STOCHASTICALLY LARGER COMPONENT OF A RANDOM VECTOR

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Summary

Definitions of different strengths are given to the notion of 'a stochastically larger component of a two-dimensional random vector.' Some of them reduce to the known definitions of stochastic order relationship when the components are stochastically independent. The definitions and the approach are related to nonparametric problems.

1. Introduction

The notion of stochastic order relationship is fundamental in statistical inferences and especially in the nonparametric theory. The notion for one-dimensional case was established in the publications [1], [2] and [3]. It is related to other notions like monotone likelihood ratio, positive (or negative) dependence, test of symmetry, increasing hazard rate, etc.

In this paper we extend the notion to two-dimensional case using a systematic way for defining what is meant by saying that 'a component X of a random vector (X, Y) is stochastically larger than the other component Y'. The formulation is stated later in this section. Some series of definitions are introduced and their characteristics are discussed in Sections 2 and 3. It is shown that a series of definitions covers the known definitions in one-dimensional case as a special case where X and Y are stochastically independent. Other possible approaches are also stated in Section 4.

For any measurable subset S in R^2 we denote by S^* its symmetric image about the line $x=y$, namely $S^*=\{(x, y); (y, x)\in S\}$. We also use the notation $R_x=\{(x, y); x\geqq y\}\subset R^2$ and $R_y=R_x^*$.

DEFINITION 1. Let $\mathcal{R}$ be a class of measurable subsets of R_x, then we say that a component X of a random vector (X, Y) is stochastically larger than Y with respect to $\mathcal{R}$ iff

$$P\{(X, Y)\in S\}\geqq P\{(X, Y)\in S^*\} \qquad \text{for all } S\in\mathcal{R}\ ,$$

and we shall write this fact simply as $X \succ Y$ ($\mathcal{R}$).

Choosing a larger class of $\mathcal{R}$ we get stronger definition of ordering. There may be many principles for choosing a class $\mathcal{R}$. A choice is justified by its usefulness in applications, which will be discussed in forthcoming papers [4] and [5].

2. Definitions

Throughout the paper (X, Y) is a random vector with the distribution function $F(x, y)$. Its marginal distributions are written as $G(x) = F(x, \infty)$ and $H(y) = F(\infty, y)$.

We denote a two-dimensional (finite or infinite) interval by $S(s_1, s_2; t_1, t_2) = \{(x, y); s_1 < x \leqq s_2, t_1 < y \leqq t_2\}$ and introduce classes of intervals:

DEFINITION 2.

$$\mathcal{R}_{3A} = \{S(a_1, a_2; b_1, b_2);\ b_1 \leqq b_2 \leqq a_1 \leqq a_2\}\ ,$$

$$\mathcal{R}_3 = \{S(a_1, a_2; b_1, b_2);\ b_1 \leqq b_2 = a_1 \leqq a_2\}\ ,$$

$$\mathcal{R}'_{2A} = \{S(a_1, \infty; b_1, b_2);\ b_1 \leqq b_2 \leqq a_1\}\ ,$$

$$\mathcal{R}''_{2A} = \{S(a_1, a_2; -\infty, b_2);\ b_2 \leqq a_1 \leqq a_2\}\ ,$$

$$\mathcal{R}_{2A} = \mathcal{R}'_{2A} \cup \mathcal{R}''_{2A}\ ,$$

$$\mathcal{R}'_2 = \{S(a_1, \infty; b_1, b_2);\ b_1 \leqq b_2 = a_1\}\ ,$$

$$\mathcal{R}''_2 = \{S(a_1, a_2; -\infty, b_2);\ b_2 = a_1 \leqq a_2\}\ ,$$

$$\mathcal{R}_2 = \mathcal{R}'_2 \cup \mathcal{R}''_2\ ,$$

$$\mathcal{R}_{1A} = \{S(a_1, \infty; -\infty, b_2);\ b_2 \leqq a_1\}\ ,$$

$$\mathcal{R}_1 = \{S(a_1, \infty; -\infty, b_2);\ b_2 = a_1\}\ .$$

We introduce also classes of band regions:

DEFINITION 3.

$$\mathcal{R}_{\mathrm{II}} = \{\{(x, y); a < x - y \leqq b\};\ 0 \leqq a \leqq b\}\ ,$$

$$\mathcal{R}_{\mathrm{I}} = \{\{(x, y); a < x - y < \infty\};\ 0 \leqq a\}\ , \qquad \mathcal{R}_0 = \{R_x\}\ .$$

Notice that in Definitions 2 and 3, the suffix of $\mathcal{R}$ corresponds almost to the number of parameters which appear in the definition. In Fig. 1 the typical members of these classes are illustrated with the slant $x = y$.

In the framework of Definitions 2 and 3 $\mathcal{R}_{3A}$ is the strongest since $X \succ Y$ ($\mathcal{R}_{3A}$) means that $P\{(X, Y) \in S\} \geqq P\{(X, Y) \in S^*\}$ for any measurable set S in R_x, or that if $f(x, y)$ is a density with respect to a

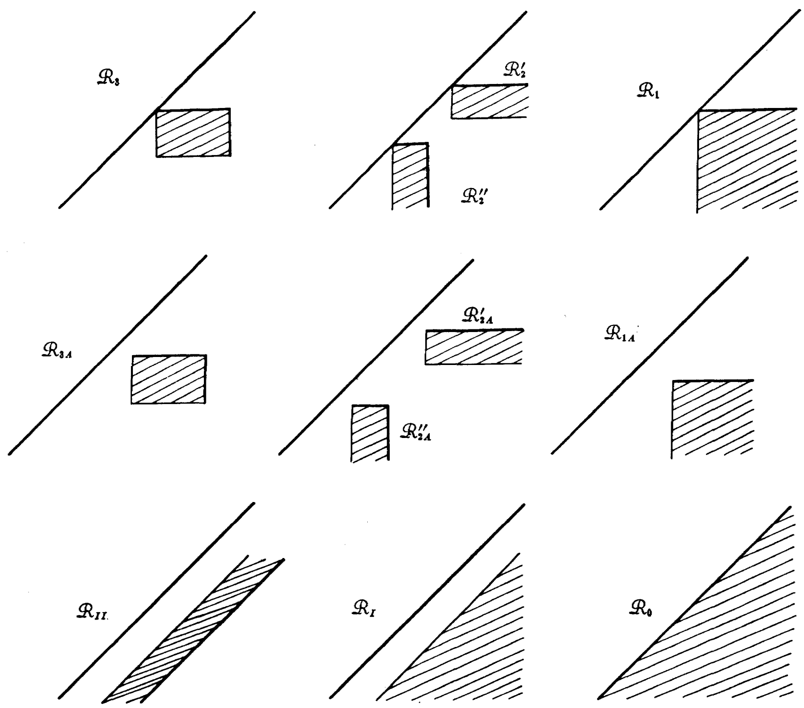

Fig. 1 Typical elements of the classes $\mathcal{R}_i$'s.

dominating symmetric measure, then $f(x, y) \geqq f(y, x)$ for any $x > y$. $\mathcal{R}_1$ is the weakest in Definition 2 and in terms of the marginal distribution functions it is equivalent to $G(t) \leqq H(t)$, $-\infty < t < \infty$. In Definition 3 the weakest $\mathcal{R}_0$ means $P(X > Y) \geqq P(Y > X)$.

PROPOSITION 2.1. For the probability distributions on R^2 the implications in Fig. 2 (shown by arrows) and only these are valid.

PROOF. If $\mathcal{R}_i \subset \mathcal{R}_j$ then the σ-algebra generated by $\mathcal{R}_i$ is a sub-

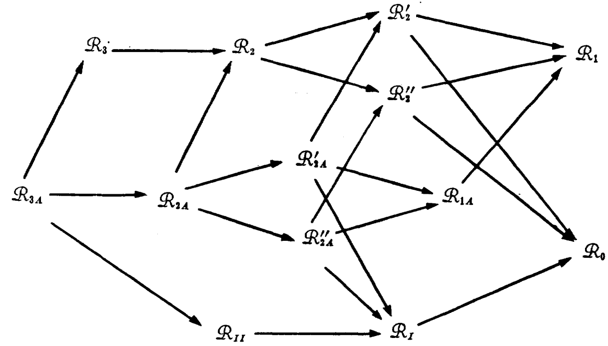

Fig. 2 Implication scheme among the definitions, the general case.

algebra of that by $\mathcal{R}_j$, and $X \succ Y$ $(\mathcal{R}_j)$ implies $X \succ Y$ $(\mathcal{R}_i)$. The if-part can be shown straightly or by considering limits and integrations. The only-if-part is shown by counter examples in Appendices A and B. Refer also to Propositions 2.2 and 2.3.

Now we consider the special case where the components of (X, Y) are stochastically independent.

PROPOSITION 2.2. If X and Y are stochastically independent then the stochastic order $\mathcal{R}_{3A}$, $\mathcal{R}_{2A}$, $\mathcal{R}'_{2A}$, $\mathcal{R}''_{2A}$ and $\mathcal{R}_{1A}$ are equivalent to $\mathcal{R}_3$, $\mathcal{R}_2$, $\mathcal{R}'_2$, $\mathcal{R}''_2$ and $\mathcal{R}_1$, respectively.

PROOF. To prove the equivalence of $\mathcal{R}_{3A}$ and $\mathcal{R}_3$ put

$$p_1 = G(b_2) - G(b_1)\,, \qquad p_2 = G(a_1) - G(b_2)\,, \qquad p_3 = G(a_2) - G(a_1)\,,$$

$$q_1 = H(b_2) - H(b_1)\,, \qquad q_2 = H(a_1) - H(b_2)\,, \qquad q_3 = H(a_2) - H(a_1)\,.$$

Then $X \succ Y$ $(\mathcal{R}_3)$ means $p_2 q_1 \geqq p_1 q_2$ and $p_3 q_2 \geqq p_2 q_3$, and these imply $p_3 q_1 \geqq p_1 q_3$ or $P\{S(a_1, a_2; b_1, b_2)\} \geqq P\{S^*(a_1, a_2; b_1, b_2)\}$. The other equivalence can be proved similarly.

PROPOSITION 2.3. In the independent case the implications scheme of Fig. 2 reduces to Fig. 3 and only these implications are possible.

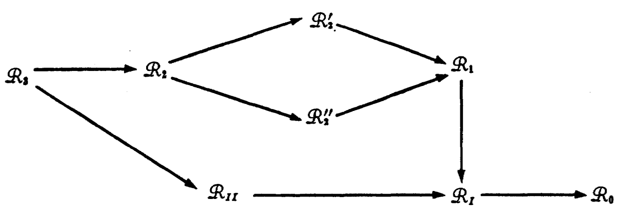

Fig. 3 Implication scheme among the definitions, the independent case.

PROOF. Firstly we prove $\mathcal{R}_1 \rightarrow \mathcal{R}_{\mathrm{I}}$. For any $a > 0$

$$\begin{aligned} P(X \geqq Y + a) &= \int H(x-a)\,dG(x) \\ &\geqq \int G(x-a)\,dG(x) \\ &= \iint_{s \geqq t+a} dG(s)\,dG(t) \\ &= \int (1 - G(t+a-0))\,dG(t) \\ &\geqq \int (1 - H(t+a-0))\,dG(t) \\ &= P(Y \geqq X + a)\,. \end{aligned}$$

Other implications follow immediately from Propositions 2.1 and 2.2. The impossibility of other implications are shown by counter examples in Appendix B. They are also counter examples for Proposition 2.1. The increasing strictness of $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ was proved by Pfanzagl [3] but we add some comments.

3. Some properties and examples

3.1. Some comparisons

Example 1. Consider the family of two-dimensional exponential-type distributions with the density

$$f(x, y; \theta)=h(x, y; \theta) \exp \{\theta_1 g(x)+\theta_2 g(y)\} ,$$

where $\theta=(\theta_1, \cdots, \theta_p)$, $h(x, y; \theta)$ is a symmetric function of x and y, and $g(\cdot)$ is nondecreasing. Then $X \succ Y$ ($\mathcal{R}_{3A}$) iff $\theta_1 \geqq \theta_2$. The trinomial distribution, the negative trinomial distribution (or the bivariate negative binomial distribution) and the bivariate beta distribution (or the Dirichlet distribution) are of this type.

Example 2. For the two-dimensional normal distribution with the mean vector $(\mu, 0)$ and the variance-covariance matrix

$$\begin{bmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{bmatrix} , \qquad \log \frac{f(x, y)}{f(y, x)}=\frac{x-y}{2(1-\rho^2)} \left\{\left(1-\frac{1}{\sigma^2}\right)(x+y)+\frac{2\mu}{\sigma^2}(1+\rho\sigma)\right\} .$$

Then $X \succ Y$ ($\mathcal{R}_{3A}$) iff $\sigma=1$ and $\mu>0$. In general $X \succ Y$ ($\mathcal{R}_1$) iff $F(t, \infty) \leqq F(\infty, t)$ and in this case $F(t, \infty)=\Phi((t-\mu)/\sigma)$ and $F(\infty, t)=\Phi(t)$, the distribution function of $N(0, 1)$, so $X \succ Y$ ($\mathcal{R}_1$) iff $\sigma=1$ and $\mu>0$. These two facts show that in the sense of all $\mathcal{R}_{3A}$, $\mathcal{R}_{2A}$, $\mathcal{R}'_{2A}$, $\mathcal{R}''_{2A}$, $\mathcal{R}_{1A}$; $\mathcal{R}_3$, $\mathcal{R}_2$, $\mathcal{R}'_2$, $\mathcal{R}''_2$ and $\mathcal{R}_1$, X and Y are comparable iff $\sigma=1$ and then the stochastic order relation does not depend on the correlation coefficient ρ.

The difference $Z=X-Y$ has distribution function $\Phi((z-\mu)/\sqrt{\sigma^2-2\rho\sigma+1})$ so that $X \succ Y$ ($\mathcal{R}_{\mathrm{II}}$, $\mathcal{R}_{\mathrm{I}}$ and $\mathcal{R}_0$) if $\mu>0$, irrespective of the values of σ and ρ.

Example 3. If $X_{(1)} \leqq \cdots \leqq X_{(n)}$ are ordered observations from any distribution then trivially $X_{(j)} \succ X_{(i)}$ ($\mathcal{R}_{3A}$) for any $j>i$.

Let (X_i, Y_i), $i=1, \cdots, n$, be a sample from the distribution $F(x, y)$ and consider the joint distribution of $X_{\max}=\max X_i$ and $Y_{\max}=\max Y_i$. A question is whether $X_i \succ Y_i$ ($\mathcal{R}_k$) implies $X_{\max} \succ Y_{\max}$ ($\mathcal{R}_k$). The answer is affirmative for $\mathcal{R}_1$, $\mathcal{R}'_2$, $\mathcal{R}''_2$, $\mathcal{R}_2$, $\mathcal{R}''_{2A}$, $\mathcal{R}_{2A}$ and $\mathcal{R}_{3A}$. For other cases $X_i \succ Y_i$ ($\mathcal{R}_3$ and $\mathcal{R}''_{2A}$) implies $X_{\max} \succ Y_{\max}$ ($\mathcal{R}_3$), and $X_i \succ Y_i$ ($\mathcal{R}_{1A}$ and $\mathcal{R}''_2$) implies $X_{\max} \succ Y_{\max}$ ($\mathcal{R}_{1A}$).

3.2. Transitivity and asymmetry

In the independent case it was shown by Pfanzagl [3] that the order relations in the sense of $\mathcal{R}_1$, $\mathcal{R}_2'$, $\mathcal{R}_2''$ and $\mathcal{R}_3$ are transitive. The order relation in the sense of $\mathcal{R}_0$ is not transitive even in the independent case. In the general case, however, the relations are not transitive, that is there exists a three-dimensional random vector (X, Y, Z) whose marginals satisfy $X \succ Y$ and $Y \succ Z$ $(\mathcal{R}_k)$ but not $X \succ Z$ $(\mathcal{R}_k)$.

Example 4. Let (X, Y, Z) have equal probabilities at three points $(2, 1, 0)$, $(0, 2, 1)$ and $(1, 0, 2)$, then $X \succ Y$ and $Y \succ Z$ ($\mathcal{R}_3$ and $\mathcal{R}_0$), but not even $X \succ Z$ ($\mathcal{R}_1$ or $\mathcal{R}_0$). So the transitivity does not hold for $\mathcal{R}_3$, $\mathcal{R}_2$, $\mathcal{R}_2'$, $\mathcal{R}_2''$, $\mathcal{R}_1$ and $\mathcal{R}_0$.

Example 5. Let (X, Y, Z) have equal probabilities at 15 points of the following table:

x	y	z	x	y	z	x	y	z
1	2	1	1	3	2	1	2	3
1	3	1	2	1	2	2	1	3
2	1	1	2	3	2	3	1	3
2	2	1	3	1	2	3	2	3
2	3	1	3	2	2			
			3	3	2			

Then $X \succ Y$ and $Y \succ Z$ $(\mathcal{R}_{3A})$ but not even $X \succ Z$ ($\mathcal{R}_{1A}$ or $\mathcal{R}_{\mathrm{I}}$). So the transitivity does not hold for $\mathcal{R}_{3A}$, $\mathcal{R}_{2A}$, $\mathcal{R}_{2A}'$, $\mathcal{R}_{2A}''$, $\mathcal{R}_{1A}$, $\mathcal{R}_{\mathrm{II}}$ and $\mathcal{R}_{\mathrm{I}}$.

It is easy to see that (X, Y) is symmetric iff $X \succ Y$ $(\mathcal{R}_{1A})$ and $Y \succ X$ $(\mathcal{R}_{1A})$, and that X and Y are identically distributed iff $X \succ Y$ $(\mathcal{R}_{\mathrm{I}})$ and $Y \succ X$ $(\mathcal{R}_{\mathrm{I}})$.

3.3. Comparison by the marginal distributions

Even if X and Y are not independent we may compare their marginal distributions, that is to determine the order of (X, Y) we may apply the previous definitions to $G(x)H(y)$ instead of $F(x, y)$. We denote this type of comparison by $X \succ Y$ ($\mathcal{R}$, mrg). Compared in the sense of $\mathcal{R}_1$ $X \succ Y$ ($\mathcal{R}_1$, mrg) is equivalent to $X \succ Y$ $(\mathcal{R}_{\mathrm{I}})$. Consider the situation where we compare two statistics $s(Z)$ and $t(Z)$ of a random sample Z. Generally two statistics are not independent, so if we treat analysis by $s(\cdot)$ and $t(\cdot)$ of the same data we should study the sampling distribution of $(s(Z), t(Z))$. If we apply each of $s(\cdot)$ and $t(\cdot)$ to different data, however, we can do with its marginal distributions.

Example 6. In a finite Markov chain with stochastic matrix P_{ij} assume that states are numbered in an ascending order of magnitudes.

Let Y denote the state in a step with probability distribution $(q_1, \cdots, q_n)$ and X the state in the next step, then

$$X \succ Y\ (\mathscr{R}_{3A}) \text{ iff } q_i P_{ij} \geqq q_j P_{ji} \qquad \text{for } i<j\ ,$$

while

$$X \succ Y\ (\mathscr{R}_3, \text{ mrg}) \text{ iff } \left(\sum_k q_k P_{kj}\right) \Big/ \left(\sum_k q_k P_{ki}\right) \geqq q_j/q_i \qquad \text{for } i<j\ .$$

4. Other approaches

A common measure to compare two random variables is the difference of their mean values. For general use we extend it by Definition 4. Let the (marginal) distribution functions of X and Y be $G(x)$ and $H(y)$, respectively.

DEFINITION 4. We call $\int (H(t)-G(t))dt$ the generalized mean difference of X and Y. We write $X \succ Y\ (\mathscr{R}_E)$ iff the generalized mean difference is nonnegative or positive infinite.

PROPOSITION 4.1. (1) If X and Y have finite mean values, then the difference of them is the generalized mean difference.
(2) If $G(t)=H(t-\theta)$ then the generalized mean difference is θ.
(3) If $\int xdG(x)=+\infty$ and the mean value of Y is finite or if the mean value X is finite and $\int ydH(y)=-\infty$ then $X \succ Y\ (\mathscr{R}_E)$.

PROPOSITION 4.2. $X \succ Y\ (\mathscr{R}_{\mathrm{I}}, \text{ mrg})$ implies $X \succ Y\ (\mathscr{R}_E)$. And $X \succ Y\ (\mathscr{R}_{\mathrm{I}}, \text{ mrg})$ implies $X \succ Y\ (\mathscr{R}_E)$, if $\int_{-\infty}^{\infty} [H(t)-G(t)]dt$ exists.

PROOF. The former statement is obvious. $X \succ Y\ (\mathscr{R}_{\mathrm{I}}, \text{ mrg})$ means that for any $a \geqq 0$

$$\int [1-G(t+a)]dH(t) \geqq \int [1-H(t+a)]dG(t) = \int G(t-a-0)dH(t)\ .$$

After some arrangements and integration

$$\int_{-\infty}^{\infty} \Big[\int [H(t+a)-G(t+a)]dH(t) \\ - \int [G(t-a-0)-H(t-a-0)]dH(t)\Big] da \geqq 0\ .$$

As the integrands are integrable, applying Fubini's theorem

$$\int\left[\int_{-\infty}^{\infty}[H(a)-G(a)]da-\int_{-\infty}^{\infty}[G(a)-H(a)]da\right]dH(t)\geqq 0$$

which means the generalized mean difference to be nonnegative.

Finally we introduce another class of sets $\mathcal{R}_S$ of middle strength. The meaning of its introduction is shown by Propositions 4.4 and 4.5.

DEFINITION 5. Let $\mathcal{R}_S$ be the class of all measurable subsets of R_x such that if $(x, y)\in S$ then any (x', y'), $x'\geqq x\geqq y\geqq y'$, belongs to it also.

PROPOSITION 4.3. $X\succ Y$ ($\mathcal{R}'_{2A}$ or $\mathcal{R}''_{2A}$) implies $X\succ Y$ ($\mathcal{R}_S$), and the latter implies $X\succ Y$ ($\mathcal{R}_1$ and $\mathcal{R}_{\mathrm{I}}$). Even in the independent case the inverse statements of these are not valid.

PROPOSITION 4.4. Let $r(x, y)$ be a function which is nondecreasing in x and nonincreasing in y. If $S\in\mathcal{R}_S$ then

$$\{(x, y); (r(x, y), r(y, x))\in S\}\in\mathcal{R}_S\ .$$

From this $r(X, Y)\succ r(Y, X)$ ($\mathcal{R}_S$) if $X\succ Y$ ($\mathcal{R}_S$). Conversely if a function $r(x, y)$ satisfies the condition $r(X, Y)\succ r(Y, X)$ ($\mathcal{R}_S$) for any (X, Y) such that $X\succ Y$ ($\mathcal{R}_S$), then $r(x, y)$ is nondecreasing in x and nonincreasing in y.

PROPOSITION 4.5. If (X, Y) has a symmetric distribution and $f(t)$ and $g(t)$ are real valued functions such that $f(t)\geqq g(t)$ for all t, then $f(X)\succ g(Y)$ ($\mathcal{R}_S$).

APPENDIX A. Counter examples in the dependent variables case.

Example A.1. To show that $X\succ Y$ ($\mathcal{R}_3$) does not imply $X\succ Y$ ($\mathcal{R}_{1A}$ nor $\mathcal{R}_{\mathrm{I}}$), so that none of ($\mathcal{R}_3$, $\mathcal{R}_2$, $\mathcal{R}'_2$, $\mathcal{R}''_2$ and $\mathcal{R}_1$) implies any of ($\mathcal{R}_{3A}$, $\mathcal{R}_{2A}$, $\mathcal{R}'_{2A}$, $\mathcal{R}''_{2A}$, $\mathcal{R}_{1A}$ or $\mathcal{R}_{\mathrm{II}}$), probabilities on 3×3 points:

y \ x	1	2	3
3	1/13	2/13	0
2	2/13	0	4/13
1	0	4/13	0

Example A.2. To show that $X\succ Y$ ($\mathcal{R}_{1A}$) does not imply $X\succ Y$ ($\mathcal{R}_0$). Probabilities on 4×4 points.

y \ x	1	2	3	4
4	1/11	1/11	1/11	0
3	1/11	1/11	0	0
2	1/11	0	1/11	0
1	0	0	0	4/11

Example A.3. To show that $X \succ Y$ ($\mathcal{R}_{2A}$) does not imply $X \succ Y$ ($\mathcal{R}_{\mathrm{II}}$). Probabilities on 4×4 points:

y \ x	1	2	3	4
4	0	0	0	0
3	0	1/4	0	0
2	0	0	0	1/4
1	0	0	1/4	1/4

APPENDIX B. Counter examples in the independent case.

Notice that counter examples in a restricted case play their role in general case.

Example B.1. To show that $X \succ Y$ ($\mathcal{R}_{\mathrm{II}}$) does not imply $X \succ Y$ ($\mathcal{R}_1$), so that none of ($\mathcal{R}_{\mathrm{II}}$, $\mathcal{R}_{\mathrm{I}}$ and $\mathcal{R}_0$) implies any of ($\mathcal{R}_3$, $\mathcal{R}_2$, $\mathcal{R}_2'$, $\mathcal{R}_2''$ or $\mathcal{R}_1$; further $\mathcal{R}_{3A}$, $\mathcal{R}_{2A}$, $\mathcal{R}_{2A}'$, $\mathcal{R}_{2A}''$ or $\mathcal{R}_{1A}$ in general case). Probabilities on 3×3 points:

y \ x	1	2	3	sum
3	3/24	3/24	3/24	3/8
2	1/24	1/24	1/24	1/8
1	4/24	4/24	4/24	4/8
sum	1/3	1/3	1/3	1

Example B.2. To show that $X \succ Y$ ($\mathcal{R}_1$) does not imply $X \succ Y$ ($\mathcal{R}_2'$ nor $\mathcal{R}_2''$; further $\mathcal{R}_{2A}'$ nor $\mathcal{R}_{2A}''$ in general case), consider the location-parameter family $F(x-\theta)$. It is ordered in the sense of $\mathcal{R}_1$, and furthermore it is ordered in the sense of $\mathcal{R}_2'$ (or $\mathcal{R}_2''$) iff $-\log(1-F(x))$ (or $-\log(F(x))$) is convex. Put

$$F(x)=\begin{cases} x^2/(1+x^2)\,, & x \geqq 0\,, \\ 0\,, & x<0\,, \end{cases}$$

then $-\log F(x)$ is convex but $-\log(1-F(x))$ is not. The example shows that $X \succ Y$ ($\mathcal{R}_2''$) does not imply $X \succ Y$ ($\mathcal{R}_2'$ therefore $\mathcal{R}_2$), and also that $X \succ Y$ ($\mathcal{R}_1$) does not imply $X \succ Y$ ($\mathcal{R}_2'$). Since $X \succ Y$ ($\mathcal{R}_2'$) iff $-Y \succ -X$ ($\mathcal{R}_2''$), the dual statements like that $X \succ Y$ ($\mathcal{R}_2'$) does not imply $X \succ Y$ ($\mathcal{R}_2''$) are valid.

For the location-parameter family of a discrete distribution like the two-point distribution a positively shifted distribution is not stochastically larger than the original in the sense of $\mathcal{R}_2'$ nor $\mathcal{R}_2''$.

Exmple B.3. To show that $X \succ Y$ ($\mathcal{R}_0$) does not imply $X \succ Y$ ($\mathcal{R}_{\mathrm{I}}$). Probabilities on 3×3 points:

y \ x	1	2	3	sum
3	4/25	0	6/25	2/5
2	4/25	0	6/25	2/5
1	2/25	0	3/25	3/5
sum	2/5	0	3/5	1

Example B.4. To show that $X \succ Y$ ($\mathcal{R}_2$) does not imply $X \succ Y$ ($\mathcal{R}_{\mathrm{II}}$), which $X \succ Y$ ($\mathcal{R}_2'$, $\mathcal{R}_2''$, $\mathcal{R}_1$ or $\mathcal{R}_{\mathrm{I}}$) does not imply a fortiori. Probabilities on 8×8 points (multiplied by 58×27): Notice that

$$P(X-Y=3)=86/(58\times 27)<P(X-Y=-3)=106/(58\times 27) .$$

y \ x	1	2	3	4	5	6	7	8	sum
8	1	1	10	2	1	10	1	32	1/27
7	1	1	10	2	1	10	1	32	1/27
6	10	10	100	20	10	100	10	320	10/27
5	1	1	10	2	1	10	1	32	1/27
4	2	2	20	4	2	20	2	64	2/27
3	4	4	40	8	4	40	4	128	4/27
2	4	4	40	8	4	40	4	128	4/27
1	4	4	40	8	4	40	4	128	4/27
sum	1/58	1/58	10/58	2/58	1/58	10/58	4/58	32/58	1

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