THE ASYMPTOTIC DISTRIBUTIONS OF THE STATISTICS BASED ON THE COMPLEX GAUSSIAN DISTRIBUTION*

TAKESI HAYAKAWA

(Received April 1, 1971; revised Nov. 8, 1971)

1. Introduction and Notations

Recently the asymptotic distribution of the statistics based on the multivariate normal samples were derived by the use of the fundamental formulas of the series of the zonal polynomials [1], [2], [7]. The purpose of this paper is to give the asymptotic distributions of the statistics based on the complex multivariate Gaussian distribution which was developed by Goodman, N. R. [3], James, A. T. [4] and Khatri, C. G. [5], [6]. To obtain these distributions, we need also the fundamental formulas of the series of the zonal polynomials of the positive definite hermitian matrix. If we do not notice in this paper, we assume that all the matrices are $m\times m$ positive definite hermitian matrices.

Let S be a positive definite hermitian matrix whose characteristic roots are $\lambda_1, \cdots, \lambda_m$ such that $\lambda_1>\cdots>\lambda_m>0$ and $\Lambda=\mathrm{diag}(\lambda_1, \cdots, \lambda_m)$ be a diagonal matrix whose diagonal elements are $\lambda_1, \cdots, \lambda_m$ in a descending order. Let $\tilde{C}_\kappa(S)$ be a zonal polynomial of S, which corresponds to the partition κ of k into not more than m parts. It can be represented by

$$\tilde{C}_\kappa(S)=\chi_{[\kappa]}(1)\chi_{\{\kappa\}}(S)\ ,$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group and $\chi_{\{\kappa\}}(S)$ is the character of the representation $\{\kappa\}$ of the general linear group [4].

Let

$${}_p\tilde{F}_q^{(m)}(a_1, \cdots, a_p, b_1, \cdots, b_q;\ S, T)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_p]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\frac{\tilde{C}_\kappa(S)\tilde{C}_\kappa(T)}{k!\tilde{C}_\kappa(I_m)}\ ,$$

where

* This research was supported by the National Science Foundation under Grant No. GU-2059, the U.S. Air Force Office of Scientific Research under Contract No. AFOSR-68-1415 and the Sakko-kai Foundation.

$$[a]_\kappa=\prod_{\alpha=1}^{m}(a-\alpha+1)_{k_\alpha}\,, \qquad (a)_x=a(a+1)\cdots(a+x-1)\,,$$

$$k=k_1+\cdots+k_m\,, \qquad k_1\geq\cdots\geq k_m\geq 0\,.$$

We denote as ${}_p\tilde{F}_q(\cdots,\cdots;S)={}_pF_q^{(m)}(\cdots,\cdots;S,I)$ if $T=I_m$.

Let S and R be positive definite hermitian matrices and T be also an hermitian matrix, then

$$\text{(1.1)} \qquad \int_{\bar{R}'=R>0}\operatorname{etr}(-RS)(\det R)^{a-m}\tilde{C}_\kappa(RT)dR=\tilde{\Gamma}_m(a,\kappa)(\det S)^{-a}\tilde{C}_\kappa(TS^{-1})\,,$$

where $\tilde{\Gamma}_m(a,\kappa)=\pi^{m(m-1)/2}\Pi_{\alpha=1}^{m}\Gamma(a+k_\alpha-(\alpha-1))$, and

$$\text{(1.2)} \qquad \int_{I>\bar{R}'=R>0}(\det R)^{a-m}\det(I-R)^{b-m}\tilde{C}_\kappa(RS)dR=\frac{\tilde{\Gamma}_m(a,\kappa)\tilde{\Gamma}_m(b)}{\tilde{\Gamma}_m(a+b,\kappa)}\tilde{C}_\kappa(S)\,,$$

where $\tilde{\Gamma}_m(a)=\pi^{m(m-1)/2}\Pi_{\alpha=1}^{m}\Gamma(a-(\alpha-1))=\tilde{\Gamma}_m(a,\kappa)/[a]_\kappa$.

Let X be an $n\times n$ arbitrary complex matrix and U be a unitary matrix on the unitary group $U(n)$ of order n, then

$$\text{(1.3)} \qquad \int_{U(n)}\operatorname{etr}(XU+\bar{U}'\bar{X}')d(U)={}_0\tilde{F}_1(n,X\bar{X}')\,,$$

where $d(U)$ is the unitary invariant measure of the unitary group with total volume unity.

We use the following notations. Let X be an $m\times n$ $(m\leq n)$ complex matrix which has a complex Gaussian distribution with mean $M_{m\times n}$ and covariance matrix Σ, then we denote as $X\sim CN_m(M,\Sigma)$. Let S be an hermitian matrix which has a complex Wishart distribution of n degrees of freedom with a non-central matrix Ω, then we denote as $S\sim CW_m(\Sigma,n,\Omega)$.

2. The fundamental formulas of the sum of the zonal polynomials

In this section, we consider only $m\times m$ positive definite hermitian matrices. Let Σ be hermitian matrix and

$$\Sigma=\Sigma^R+i\Sigma^I\,,\quad \Sigma^R=(\sigma_{\alpha\beta}^R)\,,\quad \Sigma^I=(\sigma_{\alpha\beta}^I)\,,\quad \alpha,\beta=1,2,\cdots,m\,,$$

where $\Sigma^{R'}=\Sigma^R$ and $\Sigma^{I'}=-\Sigma^I$. We here define the hermitian differential operator matrix ∂ as follows.

$$\text{(2.1)} \qquad \partial=\partial_R+i\partial_I\,,\quad \partial=(\partial_{\alpha\beta})\,,\quad \partial_R=(\partial_{\alpha\beta}^R)\,,\quad \partial_I=(\partial_{\alpha\beta}^I)\,,$$

and

$$\partial_{\alpha\beta}^R=\frac{1+\delta_{\alpha\beta}}{2}\frac{\partial}{\partial\sigma_{\alpha\beta}^R} \quad\text{and}\quad \partial_{\alpha\beta}^I=\frac{1-\delta_{\alpha\beta}}{2}\frac{\partial}{\partial\sigma_{\alpha\beta}^I}\,.$$

From the symmetry of Σ^R and the skew symmetry of Σ^I, we can see $\partial^R_{\alpha\beta}=\partial^R_{\alpha\beta}$ and $\partial^I_{\beta\alpha}=-\partial^I_{\beta\alpha}$. Hence ∂_R and ∂_I are a symmetric and a skew symmetric differential operator matrices, respectively.

Let $f(\Sigma)$ be a real valued function of an hermitian matrix Σ, and it belongs to C^∞, then we have a Taylor series expansion of $f(\Sigma)$ in the neighborhood at $\Sigma=\Sigma_0$ as follows.

$$f(\Sigma)=\operatorname{etr}((\Sigma-\Sigma_0)\partial)f(\Sigma)|_{\Sigma=\Sigma_0}. \tag{2.2}$$

We can show easily that (2.2) is same as (2.3) if S is an hermitian matrix.

$$f(S)=\operatorname{etr}((S-\Sigma_0)\partial)f(\Sigma)|_{\Sigma=\Sigma_0}. \tag{2.3}$$

The following lemmas are fundamental.

LEMMA 1. *Let κ be a partition of k into not more than m parts, i.e.,*

$$\kappa=(k_1,\cdots,k_m),\quad k=k_1+\cdots+k_m,\quad k_1\geq\cdots\geq k_m\geq 0$$

and let

$$\tilde{a}_1(\kappa)=\Sigma_{\alpha=1}^m k_\alpha(k_\alpha-2\alpha)\quad and\quad \tilde{a}_2(\kappa)=2\Sigma_{\alpha=1}^m k_\alpha(k_\alpha^2-3\alpha k_\alpha+3\alpha^2),$$

then

$$(\tilde{a}_1(\kappa)+k)\tilde{C}_\kappa(\Sigma)=\operatorname{tr}(\Lambda\partial)^2\tilde{C}_\kappa(\Sigma)|_{\Sigma=\Lambda} \tag{2.4}$$

$$\begin{aligned}&\{3\tilde{a}_1^2(\kappa)-2\tilde{a}_2(\kappa)+6k\tilde{a}_1(\kappa)-6\tilde{a}_1(\kappa)+3k^2-2k\}\tilde{C}_\kappa(\Sigma)\\&=[8(\operatorname{tr}(\Lambda\partial)^3)+3(\operatorname{tr}(\Lambda\partial)^2)^2]\tilde{C}_\kappa(\Sigma)|_{\Sigma=\Lambda},\end{aligned} \tag{2.5}$$

where $\Lambda=\operatorname{diag}(\lambda_1,\cdots,\lambda_m)$ is a diagonal matrix of latent roots of Σ.

PROOF. From (1.1), we have

$$\begin{aligned}&\frac{n^{mn}}{\tilde{\Gamma}_m(n)(\det\Sigma)^n}\int_{\bar{R}'=R>0}\operatorname{etr}(-n\Sigma^{-1}R)(\det R)^{n-m}\tilde{C}_\kappa(R)dR\\&=[n]_\kappa\left(\frac{1}{n}\right)^k\tilde{C}_\kappa(\Sigma).\end{aligned} \tag{2.6}$$

We can see easily that the L.H.S. of (2.6) is invariant under the transformation $R=UW\bar{U}'$ such that $\Sigma=U\Lambda\bar{U}'$ where $\Lambda=\operatorname{diag}(\lambda_1,\cdots,\lambda_m)$ is a diagonal latent roots matrix and $U\in U(m)$. Hence we can rewrite L.H.S. of (2.6) as (2.7),

$$\frac{n^{mn}}{\tilde{\Gamma}_m(n)(\det\Lambda)^n}\int_{\bar{W}'=W>0}\operatorname{etr}(-n\Lambda^{-1}W)(\det W)^{n-m}\tilde{C}_\kappa(W)dW. \tag{2.7}$$

Here we expand $\tilde{C}_\kappa(W)$ into a Taylor series expansion in the neighborhood at $W=\Lambda$ by the use of (2.3). Then we have a following asymptotic expansion

$$\text{(2.8)}\qquad \frac{n^{mn}}{\tilde{\Gamma}_m(n)(\det \Lambda)^n}\int_{\bar{W}'=W>0}\operatorname{etr}(-n\Lambda^{-1}W)(\det W)^{n-m}$$
$$\times\operatorname{etr}((W-\Lambda)\partial)dW\tilde{C}_\kappa(\Sigma)|_{\Sigma=\Lambda}$$
$$=\operatorname{etr}(-\Lambda\partial)\det\left(I-\frac{1}{n}\Lambda\partial\right)^{-n}\tilde{C}_\kappa(\Sigma)|_{\Sigma=\Lambda}$$
$$=\left\{1+\frac{1}{2n}\operatorname{tr}(\Lambda\partial)^2+\frac{1}{24n^2}[8\operatorname{tr}(\Lambda\partial)^3+3(\operatorname{tr}(\Lambda\partial)^2)^2]\right.$$
$$\left.+0(1/n^3)\right\}\tilde{C}_\kappa(\Sigma)|_{\Sigma=\Lambda}\,.$$

On the other hand, R.H.S. of (2.6) also have an asymptotic expansion such that

$$\text{(2.9)}\qquad \left\{1+\frac{1}{2n}(\tilde{a}_1(\kappa)+k)+\frac{1}{24n^2}\{3\tilde{a}_1^2(\kappa)-2\tilde{a}_2(\kappa)+6k\tilde{a}_1(\kappa)-6\tilde{a}_1(\kappa)\right.$$
$$\left.+3k^2-2k\}+0(1/n^3)\right\}\tilde{C}_\kappa(\Sigma)\,.$$

Hence by comparing with the both side of order $1/n$ and $1/n^2$, we have Lemma 1.

LEMMA 2.

$$\text{(2.10)}\qquad \sum_{k=r}^{\infty}\sum_{\kappa}\frac{x^k\tilde{C}_\kappa(\Sigma)}{(k-r)!}=x^r\cdot(\operatorname{tr}\Sigma)^r\operatorname{etr}(x\Sigma)\,.$$

(2.10) *holds for all integers r.*

$$\text{(2.11)}\qquad \sum_{k=0}^{\infty}\sum_{\kappa}\frac{x^k\tilde{a}_1(\kappa)\tilde{C}_\kappa(\Sigma)}{k!}=(x^2\operatorname{tr}\Sigma^2-x\operatorname{tr}\Sigma)\operatorname{etr}(x\Sigma)\,.$$

$$\text{(2.12)}\qquad \sum_{k=r}^{\infty}\sum_{\kappa}\frac{x^k\tilde{a}_1(\kappa)\tilde{C}_\kappa(\Sigma)}{(k-r)!}=\{x^{r+2}\operatorname{tr}\Sigma^2(\operatorname{tr}\Sigma)^r-x^{r+1}(\operatorname{tr}\Sigma)^{r+1}$$
$$+2rx^{r+1}\operatorname{tr}\Sigma^2(\operatorname{tr}\Sigma)^{r-1}-rx^r(\operatorname{tr}\Sigma)^r$$
$$+r(r-1)x^r\operatorname{tr}\Sigma^2(\operatorname{tr}\Sigma)^{r-2}\}\operatorname{etr}(x\Sigma)\,.$$

$$\text{(2.13)}\qquad \sum_{k=0}^{\infty}\sum_{\kappa}\frac{x^k\tilde{a}_1^2(\kappa)\tilde{C}_\kappa(\Sigma)}{k!}=\{x^4(\operatorname{tr}\Sigma^2)^2+4x^3\operatorname{tr}\Sigma^3-2x^3\operatorname{tr}\Sigma\operatorname{tr}\Sigma^2$$
$$+3x^2(\operatorname{tr}\Sigma)^2-4x^2\operatorname{tr}\Sigma^2+x\operatorname{tr}\Sigma\}\operatorname{etr}(x\Sigma)\,.$$

$$\text{(2.14)}\qquad \sum_{k=0}^{\infty}\sum_{\kappa}\frac{x^k\tilde{a}_2(\kappa)\tilde{C}_\kappa(\Sigma)}{k!}=\{2x^3\operatorname{tr}\Sigma^3+3x^2(\operatorname{tr}\Sigma)^2-3x^2\operatorname{tr}\Sigma^2$$
$$+2x\operatorname{tr}\Sigma\}\operatorname{etr}(x\Sigma)\,.$$

PROOF. Since $\operatorname{etr}(x\Sigma)=\sum_{k=0}\sum_{\kappa}(x^k\tilde{C}_\kappa(\Sigma)/k!)$, we have (2.10) by differentiation or integration on both sides, succesively. From Lemma 1, we know

$$\tilde{a}_1(\kappa)\tilde{C}_\kappa(\Sigma)=\operatorname{tr}(\Lambda\partial)^2\tilde{C}_\kappa(\Sigma)|_{\Sigma=\Lambda}-k\tilde{C}_\kappa(\Sigma)\,.$$

Multiply $x^k/k!$ on both sides and sum form $k=0$ to infinite, we have

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{x^k\tilde{a}_1(\kappa)\tilde{C}_\kappa(\Sigma)}{k!}=\operatorname{tr}(\Lambda\partial)^2\operatorname{etr}(x\Sigma)|_{\Sigma=\Lambda}-x\operatorname{tr}\Sigma\operatorname{etr}(x\Sigma)\,.$$

From the definition of ∂, the first term of R.H.S. becomes

$$\begin{aligned}\operatorname{tr}(\Lambda\partial)^2\operatorname{etr}(x\Sigma)|_{\Sigma=\Lambda}&=\sum_{\alpha,\beta=1}^{m}\lambda_\alpha\lambda_\beta\partial_{\alpha\beta}\bar{\partial}_{\alpha\beta}\exp\left(x\sum_{\alpha=1}^{m}\sigma_{\alpha\alpha}\right)\Big|_{\Sigma=\Lambda}\\&=\left\{\sum_{\alpha=1}^{m}\lambda_\alpha^2\frac{\partial^2}{\partial\sigma_{\alpha\alpha}^2}+\frac{1}{2}\sum_{\alpha<\beta}\lambda_\alpha\lambda_\beta\left(\frac{\partial^2}{\partial\sigma_{\alpha\beta}^{R^2}}+\frac{\partial^2}{\partial\sigma_{\alpha\beta}^{I^2}}\right)\right\}\operatorname{etr}(x\Sigma)|_{\Sigma=\Lambda}\\&=x^2\operatorname{tr}\Lambda^2\operatorname{etr}(x\Lambda)\,.\end{aligned}$$

Hence we obtain (2.11). (2.12) can be obtained by applying the Leibnitz formula of differentiation to (2.11). As we can show (2.13) and (2.14) by the same way as one of [7], we will omit.

3. The asymptotic distribution of the statistics based on the non-central complex Wishart matrix

Recently Fujikoshi [1], [2] has obtained the asymptotic distributions of a generalized variance and a trace of non-central Wishart Matrix. In this section, we give the asymptotic distribution of these statistics based on a complex non-central Wishart matrix by the completely same way as [1] and [2].

THEOREM 1. *Let nS be distributed with $CW_m(\Sigma, n, \Omega)$ and let's assume that Ω is a constant matrix with respect to n. Put*

$$\lambda=\sqrt{n/m}\,\log\{\det S/\det\Sigma\}\,. \tag{3.1}$$

Then we have

$$\Pr\{\lambda\leq x\}=\Phi(x)+\frac{G_1}{\sqrt{mn}}+\frac{G_2}{mn}+\frac{G_3}{mn\sqrt{mn}}+O\left(\frac{1}{n^2}\right),$$

where

$$\begin{aligned}G_1&=\frac{1}{2}(m^2-2\operatorname{tr}\Omega)\Phi^{(1)}(x)+\frac{1}{6}\Phi^{(3)}(x)\,.\\G_2&=\frac{1}{8}\{m^2(m^2+2)-4m^2\operatorname{tr}\Omega+4(\operatorname{tr}\Omega)^2\}\Phi^{(2)}(x)\end{aligned}\tag{3.2}$$

$$+\frac{1}{12}(m^2+1-2\operatorname{tr}\Omega)\Phi^{(4)}(x)+\frac{1}{72}\Phi^{(6)}(x)\,,$$

$$G_3=\frac{1}{12}\{m^2(2m^2-1)+6m\operatorname{tr}\Omega^2\}\Phi^{(1)}(x)$$

$$+\frac{1}{48}\{m^2(m^2+2)(m^2+4)-6m^2(m^2+2)\operatorname{tr}\Omega+12m^2(\operatorname{tr}\Omega)^2$$

$$-8(\operatorname{tr}\Omega)^3\}\Phi^{(3)}(x)$$

$$+\frac{1}{240}\{5m^4+20m^2+12-20(m^2+1)\operatorname{tr}\Omega+20(\operatorname{tr}\Omega)^2\}\Phi^{(5)}(x)$$

$$+\frac{1}{144}(m^2+2-2\operatorname{tr}\Omega)\Phi^{(7)}(x)+\frac{\Phi^{(9)}(x)}{1296}\,.$$

$\Phi^{(k)}(x)$ denotes the k-th derivative of the standard normal distribution function $\Phi(x)$.

PROOF. We can easily obtain the characteristic function $\varphi(t)$ of λ as follows:

$$\text{(3.3)}\qquad \operatorname{etr}(-\Omega)\left(\frac{1}{n}\right)^{it\sqrt{mn}}\frac{\tilde{\Gamma}_m(n+it\sqrt{n/m})}{\tilde{\Gamma}_m(n)}\,{}_1\tilde{F}_1(n+it\sqrt{n/m}\,,\,n\,;\,\Omega)\,.$$

Hence by expanding (3.3) as the series of order $1/\sqrt{n}$ and by applying Lemma 2, we have the asymptotic expansion of $\varphi(t)$. Therefore, by inverting this series, we obtain the result (3.2).

THEOREM 2. *Let nS be distributed with $CW_m(\Sigma, n, \Omega)$.*

Case 1. *Ω is a constant matrix with respect to n. Put $\hat{\lambda}=\sqrt{n}\cdot(\operatorname{tr}S-\operatorname{tr}\Sigma)/\tau$, where $\tau^2=\operatorname{tr}\Sigma^2$, then*

$$\text{(3.4)}\qquad \Pr\{\hat{\lambda}\le x\}=\Phi(x)-\frac{T_1}{\sqrt{n}}+\frac{T_2}{n}-\frac{T_3}{n\sqrt{n}}+O\left(\frac{1}{n^2}\right),$$

where

$$T_1=\Phi^{(1)}(x)\frac{\operatorname{tr}\Sigma\Omega}{\tau}+\frac{\operatorname{tr}\Sigma^3}{3\tau^2}\Phi^{(3)}(x)\,,$$

$$T_2=\frac{\Phi^{(2)}(x)}{\tau^2}\left(\operatorname{tr}\Sigma^2\Omega+\frac{1}{2}(\operatorname{tr}\Omega)^2\right)$$

$$+\frac{\Phi^{(4)}(x)}{12\tau^2}\{3\operatorname{tr}\Sigma^4+4\operatorname{tr}\Omega\Sigma\operatorname{tr}\Sigma^4\}+\frac{\Phi^{(6)}(x)}{18\tau^6}(\operatorname{tr}\Sigma^3)^2\,,$$

$$T_3=\frac{\Phi^{(3)}(x)}{\tau^3}\left\{\operatorname{tr}\Sigma^3\Omega+\operatorname{tr}\Sigma\Omega\operatorname{tr}\Sigma^2\Omega+\frac{1}{6}(\operatorname{tr}\Sigma\Omega)^3\right\}$$

$$+\frac{\Phi^{(5)}(x)}{\tau^5}\left\{\frac{\operatorname{tr}\Sigma^5}{5}+\frac{\operatorname{tr}\Sigma\Omega\operatorname{tr}\Sigma^4}{4}+\frac{(\operatorname{tr}\Sigma\Omega)^2\operatorname{tr}\Sigma^3}{6}+\frac{\operatorname{tr}\Sigma^3\operatorname{tr}\Sigma^2\Omega}{3}\right\}$$

$$+\frac{\Phi^{(7)}(x)}{10\tau^7}\left\{\frac{1}{12}\operatorname{tr}\Sigma^4\operatorname{tr}\Sigma^3+\frac{1}{18}\operatorname{tr}\Sigma\Omega(\operatorname{tr}\Sigma^3)^2\right\}+\frac{\Phi^{(9)}(x)}{162\tau^9}(\operatorname{tr}\Sigma^3)^3 .$$

Case 2. *$\Omega=n\theta$, where θ is a constant matrix. Put*

$$\tilde{\lambda}=\frac{\sqrt{n}}{\sigma}(\operatorname{tr}S-\operatorname{tr}(I+\theta)\Sigma)\,,\qquad \sigma^2=\operatorname{tr}(I+2\theta)\Sigma^2 .$$

Then we have

$$\Pr\{\tilde{\lambda}\le x\}=\Phi(x)-\frac{M_1}{\sqrt{n}}+\frac{M_2}{n}-\frac{M_3}{n\sqrt{n}}+O\left(\frac{1}{n^2}\right), \tag{3.5}$$

where

$$M_1=\frac{\Phi^{(3)}(x)}{3\sigma^3}\operatorname{tr}(I+3\theta)\Sigma^3\,,$$

$$M_2=\frac{\Phi^{(4)}(x)}{4\sigma^4}\operatorname{tr}(I+4\theta)\Sigma^4+\frac{\Phi^{(6)}(x)}{18\sigma^6}\{\operatorname{tr}(I+3\theta)\Sigma^3\}^2\,,$$

$$M_3=\frac{\Phi^{(5)}(x)}{5\sigma^5}\operatorname{tr}(I+5\theta)\Sigma^5+\frac{\Phi^{(7)}(x)}{12\sigma^7}\operatorname{tr}(I+3\theta)\Sigma^3\operatorname{tr}(I+4\theta)\Sigma^4$$

$$+\frac{\Phi^{(9)}(x)}{162\sigma^9}(\operatorname{tr}(I+3\theta)\Sigma^3)^3 .$$

PROOF. *Case* 1. As the characteristic function $\varphi(t)$ of $\hat{\lambda}$ is given by

$$\varphi(t)=\operatorname{etr}(-\Omega)\operatorname{etr}\left(-it\frac{\sqrt{n}}{\tau}\Sigma\right)\det\left(I-\frac{it}{\sqrt{n}\,\tau}\Sigma\right)^{-n} \tag{3.6}$$

$$\cdot\operatorname{etr}\left(\Omega\left(I-\frac{it}{\sqrt{n}\,\tau}\Sigma\right)^{-1}\right),$$

we expand this as the series of order $1/\sqrt{n}$ by using the formulae such that

$$\det\left(I-\frac{it}{\sqrt{n}\,\tau}\Sigma\right)^{-n}=\operatorname{etr}\left(\frac{\sqrt{n}}{\tau}it\Sigma\right)\left\{1-\frac{t^2}{2\tau^2}\operatorname{tr}\Sigma^2+\cdots\right\},$$

$$\left(I-\frac{it}{\sqrt{n}\,\tau}\Sigma\right)^{-1}=I+\frac{it}{\sqrt{n}\,\tau}\Sigma-\frac{t^2}{n\tau^2}\Sigma^2+\cdots,$$

where the above formulas are only valid for $|t|<\sqrt{n}\,\tau/|\lambda_1|$, where λ_1 is the maximum latent root of Σ. Hence

$$\varphi(t)=\begin{cases}\varphi(t)\,, & \text{for}\quad |t|>\dfrac{\sqrt{n}\,\tau}{|\lambda_1|}\,,\\ \exp\left(-\dfrac{t^2}{2}\right)\left[1+\dfrac{T_1^*}{\sqrt{n}}+\dfrac{T_2^*}{n}+\dfrac{T_3^*}{n\sqrt{n}}+O\left(\dfrac{1}{n^2}\right)\right]\,, & \\ & \text{for}\quad |t|<\dfrac{\sqrt{n}\,\tau}{|\lambda_1|}\,,\end{cases}$$

where

$$T_1^*=\sum_{\beta=1}^{2} l_{1\beta}(it)^{2\beta-1}\,,\quad T_2^*=\sum_{\beta=1}^{3} l_{2\beta}(it)^{2\beta}\,,\quad T_3^*=\sum_{\beta=2}^{5} l_{3\beta}(it)^{2\beta-1}\,,$$

and $l_{j\beta}$'s, $j=1, 2, 3$ are the corresponding coefficients of $\Phi^{(2\beta-1)}(x)$ or $\Phi^{(2\beta)}(x)$ in T_j's.

Since, for arbitrary positive integer k,

$$\frac{1}{2\pi}\int_{|t|<\sqrt{n}\tau/|\lambda_1|}\exp(-itx)\exp\left(-\frac{t^2}{2}\right)(it)^k dt$$
$$=(-1)^k\Phi^{(k)}(x)-\frac{1}{2\pi}\int_{|t|>\sqrt{n}\tau/|\lambda_1|}\exp(-itx)\exp\left(-\frac{t^2}{2}\right)(it)^k dt\,,$$

the second term is estimated by

$$\left|\frac{1}{2\pi}\int_{|t|>\sqrt{n}\tau/|\lambda_1|}\exp(-itx)\exp\left(-\frac{t^2}{2}\right)(it)^k dt\right|$$
$$\leq\frac{1}{\pi}\int_{t>\sqrt{n}\tau/|\lambda_1|}\exp\left(-\frac{t^2}{2}\right)t^k dt$$
$$\begin{cases}\leq\exp\left(-\dfrac{n\tau^2}{2\lambda_1^2}\right)\left[\left(\dfrac{\sqrt{n}\,\tau}{|\lambda_1|}\right)^{k-1}+(k-1)\left(\dfrac{\sqrt{n}\,\tau}{|\lambda_1|}\right)^{k-3}+\cdots\right.\\ \qquad\left.+(k-1)(k-3)\cdots3\cdot1\left(\dfrac{\sqrt{n}\,\tau}{|\lambda_1|}\right)^{-1}\right] \qquad \text{for}\quad k=\text{even},\\ =\exp\left(-\dfrac{n\tau^2}{2\lambda_1^2}\right)\left[\left(\dfrac{\sqrt{n}\,\tau}{|\lambda_1|}\right)^{k-1}+(k-1)\left(\dfrac{\sqrt{n}\,\tau}{|\lambda_1|}\right)^{k-3}+\cdots\right.\\ \qquad\left.+(k-1)(k-3)\cdots2\right] \qquad \text{for}\quad k=\text{odd}.\end{cases}$$

This implies that the second terms is $O(1/n^l)$ for arbitrary positive integer l.

Next we estimate $\varphi(t)$ for $|t|>\sqrt{n}\tau/|\lambda_1|$. Let w_α's $\alpha=1, 2, \cdots, m$, be the diagonal elements of $H'\Omega H$ such that $H\Sigma H'=\Lambda$, then

$$\left|\operatorname{etr}\left\{\Omega\left(I-\frac{it}{\sqrt{n}\,\tau}\Sigma\right)^{-1}\right\}\right|=\left|\exp\left\{\sum_{\alpha=1}^{m} w_\alpha\left(1-\frac{it}{\sqrt{n}\,\tau}\lambda_j\right)^{-1}\right\}\right|$$

$$\leq \exp\left\{\sum_{\alpha=1}^{m} w_\alpha\left(1+\frac{t^2}{n\tau^2}\lambda_j^2\right)^{-1/2}\right\}$$

$$\leq \exp\left\{\sum_{\alpha=1}^{m} w_\alpha\left(1+\frac{\lambda_j^2}{\lambda_1^2}\right)^{-1/2}\right\}=\text{Constant}.$$

Hence

$$\left|\frac{1}{2\pi}\int_{|t|>\sqrt{n}\tau/|\lambda_1|}\exp(-itx)\varphi(t)dt\right|$$

$$\leq \text{Const.}\int_{t>\sqrt{n}\tau/|\lambda_1|}\left|\det\left(I-\frac{it}{\sqrt{n}\,\tau}\Sigma\right)^{-n}\right|dt$$

$$=\text{Const.}\int_{t>\sqrt{n}\tau/|\lambda_1|}\left|\prod_{\alpha=1}^{m}\left(1-\frac{it}{\sqrt{n}\,\tau}\lambda_\alpha\right)^{-n}\right|dt$$

$$\leq \text{Const.}\int_{t>\sqrt{n}\tau/|\lambda_1|}\left|\prod_{\alpha=1}^{m}\left(1+\frac{\lambda_\alpha^2}{n\tau^2}t^2\right)^{-n/2}\right|dt$$

$$\leq \text{Const.}\int_{t>\sqrt{n}\tau/|\lambda_1|}\left(1+\frac{\lambda_m^2}{n\tau^2}t^2\right)^{-mn/2}dt$$

$$\leq \text{Const.}\left(\frac{|\lambda_1|}{\sqrt{n}\,\tau}\right)\int_{t>\sqrt{n}\tau/|\lambda_1|}t\left(1+\frac{\lambda_m^2}{n\tau^2}t^2\right)^{-mn/2}dt$$

$$=\text{Const.}\left(\frac{\sqrt{n}\,\tau}{mn-2}\right)\left(\frac{|\lambda_1|}{\lambda_m^2}\right)\left(1+\frac{\lambda_m^2}{\lambda_1^2}\right)^{-(mn/2-1)}$$

$$=O\left(\frac{1}{n^l}\right),$$

for arbitrary positive integer l.

Thus, summarizing the above consideration, we have (3.4), formally.

Case 2. This is obtained by the similar way as Case 1.

4. The likelihood ratio criterion in a linear model

In the linear hypothesis we have a following canonical model. Let the each column vectors of $Y=[y_1, y_2, \cdots, y_N]_{m\times N}$ be independently distributed with the complex normal distribution with the same covariance matrix Σ. The hypothesis H and the alternative K are specified by

$$\begin{aligned}
&H:\ E(y_\alpha)=0\,, &&\alpha=1,2,\cdots,q_1 \text{ and } q_2+1,\cdots,N. \quad (q_1\leq q_2)\\
(4.1)\quad &K:\ E(y_\alpha)\neq 0\,, &&\text{for some } \alpha\ (1\leq\alpha\leq q_1) \text{ and}\\
& E(y_\alpha)=0\,, &&\text{for some } \alpha=q_2+1,\cdots,N.
\end{aligned}$$

The likelihood ratio criterion for this test is expressed by

$$\Lambda=\left(\frac{\det A}{\det (A+B)}\right)^{N}, \tag{4.2}$$

where $A=\sum_{\alpha=q_1+1}^{N} y_\alpha \bar{y}'_\alpha$ and $B=\sum_{\alpha=1}^{q_1} y_\alpha \bar{y}'_\alpha$. Under K, $A\sim CW_m(\Sigma, N-q_2)$ and $B\sim CW_m(\Sigma, q_1, \Omega)$, $\Omega=\Gamma\bar{\Gamma}'\Sigma^{-1}$, where $\Gamma=E[y_1, \cdots, y_{q_1}]$.

Put $\lambda=-\rho\log\Lambda$, where $\rho N=2n-2q_2+q_1-m$. Then the characteristic function $\phi(t)$ of λ is given by

$$\begin{aligned}(4.3)\qquad &\frac{\tilde{\Gamma}_m(n+(q_1+m)/2)\tilde{\Gamma}_m(n(1-2it)-(q_1-m)/2)}{\tilde{\Gamma}_m(n-(q_1-m)/2)\tilde{\Gamma}_m(n(1-2it)+(q_1+m)/2)}\operatorname{etr}(-\Omega)\\ &\quad\cdot {}_1F_1\left(n+\frac{1}{2}(q_1+m),\ n(1-2it)+\frac{1}{2}(q_1+m);\ \Omega\right)\\ &=\phi_1(t)\phi_2(t).\end{aligned}$$

We expand (4.3) as before.

$$\phi_1(t)=x^{+q_1m}\left\{1+\frac{mq_1}{24n^2}(m^2+q_1^2-2)(x^2-1)+O\left(\frac{1}{n^3}\right)\right\}\operatorname{etr}(-\Omega),$$

$$\begin{aligned}\phi_2(t)=\operatorname{etr}(x\Omega)\Big[1-\frac{1}{2n}\{x^3\operatorname{tr}\Omega^2+x^2((q_1+m)\operatorname{tr}\Omega-\operatorname{tr}\Omega^2)-x(q_1+m)\operatorname{tr}\Omega\}\\ +\frac{1}{24n^2}\sum_{\alpha=2}^{6}A_\alpha x^\alpha+O\left(\frac{1}{n^3}\right)\Big],\end{aligned}$$

where

$$\begin{aligned}&A_2=-24B^2\operatorname{tr}\Omega+12B^2(\operatorname{tr}\Omega)^2+12B\operatorname{tr}\Omega^2,\\ &A_3=24B^2\operatorname{tr}\Omega-12(2B^2+1)(\operatorname{tr}\Omega)^2-48B\operatorname{tr}\Omega^2+12B\operatorname{tr}\Omega\operatorname{tr}\Omega^2+8\operatorname{tr}\Omega^3,\\ (4.4)\quad &A_4=12(B^2+1)(\operatorname{tr}\Omega)^2+36B\operatorname{tr}\Omega^2-24\operatorname{tr}\Omega^3-24B\operatorname{tr}\Omega\operatorname{tr}\Omega^2+3(\operatorname{tr}\Omega^2)^2,\\ &A_5=16\operatorname{tr}\Omega^3+12B\operatorname{tr}\Omega\operatorname{tr}\Omega^2-6(\operatorname{tr}\Omega^2)^2,\\ &A_6=3(\operatorname{tr}\Omega^2)^2.\\ &x=(1-2it)^{-1},\qquad B=\frac{1}{2}(q_1+m).\end{aligned}$$

Therefore, we obtain the asymptotic expansion of $\phi(t)$ with respect to the order $1/n$ as follows.

$$\begin{aligned}(4.5)\quad \phi(t)=x^{q_1m}\operatorname{etr}((1-x)\Omega)\Big[1+\frac{1}{2n}\{x^3\operatorname{tr}\Omega^2+x^2((q_1+m)\operatorname{tr}\Omega-\operatorname{tr}\Omega^2)\\ -x(q_1+m)\operatorname{tr}\Omega\}+\frac{1}{24n^2}\{-q_1m(m^2+q_1^2-2)\end{aligned}$$

$$+(q_1m(m^2+q_1^2-2)+A_2)x^2+\sum_{\alpha=3}^{6} A_\alpha x^\alpha\} +O\left(\frac{1}{n^3}\right)\Big] .$$

Since we know that $\exp\{(1-x)\operatorname{tr}\Omega\}x^{q_1m+\beta}$ is a characteristic function of the χ^2 variable with $2(q_1m+\beta)$ degrees of freedom and with non-central parameter $\delta^2=\operatorname{tr}\Omega$, by inverting (4.5) we have a following theorem.

THEOREM 3. *In the linear statistical testing hypothesis model* (4.1), *we have the asymptotic distribution of Λ under the alternative K as follows* :

$$\begin{aligned}(4.6)\quad \Pr\{-\rho\log\Lambda\le x\} &=\Pr\{\chi^2_{2q_1m}(\delta^2)\le x\}\\ &+\frac{1}{2n}\{\operatorname{tr}\Omega^2\Pr\{\chi^2_{2q_1m+6}(\delta^2)\le x\}\\ &+((q_1+m)\operatorname{tr}\Omega-\operatorname{tr}\Omega^2)\Pr\{\chi^2_{2q_1m+4}(\delta^2)\le x\}\\ &-(q_1+m)\operatorname{tr}\Omega\Pr\{\chi^2_{2q_1m+2}\le x\}\}\\ &+\frac{1}{24n^2}\Big\{-q_1m(m^2+q_1^2-2)\Pr\{\chi^2_{2q_1}\le x\}\\ &+(q_1m(m^2+q_1^2-2)+A_2)\Pr\{\chi^2_{2q_1+4}\le x\}\\ &+\sum_{\alpha=3}^{6}A_\alpha\Pr\{\chi^2_{2q_1+2\alpha}\le x\}\Big\}+O\left(\frac{1}{n^3}\right),\end{aligned}$$

where $\rho N=2n=2N-2q_2+q_1-m$, and A_α and B are given in (4.4).

5. The likelihood ratio test for the independence

Let S be distributed with $CW_m(\Sigma, N)$ and let's partition S and Σ into m_1 and m_2 rows and columns ($m_1\le m_2$) as

$$S=\begin{bmatrix} S_{11} & S_{12}\\ \bar{S}'_{12} & S_{22}\end{bmatrix},\qquad \Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \bar{\Sigma}'_{12} & \Sigma_{22}\end{bmatrix}.$$

The likelihood ratio test for the independence H: $\Sigma_{12}=0$ against all alternatives K: $\Sigma_{12}\ne 0$ is given by

$$(5.1)\qquad \Lambda=\left(\frac{\det S}{\det S_{11}\det S_{22}}\right)^N .$$

LEMMA 5. *Under the alternative K, the moment of Λ^h is expressed as*

$$(5.2)\qquad \frac{\tilde{\Gamma}_{m_1}(N)\tilde{\Gamma}_{m_1}(N-m_2+Nh)}{\tilde{\Gamma}_{m_1}(N-m_2)\tilde{\Gamma}_{m_1}(N+Nh)}\det(I-\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}'_{12})^N \cdot {}_2\tilde{F}_1(N, N, N+Nh; \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}'_{12}) .$$

PROOF. The proof of this lemma is different from [7]. Let $\Sigma^{-1} = D$ and let's partition D as

$$D=\begin{bmatrix} D_{11} & D_{12} \\ \bar{D}'_{12} & D_{22} \end{bmatrix}.$$

Then the expectation of Λ^h is written as follows

$$E(\Lambda^h)=\frac{1}{\tilde{\Gamma}_m(N)(\det \Sigma)^N}\int_{\bar{S}'=S>0} \operatorname{etr}(-\Sigma^{-1}S)(\det S)^{N-m}\left(\frac{\det S}{\det S_{11}\det S_{22}}\right)^{Nh} dS$$
$$=\frac{1}{\tilde{\Gamma}_m(N)(\det \Sigma)^N}\int_{\bar{S}'_{11}=S_{11}>0} dS_{11}\operatorname{etr}(-D_{11}S_{11})(\det S_{11})^{N-m}$$
$$\cdot\int_{\bar{S}'_{22}=S_{22}>0} dS_{22}\operatorname{etr}(-D_{22}S_{22})(\det S_{22})^{N-m}$$
$$\cdot\int_{S_{12}} \operatorname{etr}\{-(\bar{D}'_{12}S_{12}+D_{12}\bar{S}'_{12})\}\det(I-S_{11}^{-1/2}S_{12}S_{22}^{-1}\bar{S}'_{12}S_{11}^{-1/2})^{N+Nh-m}dS_{12}.$$

Let $S_{12}=S_{11}^{1/2}WS_{22}^{1/2}$, then $dS_{12}=(\det S_{11})^{m_2}(\det S_{22})^{m_1}dW$. Hence

(5.3)

$$E(\Lambda^h)=\frac{1}{\tilde{\Gamma}_m(N)(\det \Sigma)^N}\int_{\bar{S}'_{11}=S_{11}>0} dS_{11}\operatorname{etr}(-D_{11}S_{11})(\det S_{11})^{N-m_1}$$
$$\cdot\int_{\bar{S}'_{22}=S_{22}>0} dS_{22}\operatorname{etr}(-D_{22}S_{22})(\det S_{22})^{N-m_2}$$
$$\cdot\int_W \operatorname{etr}\{-(S_{22}^{1/2}\bar{D}'_{12}S_{11}^{1/2}W+S_{11}^{1/2}D_{12}S_{22}^{1/2}\bar{W}')\}\det(I_{m_1}-W\bar{W}')^{N+Nh-m}dW.$$

Since $\det(I-W\bar{W}')$ is invariant under the transformation W to WU, $U\in U(m_2)$, we first project $\operatorname{etr}\{-(S_{22}^{1/2}\bar{D}'_{12}S_{11}^{1/2}W+S_{11}^{1/2}D_{12}S_{22}^{1/2}\bar{W}')\}$ into the space of $\phi(W\bar{W}')$ and we integrate $\phi(W\bar{W}')$ on the whole space such that $W\bar{W}'>0$. Therefore, by using (1.3),

$$\int_W \operatorname{etr}\{-(S_{22}^{1/2}\bar{D}'_{12}S_{11}^{1/2}W+S_{11}^{1/2}D_{12}S_{22}^{1/2}\bar{W}')\}\det(I-W\bar{W}')^{N+Nh-m}dW$$
$$=\int_W dW\int_{U(m_2)} \operatorname{etr}\{-(S_{22}^{1/2}\bar{D}'_{12}S_{11}^{1/2}WU+S_{11}^{1/2}D_{12}S_{22}^{1/2}\bar{U}'\bar{W}')\}$$
$$\cdot\det(I-W\bar{W}')^{N+Nh-m}d(U)$$
$$=\int_W \det(I-W\bar{W}')^{N+Nh-m}{}_0\tilde{F}_1(m_2;\,S_{11}^{1/2}D_{12}S_{22}\bar{D}'_{12}S_{11}^{1/2}W\bar{W}')dW.$$

Hence by applying the Hsu's lemma in a complex case and (1.2), the above integral can be written as

(5.4)

$$\frac{\pi^{m_1m_2}}{\tilde{\Gamma}_{m_1}(m_2)}\int_{\bar{R}'=R>0}(\det R)^{m_2-m_1}\det(I-R)^{N+Nh-m}{}_0\tilde{F}_1(m_2;\,S_{11}^{1/2}D_{12}S_{22}\bar{D}'_{12}S_{11}^{1/2}R)dR$$

$$=\pi^{m_1m_2}\frac{\tilde{\Gamma}_{m_1}(N-m_2+Nh)}{\tilde{\Gamma}_{m_1}(N+Nh)}\,{}_0\tilde{F}_1(N+Nh;\ S_{11}D_{12}S_{22}\bar{D}'_{12})\,.$$

Thus inserting (5.4) into (5.3), we have the moments of Λ by integration with respect to S_{11} and S_{22} as follows

$$\pi^{m_1m_2}\frac{\tilde{\Gamma}_{m_1}(N)\tilde{\Gamma}_{m_2}(N)\tilde{\Gamma}_{m_1}(N-m_2+Nh)}{\tilde{\Gamma}_{m}(N)\tilde{\Gamma}_{m_1}(N+Nh)(\det\Sigma)^N}(\det D_{11})^{-N}(\det D_{22})^{-N}$$
$$\cdot{}_2\tilde{F}_1(N,N;\ N+Nh;\ D_{11}^{-1}D_{12}D_{22}^{-1}\bar{D}'_{12})\,.$$

Since

$$\det\Sigma=(\det D_{11})^{-1}(\det D_{22})^{-1}\det(I-\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}'_{12})\,,$$

$$\tilde{\Gamma}_m(N)=\pi^{m_1m_2}\tilde{\Gamma}_{m_1}(N-m_2)\tilde{\Gamma}_{m_2}(N)\,,\qquad m=m_1+m_2\,,$$

we have (5.2).

THEOREM 4. *The asymptotic distribution of the likelihood ratio criterion for the testing of independence between two sets of variates is expresses as follows.*

Let $\lambda=-(\rho/\tau\sqrt{n})\{\log\Lambda-\log\{\det\Sigma/(\det\Sigma_{11})(\det\Sigma_{22})\}^N\}$, *where* Λ *is* (5.1), $2n=\rho N=N-(m_1+m_2)$ *and* $\tau=2\sqrt{\operatorname{tr}P}$, $P=\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}'_{12}$. *Then*

$$\Pr\{\lambda\le x\}=\Phi(x)-\frac{1}{\sqrt{n}}\left\{\frac{m_1m_2\Phi^{(1)}(x)}{\tau}-\frac{2(\operatorname{tr}P^2-\operatorname{tr}P)\Phi^{(3)}(x)}{\tau^3}\right\}$$
$$+\frac{1}{6n}\sum_{\alpha=1}^{3}\frac{A_\alpha\Phi^{(2\alpha)}(x)}{\tau^{2\alpha}}+O\left(\frac{1}{n\sqrt{n}}\right),$$

where

$$A_1=3[m_1m_2+(m_1m_2)^2+(\operatorname{tr}P)^2-2m\operatorname{tr}P]$$

$$A_2=2[10\operatorname{tr}P^3-3(5+2m_1m_2)\operatorname{tr}P^2+6(m_1m_2+1)\operatorname{tr}P]$$

$$A_3=12[(\operatorname{tr}P^2)^2+(\operatorname{tr}P)^2-2\operatorname{tr}P\operatorname{tr}P^2]\,.$$

PROOF. The proof is done completely the same way as the one of Theorem 3.

Acknowledgements

The author heartly appreciates to referee who has pointed out the errors of the original versions.

INSTITUTE OF STATISTICAL MATHEMATICS AND UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL

REFERENCES

[1] Fujikoshi, Y. (1968). Asymptotic expansion of the distribution of the generalized variance in the non-central case, *J. Sci. Hiroshima Univ. Ser. A-I.*, **32**, 293-299.

[2] Fujikoshi, Y. (1970). Asymptotic expansions of the distributions of some statistics in multivariate analysis, *J. Sci. Hiroshima Univ. Ser. A-I.*, **34**, 73-144.

[3] Goodman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution (An Introduction), *Ann. Math. Statist.*, **34**, 152-176.

[4] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples, *Ann. Math. Statist.*, **35**, 475-501.

[5] Khatri, C. G. (1965). Classical statistical analysis based on certain multivariate complex Gaussian distribution, *Ann. Math. Statist.*, **36**, 98-114.

[6] Khatri, C. G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors, *Ann. Math. Statist.*, **37**, 468-479.

[7] Sugiura, N. and Fujikoshi, Y. (1969). Asymptotic expansions of the non-central distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence, *Ann. Math. Statist.*, **40**, 942-952.