ON TESTING PROBLEMS CONCERNING MEAN OF MULTIVARIATE COMPLEX GAUSSIAN DISTRIBUTION

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Introduction

Let $\xi=(\xi_1,\cdots,\xi_p)'$ be a complex p-dimensional Gaussian random variable with complex mean $E(\xi)=\alpha=(\alpha_1,\cdots,\alpha_p)'$ and positive definite Hermitian complex covariance matrix $\Sigma=E(\xi-\alpha)(\xi-\alpha)^*$ where $(\xi-\alpha)^*$ is the adjoint of $(\xi-\alpha)$. The probability density function of ξ (with respect to the Lebesgue measure in the $2p$ dimensional Euclidian space of of real and immaginary parts of ξ) is given by

$$(1)\qquad p(\xi\,|\,\alpha,\Sigma)=\pi^{-p}(\det\Sigma)^{-1}\exp\{-(\xi-\alpha)^*\Sigma^{-1}(\xi-\alpha)\}$$

with $E(\xi-\alpha)(\xi-\alpha)'=0$. The problems considered here are the following:
(A) To test the null hypothesis $H_{10}:\alpha_1=\cdots=\alpha_p=0$ against alternatives $H_{11}:\alpha_1=\cdots=\alpha_{p_1}=0$ when $p_1<p$ and α, Σ are both unknown.
(B) To test the null hypothesis $H_{20}:\alpha_1=\cdots=\alpha_{p_1}=0$ against the alternatives $H_{21}:\alpha\neq 0$. When α, Σ are both unknown and $p_1<p$.
(C) To test the null hypothesis $H_{30}:\alpha_1=\cdots=\alpha_{p_1+p_2}=0$ against the alternatives $H_{31}:\alpha_1=\cdots=\alpha_{p_1}=0$ when $p_1+p_2<p$ and α, Σ are both unknown.

We will find here the likelihood ratio tests of these problems and examine their optimum properties. The real counterparts of these problems are wellknown in the statistical literature (see for example Giri [1], [2] and Stein [4], Giri, Kiefer and Stein [5]). As it has been shown there, the computation in this paper holds for the real normal population.

1. Likelihood ratio tests

Let $\xi_\beta=(\xi_{\beta_1},\cdots,\xi_{\beta_p})'$, $\beta=1,\cdots,N$ be N random observations on ξ. Write $N\bar{\xi}=\sum_1^N \xi_\beta$ and $S=\sum_1^N(\xi_\beta-\bar{\xi})(\xi_\beta-\bar{\xi})^*$. We will assume throughout that $N>p$ so that S is positive definite Hermitian with probability one. Write $\sigma_i=\sum_1^i p_j$ with $\sigma_0=0$ and $\sum_1^k p_j=p$. Let

$$\xi_\beta=(\xi_{\beta(1)},\cdots,\xi_{\beta(k)})', \qquad \xi_{\beta[i]}=(\xi_{\beta(1)},\cdots,\xi_{\beta(i)})'$$

$$\alpha=(\alpha_{(1)},\cdots,\alpha_{(k)})', \qquad \alpha_{[i]}=(\alpha_{(1)},\cdots,\alpha_{(i)})'$$

$$\bar\xi=(\bar\xi_{(1)},\cdots,\bar\xi_{(k)})', \qquad \bar\xi_{[i]}=(\bar\xi_{(1)},\cdots,\bar\xi_{(i)})\ ;$$

$$S=\begin{pmatrix} S_{(11)} & \cdots & S_{(1k)} \\ \cdots & \cdots & \cdots \\ S_{(k1)} & \cdots & S_{(kk)} \end{pmatrix}, \qquad S_{[ii]}=\begin{pmatrix} S_{(11)} & \cdots & S_{(1i)} \\ \cdots & \cdots & \cdots \\ S_{(i1)} & \cdots & S_{(ii)} \end{pmatrix}$$

$$\Sigma=\begin{pmatrix} \Sigma_{(11)} & \cdots & \Sigma_{(1k)} \\ \cdots & \cdots & \cdots \\ \Sigma_{(k1)} & \cdots & \Sigma_{(kk)} \end{pmatrix}, \qquad \Sigma_{[ii]}=\begin{pmatrix} \Sigma_{(11)} & \cdots & \Sigma_{(1i)} \\ \cdots & \cdots & \cdots \\ \Sigma_{(i1)} & \cdots & \Sigma_{(1i)} \end{pmatrix}$$

where $\xi_{\beta(i)}$, $\alpha_{(i)}$, $\bar\xi_{(i)}$ are $p_i\times 1$ subvectors of ξ_β, α, $\bar\xi$ respectively and $S_{(ii)}$, $\Sigma_{(ii)}$ are $p_i\times p_i$ submatrices of S, Σ respectively. We will also write for $i>j$, $S_{[ij]}=(S_{(i1)},\cdots,S_{(ij)})$ and $S_{[ji]}=(S_{(1i)},\cdots,S_{(ji)})$. $\Sigma_{[ij]}$ and $\Sigma_{[ji]}$ will be used to denote similar vectors in terms of Σ's. Furthermore we define $C_1^2,\cdots,C_k^2$ and $R_1,\cdots,R_k$ by

$$\sum_1^i C_j^2=N\bar\xi_{[i]}^*S_{[ii]}^{-1}\bar\xi_{[i]}\,, \tag{1.2}$$

$$\begin{aligned}\sum_1^i R_j&=\sum_1^i C_j^2\Big/\Big(1+\sum_1^i C_j^2\Big)\\ &=N\bar\xi_{[i]}^*S_{[ii]}^{-1}\bar\xi_{[i]}/(1+N\bar\xi_{[i]}^*S_{[ii]}^{-1}\bar\xi_{[i]})\\ &=N\bar\xi_{[i]}^*(S_{[ii]}+N\bar\xi_{[i]}\bar\xi_{[i]}^*)^{-1}\bar\xi_{[i]}\end{aligned}$$

and $\delta_1,\cdots,\delta_k$ by

$$\sum_1^i \delta_j=N\alpha_{[i]}^*\Sigma_{[ii]}^{-1}\alpha_{[i]}\ . \tag{1.3}$$

Since Σ and S are hermitian positive definite, $R_i\geqq 0$, $\delta_i\geqq 0$ for all i. The last equality of (1.2) follows from the following lemma.

LEMMA 1. *For any Hermitian positive definite matrix S of dimension $p\times p$ and for any complex p-vector ξ*

$$\xi^*(S+\xi\xi^*)^{-1}\xi=\frac{\xi^*S^{-1}\xi}{1+\xi^*S^{-1}\xi}\ .$$

PROOF.

$$(S+\xi\xi^*)^{-1}=S^{-1}-(S+\xi\xi^*)^{-1}\xi\xi^*S^{-1}\ .$$

Hence,

$$\xi^*(S+\xi\xi^*)^{-1}\xi=\xi^*S^{-1}\xi-\xi^*(S+\xi\xi^*)^{-1}\xi(\xi^*S^{-1}\xi)\ .$$

Therefore we get

$$\xi^*(S+\xi\xi^*)^{-1}\xi=\frac{\xi^*S^{-1}\xi}{1+\xi^*S^{-1}\xi} .$$

From Giri [3] the joint distribution of $R_1,\cdots,R_k$ is given by

$$(1.4)\qquad p(R_1,\cdots,R_k)=\Gamma(N)\Big[\Gamma\Big(N-\sum_i^k p_i\Big)\prod_1^k(p_i)\Big]^{-1} \cdot\Big(1-\sum_1^k R_i\Big)^{N-\sum_1^k p_i-1}\prod_1^k R_i^{p_i-1} \cdot\exp\Big\{-\sum_1^k\delta_j\Big\}+\sum_1^k R_j\sum_{i>j}\delta_i\prod_1^k\Phi(N-\sigma_{i-1},p_i;R_i\delta_i)$$

where $\Phi(a,b;x)$ is the confluent hypergeometric function given by

$$\phi(a,b;x)=1+\frac{a}{b}x\frac{a(a+1)}{b(b+1)}\frac{x^2}{2!}+\cdots .$$

Further it has been shown that the marginal distribution of $R_1,\cdots,R_j$ is obtained from (1.4) by replacing k by j. If $\delta_i=0$ for all i then the joint distribution of $R_1,\cdots,R_k$ is a multivariate beta i.e.

$$(1.5)\qquad p(R_1,\cdots,R_k)=\Gamma(N)\Big[\Gamma\Big(N-\sum_1^k p_i\Big)\prod_1^k\Gamma(p_i)\Big]^{-1} \cdot\Big(1-\sum_1^k R_i\Big)^{N-\sum_1^k p_i-1}\prod_1^k R_i^{p_i-1} .$$

In this particular case, it is easy to see that $(1-R_1-R_2)(1-R_1)^{-1}$ is a beta random variable with parameter $N-p_1-p_2$ and p_2 and is distributed independently of R_1. Also $1-R_1$ is beta with parameter $N-p_1$ and p_1.

The likelihood of the observations $\xi_1,\cdots,\xi_N$ is given by

$$(1.6)\qquad L=\pi^{-Np}(\det\Sigma)^{-N}\exp\Big\{-\sum_1^N(\xi_\beta-\alpha)^*\Sigma^{-1}(\xi_\beta-\alpha)\Big\} .$$

Case A. With the above notations k takes the value 2 for this case. Under $H_{10}:\alpha=0$ and under the alternatives $H_{11}:\alpha_{(1)}=0$.

The likelihood ratio test criterion for testing H_{10} against H_{11} is (from (1.9) below)

$$(1.7)\qquad \begin{aligned}\lambda&=\max_{H_{11}}L\Big/\max_{H_{10}}L\\ &=\frac{|S_{(11)}+N\bar\xi_{(1)}\bar\xi^*_{(1)}|^{-N}|S_{(22)}-S_{(21)}S^{-1}_{(11)}S_{(12)}|^{-N}}{|S+N\bar\xi\bar\xi^*|^{-N}}\\ &=(1+N\bar\xi^*_{(1)}S^{-1}_{(11)}\bar\xi_{(1)})^{-N}/(1+N\bar\xi^*S^{-1}\bar\xi)^{-N}\\ &=(1-R_1-R_2)^{-N}/(1-R_1)^{-N} .\end{aligned}$$

Thus the likelihood ratio test is to reject H_{10} if $(1-R_1-R_2)(1-R_1)^{-1}$ is less than a constant depending on the size of the test. Under H_{10}, $(1-R_1-R_2)(1-R_1)^{-1}$ is distributed as beta (central) with parameters $N-p_1-p_2$ and p_2.

Case B. Here also $k=2$. Under $H_{20}: \alpha_{(1)}=0$ and under the alternatives $H_{21}: \alpha\neq 0$. Now

$$
\begin{aligned}
(1.8)\qquad \lambda &= \max_{H_{20}} L \Big/ \max_{H_{21}} L \\
&= \frac{|S_{(11)}+N\bar{\xi}_{(1)}\bar{\xi}^*_{(1)}|^{-N}|S_{(22)}-S_{(21)}S^{-1}_{(11)}S_{(12)}|^{-N}}{|S|^{-N}} \\
&= (1+N\bar{\xi}^*_{(1)}S^{-1}_{(11)}\bar{\xi}_{(1)})^{-N}=(1-R_1)^N .
\end{aligned}
$$

Hence the likelihood ratio test of H_{20} against H_{21} is to reject H_{20} if $1-R_1$ is less than a constant and under H_{20}, $1-R_1$ is beta with parameter $N-p_1$ and p_1.

Case C. Here k is equal to 3. Under $H_{30}: \alpha_{[2]}=0$ and under the alternatives $H_{31}: \alpha_{(1)}=0$. Writing

$$
\begin{aligned}
(1.9)\qquad & \sum_1^N (\xi_\beta-\alpha)^*\Sigma^{-1}(\xi_\beta-\alpha) \\
&= \sum_1^N (\xi_{\beta[2]}-\alpha_{[2]})^*\Sigma^{-1}_{[22]}(\xi_{\beta[2]}-\alpha_{[2]}) \\
&\quad + \sum_1^N (\xi_{\beta(3)}-\alpha_{(3)}-\Sigma_{[32]}\Sigma^{-1}_{[22]}(\xi_{\beta[2]}-\alpha_{[2]}))^*(\Sigma_{(33)}-\Sigma_{[32]}\Sigma^{-1}_{[22]}\Sigma_{[23]})^{-1} \\
&\quad \cdot (\xi_{\beta(3)}-\alpha_{(3)}-\Sigma_{[32]}\Sigma^{-1}_{[22]}(\xi_{\beta[2]}-\alpha_{[2]})) ,
\end{aligned}
$$

we get

$$
\max_{H_{30}} L=|S|^{-N}(1+N\bar{\xi}^*_{[2]}S^{-1}_{[22]}\bar{\xi}_{[2]})^{-N}\exp(-Np)\Pi^{-Np} .
$$

Similarly,

$$
\max_{H_{31}} L=|S|^{-N}(1+N\bar{\xi}^*_{(1)}S^{-1}_{(11)}\bar{\xi}_{(1)})^{-N}\exp(-Np)\Pi^{-Np} .
$$

Thus the likelihood ratio test is to reject H_{30} whenever $(1-R_1-R_2)(1-R_1)^{-1}$ is less than a constant, and $(1-R_1-R_2)(1-R_1)^{-1}$ is beta with parameters $(N-p_1-p_2)$, p_2 when H_{30} is true.

2. Some optimum properties

Case A. The problem of testing H_{10} against H_{11} remains invariant under the group of transformations G of $p\times p$ nonsingular complex matrices

$$g=\begin{pmatrix} g_{11} & 0 \\ g_{12} & g_{22} \end{pmatrix}$$

operating as $(\xi, \alpha, \Sigma)\rightarrow(g\xi, g\alpha; g\Sigma g^*)$ where g_{11} is the $p_1\times p_1$ submatrix of g. The maximal invariant Giri [2] in the sample space of $\xi_1,\cdots,\xi_N$ is (R_1, R_2) and the corresponding maximal invariant in the parametric space of (α, Σ) is (δ_1, δ_2). From (1.4) the conditional distribution of R_2 given R_1 is given by

$$\begin{aligned}(2.1) \qquad p(R_2|R_1)=&\Gamma(N-p_1)[\Gamma(p_2)\Gamma(N-p_1-p_2)]^{-1}R_2^{p_2-1}\\ &\cdot(1-R_1-R_2)^{N-p_1-p_2-1}(1-R_1)^{-(N-p_1-1)}\\ &\cdot\exp\{-\delta_2(1-R_1)\}\Phi(N-p_1, p_2; R_2\delta_2)\ .\end{aligned}$$

Thus the ratio of this conditional density under H_{11} to this density under H_{10} is

$$(2.2) \qquad p_{H_{11}}(R_2|R_1)/p_{H_{10}}(R_2|R_1)=\exp(-\delta_2(1-R_1)\Phi((N-p_1, p_2; R_2\delta_2)\ .$$

Thus it is easy to note that conditionally given R_1 the test which rejects H_{10} for large values of R_2 or equivalently for small values of $1-R_2(1-R_1)^{-1}=(1-R_1-R_2)(1-R_1)^{-1}$ (the likelihood ratio test) is uniformly most powerful invariant.

Case B. This problem remains invariant under the following group of transformations

$$(g, b)\xi_i=g\xi_i+b\ , \qquad g\in G;\ i=1,\cdots,N\ ,$$

γ and b is any complex vector with $b_{(1)}=0$, $b_{(1)}$ is defined similar to $\alpha_{(1)}$. The maximal invariant in the sample space is R_1 and the corresponding maximal invariant in the parametric space is δ_1.

$$\begin{aligned}(2.3) \qquad p(R_1)=&\Gamma(N)[\Gamma(N-p_1)\Gamma(p_1)]^{-1}R_1^{p_1-1}(1-R_1)^{N-p_1-1}\\ &\cdot\exp[-\delta_1]\Phi(N, p_1; R_1\delta_1)\ .\end{aligned}$$

Now the ratio of the density of R_1 under H_{21} to its density under H_{20} is given by

$$(2.4) \qquad p_{H_{21}}(R_1)/p_{H_{20}}(R_1)=\exp\{-\delta_1\}\Phi(N, p_1; R_1\delta_1)\ .$$

Thus for testing H_{20} against H_{21} the test which rejects H_{20} for large values of R_1 or equivalently for small values of $1-R_1$ (the likelihood ratio test) is uniformly most powerful invariant.

Case C. Here $K=3$; $H_{30}:\alpha_{[2]}=0$ and $H_{31}:\alpha_{(1)}=0$. This problem remains invariant under the following group of transformations

$$(g_T, C)\xi_i=g_T\xi_i+C \qquad \text{for all } i;\ g_T\in G_T\ ,$$

where C is a complex p-vector with $C_{[2]}=0$ ($C_{[2]}$ is defined similar to $\alpha_{[2]}$) and G_T is the group of $p\times p$ nonsingular complex matrices of the form

$$g_T=\begin{pmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

with g_{ii} being the $p_i\times p_i$ submatrix of g_T. The maximal invariant in the sample space is (R_1R_2) and the corresponding maximal invariant in the sample space is (δ_1, δ_2). It terms of δ_1 and δ_2, $H_{30}: \delta_1=\delta_2=0$ and $H_{31}: \delta_1=0$. It is now clear that the conclusion given in case A carry over without change to this case.

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