdots, \boldsymbol{b}_{1k}^{(n)}*)' ,$$

where $\boldsymbol{\beta}^*_{2i}$ is a $p-q\times 1$ and $\boldsymbol{b}_{1i}^{(n)}*$ is a $q\times 1$ vector. With the above notation, and following the method of Hájek [2], we shall prove the following

LEMMA 5.1. *Under conditions* (C1), (C2) *and* (C4) *the distributions* P_{ni} *are contiguous to* P_{0i}.

PROOF. The lemma will be proved if we show that
(i) $\lim_{n\to\infty}\max_{1\le j\le n} P_{0i}(|r_{ij}-1|>\varepsilon)=0$ for every $\varepsilon>0$, and
(ii) $\mathcal{L}(Q_{ni}|P_{0i})\to N(-1/4\cdot\sigma_i^2, \sigma_i^2)$.
For (i), write

$$r_{ij}=f_i(V_{ij}-h_{ij}^{(n)})/f_i(V_{ij}) \tag{5.3}$$

where

$$V_{ij}=y_{ij}-\beta_{2i}^{*\prime}x_2^{(j)}, \qquad h_{ij}^{(n)}=n^{-1/2}b_{1i}^{(n)*\prime}x_1^{(j)}. \tag{5.4}$$

It may be noted that

$$h_{ij}^{(n)}\to 0 \quad\text{as } n\to\infty \qquad\text{and}\qquad \sum_{j=1}^{n} h_{ij}^{(n)^2}=n^{-1}b_{1i}^{(n)*\prime}X_nX_n'b_{1i}^{(n)*}. \tag{5.5}$$

We may take $h_{ij}^{(n)}\neq 0$. Then

$$\max_{1\le j\le n} P_0(|r_{ij}-1|>\varepsilon)\leqq\max_{1\le j\le n}\varepsilon^{-1}E_0|r_{ij}-1|$$
$$\leqq\max_{1\le j\le n}\varepsilon^{-1}|h_{ij}^{(n)}|\int_{-\infty}^{\infty}|h_{ij}^{(n)^{-1}}\{f_i(v-h_{ij}^{(n)})-f_i(v)\}|\,dv.$$

Now

$$|h_{ij}^{(n)^{-1}}\{f_i(v-h_{ij}^{(n)})-f_i(v)\}|\leqq|h_{ij}^{(n)^{-1}}|\int_{v-h_{ij}^{(n)}}^{v}|f_i'(t)|\,dt$$

and

$$|h_{ij}^{(n)^{-1}}|\int_{-\infty}^{\infty}|f_i(v-h_{ij}^{(n)})-f_i(v)|\,dv\leqq\int_{-\infty}^{\infty}|f_i'(v)|\,dv,$$
for all $j=1, 2,\cdots, n$.

Hence, from (5.5) we have

$$\max_{1\le j\le n} P_0\{|r_{ij}-1|>\varepsilon\}\leqq\max_{1\le j\le n}\varepsilon^{-1}|h_{ij}^{(n)}|\int_{-\infty}^{\infty}|f_i'(v)|\,dv\to 0. \tag{5.6}$$

To prove (ii), define

$$D_{ni}'=(f_i'(V_{i1})/f_i(V_{i1}),\cdots, f_i'(V_{in})/f(V_{in})]. \tag{5.7}$$

$$S_{ni}=n^{-1}b_{1i}^{(n)*\prime}X_n^{(1)}D_{ni}=n^{-1}b_{1i}^{(n)*\prime}(T_n^{(1)}L_n^{(1)}+T_n^{(12)}L_n^{(2)})D_{ni}. \tag{5.8}$$

$$s_i(x)=f_i^{1/2}(x). \tag{5.9}$$

We can rewrite Q_{ni} in the form

$$Q_{ni}=\sum_{j=1}^{n}[\{s_i(V_{ij}-h_{ij}^{(n)})/s_i(V_{ij})\}-1]. \tag{5.10}$$

It has been shown by Hájek [2] that

$$E_0(Q_{ni}) \sim -\frac{1}{4}\left(\sum_{j=1}^{n} h_{ij}^{(n)^2}\right)\sigma_{ii} = -\frac{1}{4}\sigma_{ni}^2\ , \qquad \text{say,} \tag{5.11}$$

where σ_{ii} has been defined in (3.9) and the sign $\sim$ denotes that the ratio of both sides tends to 1 as $n\to\infty$. We will now show that the variance of $n^{1/2}S_{ni}$ is σ_{ni}^2;

$$V(n^{1/2}S_{ni}) = n^{-1}\boldsymbol{b}_{1i}^{(n)*\prime}X_n^{(1)}X_n^{(1)\prime}\boldsymbol{b}_{1i}^{(n)*}\sigma_{ii} = \left(\sum_{j=1}^{n} h_{ij}^{(n)^2}\right)\sigma_{ii} = \sigma_{ni}^2\ . \tag{5.12}$$

Hence, under condition (C2)

$$\mathcal{L}(n^{1/2}S_{ni}\,|\,P_0) \to N(0,\sigma_i^2) \qquad \text{as } n\to\infty\ , \tag{5.13}$$

where

$$\sigma_i^2 = \lim_{n\to\infty}\sigma_{ni}^2\ . \tag{5.14}$$

Following as in Hájek [2] it can be shown that

$$E_0(Q_{ni} - E_0Q_{ni} - n^{1/2}S_{ni})^2 \to 0\ . \tag{5.15}$$

Hence, from (5.11) and (5.15)

$$\mathcal{L}(Q_{ni}\,|\,P_{0i}) \to N(-1/4\cdot\sigma_i^2,\sigma_i^2) \qquad \text{as } n\to\infty\ , \tag{5.16}$$

and the proof of the lemma is therefore complete. We shall now apply the contiguity principle to obtain the limit distribution of $L_n^{(1)}\boldsymbol{Z}_i^{(n)}$ under the sequence of alternatives K_n defined by (2.8). In this connection we first state a lemma which can be obtained as a corollary of Lemma 4.2 of Hájek [2] given by LeCam [3].

LEMMA 5.2. *If P_{ni} is contiguous to P_{0i} and*
(i) $\mathcal{L}(n^{1/2}S_{ni}^{(1)}\,|\,P_{0i}) \to N(0,\sigma_i^{(1)^2})$,
(ii) $\mathcal{L}(n^{1/2}S_{ni}^{*}\,|\,P_{0i}) \to N(a_i,b_i^2)$,
(iii) $\mathcal{L}(n^{1/2}S_{ni}^{*},n^{1/2}S_{ni}) \to$ *Bivariate normal with correlation coefficient* ρ_i.
Then

$$\mathcal{L}(n^{1/2}S_{ni}^{*}\,|\,P_{ni}) \to N(a_i + b_i\sigma_i^{(1)}\rho_i,b_i^2)\ .$$

THEOREM 5.1. *Under conditions* (C1), (C2) *and* (C4),

$$\mathcal{L}(L_n^{(1)}Z_i^{(n)}\,|\,P_{ni}) \to N\left(\lim n^{-1/2}T_n^{(1)\prime}\boldsymbol{b}_{1i}^{(n)*}\left[\int_0^1\phi_i(u)\psi_i(u)du\right],\sigma_{ii}I_q\right)\ .$$

PROOF. Let

$$S_{ni}^{(1)} = n^{-1}\boldsymbol{b}_{1i}^{(n)*\prime}T_n^{(1)}L_n^{(1)}\boldsymbol{D}_{ni}\ . \tag{5.17}$$

Under condition (C2) it follows from (5.8) and (5.13) that

(5.18) $$\mathcal{L}(n^{1/2}S_{ni}^{(1)} \mid P_{0i}) \rightarrow N(0, \sigma_i^{(1)^2}) ,$$

where

(5.19) $$\sigma_i^{(1)^2} = \lim \sigma_{ni}^{(1)^2} \equiv \lim n^{-1} b_{1i}^{(n)*\prime} T_n^{(1)} T_n^{(1)\prime} b_{1i}^{(n)*} \sigma_{ii} .$$

Let

(5.20) $$S_{ni}^{*} = n^{-1} b_{1i}^{(n)*\prime} T_n^{(1)} L_n^{(1)} W_{ni}$$

where W_{ni} has been defined in (4.6). The covariance between $n^{1/2}S_{ni}^{(1)}$ and $n^{1/2}S_{ni}^{*}$ is given by

(5.21) $$\text{Cov.}\,(n^{1/2}S_{ni}^{(1)}, n^{1/2}S_{ni}^{*}) = n^{-1} b_{1i}^{(n)*\prime} (T_n^{(1)} T_n^{(1)\prime}) b_{1i}^{(n)*} \left[\int_0^1 \phi_i(u)\psi_i(u)du \right] .$$

It can be seen that under condition (C2), the bivariate central limit theorem applies to $(n^{1/2}S_{ni}^{(1)}, n^{1/2}S_{ni}^{*})$. Consequently, from (4.9) and Lemma 5.2,

(5.22) $$\mathcal{L}(n^{-1} b_{1i}^{(n)*\prime} T_n^{(1)} L_n^{(1)} Z_i^{(n)} \mid P_{ni}) \rightarrow N\Big(\lim_{n\rightarrow\infty} n^{-1} b_{1i}^{(n)*\prime} (T_n^{(1)} T_n^{(1)\prime}) b_{1i}^{(n)*} \rho_i , \lim_{n\rightarrow\infty} n^{-1} b_{1i}^{(n)*} (T_n^{(1)} T_n^{(1)\prime}) b_{1i}^{(n)} \sigma_{ii} \Big) ,$$

where

(5.23) $$\rho_i = \int_0^1 \phi_i(u)\psi_i(u)du .$$

Hence,

(5.24) $$\mathcal{L}(L_n^{(1)} Z_i^{(n)} \mid P_{ni}) \rightarrow N\Big(\lim_{n\rightarrow\infty} n^{-1/2} T_n^{(1)\prime} b_{1i}^{(n)*} \rho_i , \sigma_{ii} I_q \Big) .$$

COROLLARY 5.1. *Under conditions* (C1), (C2) *and* (C4)

$$\mathcal{L}(\bar{C}_n \mid P_n) \rightarrow \chi^2_{kp}(\delta^{*2})$$

where

(5.25) $$\delta^{*2} = n^{-1} \operatorname{tr} D_\rho \Sigma^{-1} D_\rho b_1^{(n)} (T_n^{(1)} T_n^{(1)\prime}) b_1^{(n)\prime} , \qquad D_\rho = \operatorname{diag}(\rho_1, \cdots, \rho_k) ,$$

Hence the Pitman efficiency of $\bar{C}_n$ tests relative to the classical test C_n is the ratio of the two non-centrality parameters (5.25) and (2.9).

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CORRECTION TO "ASYMPTOTICALLY MOST POWERFUL RANK TESTS FOR REGRESSION PARAMETERS IN MANOVA"

M. S. SRIVASTAVA

In the above entitled paper in Vol. 24 (1972), pp. 285–297 which is a multivariate generalization of the results in paper "Asymptotically most powerful rank tests", *J. Statist. Res.*, Vol. 7 (1973), pp. 1–11, the results of Section 4 (in both the papers) holds under the assumption that $\beta_1=0$ and $\beta_2=0$. To obtain the results when $\beta_1=0$ but $\beta_2\neq 0$, we proceed as follows. For simplicity of presentation, we consider the univariate regression model, that is when $k=1$.

Let

$$\tilde{\boldsymbol{\beta}}'=\boldsymbol{\beta}'T_n=(\tilde{\beta}_1, \cdots, \tilde{\beta}_p) .$$

We shall assume that either

$$\tilde{\beta}_i=O(1) , \quad i=1, \cdots, p \quad \text{or} \quad \lim_{n\to\infty} \max_{1\leq i\leq n} \boldsymbol{x}^{(i)\prime}\boldsymbol{x}^{(i)}=0 . \tag{C5}$$

Let P_0 and P_{00} denote the distributions under $\boldsymbol{\beta}_1=0$ and $\boldsymbol{\beta}=\mathbf{0}$ respectively. Then it is shown in Section 4 that $\mathcal{L}(L_n^{(1)}\boldsymbol{Z}_n|P_{00})\rightarrow N(0, \sigma_1^2 I_q)$, *not* under P_0 as claimed there. However, under conditions (C2) and (C5) it can easily be shown that P_0 is centigious to P_{00}. Thus, following as in the paper, we get

$$\mathcal{L}(\boldsymbol{\beta}_2'X_2\boldsymbol{Z}_n|P_0)\rightarrow N(\lim v_{12},\ \sigma_{11}\lim \tau_{1n}^2) ,$$

where

$$v_{12}=\boldsymbol{\beta}_2'X_n^{(2)}X_n^{(2)\prime}\boldsymbol{\beta}_2\int_0^1 \phi_1(u)\psi_1(u)du$$

and

$$\tau_{1n}^2=\boldsymbol{\beta}_2'X_n^{(2)}X_n^{(2)\prime}\boldsymbol{\beta}_2 .$$

Hence

$$\mathcal{L}(L_n^{(1)}\boldsymbol{Z}_n|P_0)\rightarrow N\left(\lim L_n^{(1)}X_n^{(2)\prime}\boldsymbol{\beta}_2\int_0^1 \phi_1(u)\psi_1(u)du,\ \sigma_{11}I_q\right) .$$

But

$$L_n^{(1)} X_n^{(2)\prime} \boldsymbol{\beta}_2 = L_n^{(1)} L_n^{(2)\prime} T_n^{(2)\prime} \boldsymbol{\beta}_2 = 0 .$$

Hence

$$\mathcal{L}(L_n^{(1)} \boldsymbol{Z}_n | P_0) \to N(0, \sigma_{11} I_q) .$$

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