ON THE DISTRIBUTION OF THE MULTIVARIATE QUADRATIC FORM IN MULTIVARIATE NORMAL SAMPLES*

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1. Introduction

This paper considers the derivation of the probability density function (p.d.f.) of the latent roots of a multivariate non-central quadratic form in multivariate normal samples. The probability density function of the multivariate central quadratic form has been obtained by three types of representation; power series type, (Hayakawa [2]), Γ-series type, (Khatri [8]), and Laguerre series type (Shah [12]). In this paper, we discuss also three types of representation, i.e.

(i) power series type,
(ii) Γ-series type,
(iii) mixture type,

of the p.d.f. of the latent roots of a multivariate non-central quadratic form. To consider these representations, we use a new polynomial $P_\kappa(T, A)$ which was proposed by Hayakawa [3], and we will give some properties of $P_\kappa(T, A)$, the exact expression of $P_\kappa(T, A)$ up to $k=4$ and the exact expression of $\sum_\kappa P_\kappa(T, A)$. We also derive the p.d.f. of the trace of a multivariate non-central quadratic form by the use of $P_\kappa(T, A)$ and compare the results of Ruben [10], [11] and Kotz et al. [9] with our results. Finally, we also discuss the p.d.f.'s for the case of the complex variables.

2. Notations and some useful results

Let T and U be $m\times n$ $(m\leqq n)$ real arbitrary matrices each of rank m, and let A be an $n\times n$ positive definite symmetric matrix. Hayakawa [3] defined a new polynomial $P_\kappa(T, A)$ as follows.

$$(1)\qquad \operatorname{etr}(-TT')P_\kappa(T, A) = ((-1)^k/\pi^{mn/2})\int_U \operatorname{etr}(-2iTU')\operatorname{etr}(-UU')C_\kappa(UAU')dU\ ,$$

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where κ is a partition of k into not more than m parts, i.e., $\kappa=(k_1, k_2, \cdots, k_m)$, $k=k_1+k_2+\cdots+k_m$, $k_1\geqq k_2\geqq\cdots\geqq k_m\geqq 0$, and $C_\kappa(UAU')$ is a zonal polynomial of UAU' corresponding to a partition κ of k, James [6], and i is an imaginary number.

From the definition (1), we have

$$(2)\qquad P_\kappa(T, A)=\operatorname{etr}(TT')\frac{(-1)^k}{\pi^{mn/2}}\int_U \operatorname{etr}(-UU'-2iTU')C_\kappa(UAU')dU$$
$$=\frac{(-1)^k}{\pi^{mn/2}}\int_U \operatorname{etr}\{-(U+iT)(U+iT)'\}C_\kappa(UAU')dU$$
$$=E_V[C_\kappa\{(V-iT)A(V-iT)'\}]\ ,$$

where the expectation is done with respect to the p.d.f. $\operatorname{etr}(-VV')/\pi^{(mn)/2}$. Therefore, by the similar argument as the derivation of the characteristic function of normal random variable we have the explicit expressions of $P_\kappa(T, A)$'s.

The exact expressions of $P_\kappa(T, A)$ up to $k=4$ are given in Appendix.

Here we give some properties of $P_\kappa(T, A)$, which may be used for our discussion.

$$(3)\qquad P_\kappa(0, A)=(-1)^k(m/2)_\kappa C_\kappa(A)\ ,$$

$$(4)\qquad P_\kappa(T, I_n)=H_\kappa(T)\ ,$$

$$(5)\qquad |P_\kappa(T, A)|\leqq \operatorname{etr}(TT')(m/2)_\kappa C_\kappa(A)\ ,$$

where

$$(a)_\kappa=\prod_{\alpha=1}^{m}(a-((\alpha-1)/2))_{k_\alpha}\ ,\qquad (a)_n=a(a+1)\cdots(a+n-1)\ ,$$

and $H_\kappa(T)$ is a generalized Hermite polynomial of matrix argument T.

The generating function of $P_\kappa(T, A)$ is given as follows.

$$(6)\qquad \int_{0(m)}\int_{0(m)} \operatorname{etr}(-UH_2AH_2'U'+2H_1UH_2A^{1/2}T')d(H_1)d(H_2)$$
$$=\sum_{k=0}^{\infty}\sum_{\kappa}(P_\kappa(T, A)C_\kappa(UU'))/(k!(n/2)_\kappa C_\kappa(I_m))\ ,$$

and the right hand side (R.H.S.) of (6) converges absolutely with respect to U. $d(H_1)$ and $d(H_2)$ are the normalized orthogonal invariant measures on the orthogonal group $0(m)$ and $0(n)$, respectively.

$P_\kappa(T, A)$ is connected with $H_\kappa(T)$ by the following way.

$$(7)\qquad \int_{0(n)} P_\kappa(TH, A)d(H)=\int_{0(n)} P_\kappa(T, HAH')d(H)$$
$$=((C_\kappa(A))/(C_\kappa(I_n)))H_\kappa(T)\ ,$$

and it is obvious that $P_\kappa(T, A)$ is a homogeneous polynomial with respect to A, not to T. The detail discussion of the generalized Hermite polynomial $H_\kappa(T)$ and (3), (4), (5), (6), (7) may be found in Hayakawa [3].

PROPOSITION 1. Let T and S be $m\times n$ $(m\leqq n)$ matrices and let A and B be posititive definite symmetric matrices, then

$$
\begin{aligned}
(8)\quad & \int_{0(m)}\int_{0(n)} (1/(\det(I-u^2H_2AH_2'B)^{m/2}))\\
&\cdot \operatorname{etr}\{2uH_1TA^{1/2}H_2'B^{1/2}S'-u^2TA^{1/2}H_2'BH_2A^{1/2}T'\\
&+u^2(S-uH_1TA^{1/2}H_2'B^{1/2})B^{1/2}H_2(u^2H_2'BH_2-A^{-1})^{-1}H_2'B^{1/2}\\
&\cdot (S-uH_1TA^{1/2}H_2'B^{1/2})'\}d(H_1)d(H_2)\\
&=\sum_{k=0}^{\infty}\sum_{\kappa}((P_\kappa(T, A)P_\kappa(S, B))/(k!(n/2)_\kappa C_\kappa(I_m)))u^{2k}\,,\\
&\qquad\qquad \text{for } \max\{\|uA\|, \|uB\|\}<1\,,
\end{aligned}
$$

where $\|A\|$ means the maximum value of all the absolute values of the latent roots of A.

PROOF. By inserting (1) into the R.H.S. of (8), we have

$$
\begin{aligned}
\text{R.H.S.} &=\operatorname{etr}(TT'+SS')(1/\pi^{mn})\\
&\cdot\int_U\int_V \operatorname{etr}(-UU'-VV'-2iTU'-2iSV')\\
&\cdot\sum_{k=0}^{\infty}\sum_{\kappa}((C_\kappa(UAU')C_\kappa(VBV'))/(k!(n/2)_\kappa C_\kappa(I_m)))u^{2k}dUdV\\
&=\operatorname{etr}(TT'+SS')\\
&\cdot\int_{0(m)}\int_{0(n)}\int_U\int_V (1/\pi^{mn})\operatorname{etr}(-UU'-VV'-2iTU'\\
&\quad -2iSV'-2uA^{1/2}U'H_1VB^{1/2}H_2)dUdVd(H_1)d(H_2)\\
&=\int_{0(m)}\int_{0(n)}\operatorname{etr}(SS')/(\det(I-u^2AH_2'BH_2)^{m/2})\\
&\cdot\operatorname{etr}\{-(S-uH_1TA^{1/2}H_2'B^{1/2})(I-u^2B^{1/2}H_2'AH_2B^{1/2})^{-1}\\
&\quad\cdot(S-uH_1TA^{1/2}H_2'B^{1/2})'\}d(H_1)d(H_2)\,.
\end{aligned}
$$

By noting that

$$(I-u^2B^{1/2}H_2'AH_2B^{1/2})^{-1}=I-u^2B^{1/2}H_2'(u^2H_2BH_2'-A^{-1})^{-1}H_2B^{1/2}\,,$$

we have (8) easily.

COROLLARY. *If we set $B=I_n$ in* (8), *we have*

$$
\begin{aligned}
(9)\quad & (1/(\det(I-u^2A)^{m/2}))\int_{0(m)}\int_{0(n)}\operatorname{etr}\{2uH_1TA^{1/2}H_2'S'-u^2TAT'\\
&+u^2(S-uH_1TA^{1/2}H_2')H_2(u^2I-A^{-1})^{-1}
\end{aligned}
$$

$$\cdot H_2'(S-uH_1TA^{1/2}H_2')'\}d(H_1)d(H_2)$$
$$=\sum_{k=0}^{\infty}\sum_{\kappa}((P_\kappa(T, A)H_\kappa(S))/(k!(n/2)_\kappa C_\kappa(I_m)))u^{2k}\,,$$
$$\text{for } \max\{|u|, \|uA\|\}<1\,.$$

If we set $A=B=I_n$ in (8), we have

$$(10)\qquad (1/(1-u^2)^{mn/2})\int_{0(m)}\int_{0(n)}\operatorname{etr}\{-(u^2/(1-u^2))(SS'+TT')$$
$$+(2u/(1-u^2))H_1TH_2'S'\}d(H_1)d(H_2)$$
$$=\sum_{k=0}^{\infty}\sum_{\kappa}((H_\kappa(T)H_\kappa(S))/(k!(n/2)_\kappa C_\kappa(I_m)))u^{2k}\,,\qquad \text{for } |u|<1\,.$$

This is a generalized Mehler's formula of the generalized Hermite polynomials.

Note. By setting $T=0$ in (10) and using (4) and (5), we may have (19),

$$\sum_{\kappa} H_\kappa(S)=(-1)^k L_k^{mn/2-1}(\operatorname{tr} SS')\,,$$

by comparing the coefficient of u^{2k} on both side of (10).

PROPOSITION 2.

$$(11)\qquad \sum_{\kappa} P_\kappa(T, A)=(-1)^k k!\Big[A_k+(1/2)\sum_{l_1+l_2=k}A_{l_1}A_{l_2}$$
$$+(1/(3!))\sum_{l_1+l_2+l_3=k}A_{l_1}A_{l_2}A_{l_3}+\cdots+(A_1^k/(k!))\Big]\,,$$

where l_α's are positive integers greater than or equal to 1, and

$$A_l=(m/(2l))\operatorname{tr}A^l-\operatorname{tr}TA^lT'\,,\qquad l=1, 2,\cdots, k\,,$$

and

$$A_0\equiv 0\,,\qquad \text{for convenience}\,.$$

PROOF. From the definition of $P_\kappa(T, A)$, we construct the generating function of $(-1)^k\sum_{\kappa}P_\kappa(T, A)$.

$$(12)\qquad \sum_{k=0}^{\infty}((-x)^k/(k!))\sum_{\kappa}P_\kappa(T, A)$$
$$=(\operatorname{etr}(TT')/\pi^{mn/2})\int_U \operatorname{etr}(-2iTU')\operatorname{etr}(-UU')\operatorname{etr}(xUAU')dU$$
$$=\det(I-xA)^{-m/2}\operatorname{etr}\{T(I-(I-xA)^{-1})T'\}\,,\qquad \|xA\|<1\,.$$

We expand the R.H.S. of (12) with respect to x by noting that $\|xA\|<1$ using

$$\log \det (I-xA) = -[x \operatorname{tr} A + (x^2/2) \operatorname{tr} A^2 + \cdots + (x^k/k) \operatorname{tr} A^k + \cdots]$$

and

$$(I-xA)^{-1} = I + xA + x^2A^2 + \cdots + x^kA^k + \cdots .$$

Hence

$$\text{(13)} \qquad \begin{aligned} \text{R.H.S. of (12)} &= \exp[-(m/2) \log \det (I-xA)] \\ &\quad \cdot \operatorname{etr}(-xTA(I-xA)^{-1}T') \\ &= \exp\left[\sum_{k=1}^{\infty} x^k \operatorname{tr}((m/(2k))A^k - TA^kT')\right] . \end{aligned}$$

Here we set

$$\text{(14)} \qquad A_l = \operatorname{tr}((m/(2l))A^l - TA^lT') , \qquad l=1, 2, 3, \cdots$$

and

$$\text{(15)} \qquad A_0 \equiv 0 , \qquad \text{for convenience} .$$

We can obtain the value of $(-1)^k \sum_{\kappa} P_\kappa(T, A)$ by comparing the coefficients of x^k on the two side of (13). The coefficient of x^k on the right hand side is $(-1)^k \sum_{\kappa} P_\kappa(T, A)$. Putting $g(x)=\sum_{j=0}^{\infty} A_j x^j$, then

$$\begin{aligned} \text{R.H.S.} &= \exp(g(x)) \\ &= 1 + g(x) + (1/2)g(x)^2 + \cdots + (1/k!)g(x)^k + \cdots . \end{aligned}$$

Picking up the terms of x^k from each element in the above series, completes the proof.

Examples.

$$\text{(16)} \qquad k=1, \quad P_1(T, A) = (-1)A_1 = (-m/2) \operatorname{tr} A + \operatorname{tr} TAT' .$$

$$\text{(17)} \qquad \begin{aligned} k=2, \quad \sum_{\kappa} P_\kappa(T, A) &= 2![A_2 + (1/2)A_1^2] \\ &= 2![\operatorname{tr}((m/4)A^2 - TA^2T') \\ &\qquad + (1/2)\{\operatorname{tr}((m/2)A - TAT')\}^2] . \end{aligned}$$

$$\text{(18)} \qquad \begin{aligned} k=3, \quad \sum_{\kappa} P_\kappa(T, A) &= (-1)^3 3![A_3 + A_1A_2 + (1/(3!))A_1^3] \\ &= -3![\operatorname{tr}((m/6)A^3 - TA^3T') + \operatorname{tr}((m/2)A - TAT') \\ &\qquad \cdot \operatorname{tr}((m/4)A^2 - TA^2T') \\ &\qquad + (1/(3!))\{\operatorname{tr}((m/2)A - TAT')\}^3] . \end{aligned}$$

Remark. If we set $A=I_n$, then we have immediately from Hayakawa [14],

(19) $$\sum_{\kappa} P_{\kappa}(T, I_n)=\sum_{\kappa} H_{\kappa}(T)=(-1)^k L_k^{(mn/2)-1}(\operatorname{tr} TT') ,$$

where $L_k^{\alpha}(z)$ is a univariate Laguerre polynomial defined by

$$\sum_{k=0}^{\infty} ((t^k)/(k!))L_k^{\alpha}(z)=(1/(1-t)^{\alpha+1}) \exp(-(zt)/(1-t)) , \qquad |t|<1 .$$

3. The p.d.f. of the latent roots of $\Sigma^{-1/2}XAX'\Sigma^{-1/2}$

Let X be an $m\times n$ $(m\leqq n)$ matrix whose density function is given by

(20) $$(1/(\pi^{mn/2}(\det 2\Sigma)^{n/2}(\det B)^{m/2})) \operatorname{etr}(-(1/2)\Sigma^{-1}(X-M)B^{-1}(X-M)') ,$$

where Σ is an $m\times m$ p.d.s. matrix, B is an $n\times n$ p.d.s. matrix, and M is an $m\times n$ $(m\leqq n)$ matrix such that $E(X)=M$ and rank $M=m$. Let A be an $n\times n$ p.d.s. matrix.

THEOREM 1 (*Power series representation*). *Let X be distributed with p.d.f.* (20), *then the p.d.f. of the latent roots $\Lambda=\operatorname{diag}(\lambda_1,\cdots,\lambda_m)$ of $\Sigma^{-1/2}\cdot XAX'\Sigma^{-1/2}$ is given by*

(21) $$\begin{aligned}&((\pi^{m^2/2} \operatorname{etr}(-(1/2)MB^{-1}M'\Sigma^{-1}))/(\Gamma_m(n/2)\Gamma_m(m/2)(\det 2AB)^{m/2}))\\ &\cdot (\det \Lambda)^{(n-m-1)/2}\prod_{i<j}(\lambda_i-\lambda_j)\\ &\cdot \sum_{k=0}^{\infty}\sum_{\kappa}(P_{\kappa}((1/\sqrt{2})\Sigma^{-1/2}MB^{-1}A^{-1/2}C^{1/2}, C^{-1})C_{\kappa}((1/2)\Lambda))/\\ &(k!(n/2)_{\kappa}C_{\kappa}(I_m)) ,\end{aligned}$$

where

$$\Gamma_m(a)=\pi^{m(m-1)/4}\prod_{\alpha=1}^{m}\Gamma(a-(\alpha-1)/2) \qquad \text{and} \qquad C=A^{1/2}BA^{1/2}$$

and R.H.S. of (21) *converges absolutely.*

PROOF. See the proof of Theorem 8 of Hayakawa [3].

Note. We can obtain the distribution function (or p.d.f.) of the maximum latent root λ_1 of Λ by using (21) and

$$\begin{aligned}&\int_{xI>\Lambda>0}(\det \Lambda)^{(n-m-1)/2}C_{\kappa}(\Lambda)\prod_{i<j}(\lambda_i-\lambda_j)d\Lambda\\ &=((\Gamma_m(m/2)/\pi^{m^2/2})(\Gamma_m(n/2;\kappa)\Gamma_m((m+1)/2))/\\ &(\Gamma_m((n+m+1)/2;\kappa)))x^{(mn/2)+k}C_{\kappa}(I_m) .\end{aligned}$$

The exact expression is given by Theorem 10 of [3] by the appropriate change of parameter.

THEOREM 2 (*Γ-type representation*). *Under the same conditions of*

Theorem 1, *the p.d.f. of the latent roots* $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$ *of* $\Sigma^{-1/2}XAX' \cdot \Sigma^{-1/2}$ *is given by, for* $\|AB\| < p$,

$$(22) \quad ((\pi^{m^2/2}\,\mathrm{etr}(-(1/2)\Sigma^{-1}MB^{-1}M'))/(\Gamma_m(n/2)\Gamma_m(m/2)(\det 2AB)^{m/2})) \cdot \mathrm{etr}(-(1/(2p))\Lambda)(\det \Lambda)^{(n-m-1)/2} \prod_{i<j}(\lambda_i - \lambda_j) \cdot \sum_{k=0}^{\infty}\sum_{\kappa}(P_\kappa((1/\sqrt{2})\Sigma^{-1/2}MB^{-1}A^{-1/2}(C^{-1}-p^{-1}I)^{-1/2}, C^{-1}-p^{-1}I)C_\kappa((1/2)\Lambda))/(k!(n/2)_\kappa C_\kappa(I_m)) ,$$

where $C = A^{1/2}BA^{1/2}$ *and* p *is an arbitrary positive number.*

PROOF. The proof is done completely same way as the one of Theorem 1.

COROLLARY 1. *The moments of the determinant,* $\det\Lambda = \det(\Sigma^{-1/2} \cdot XAX'\Sigma^{-1/2})$, *is given by*

$$(23) \quad E[(\det \Sigma^{-1/2}XAX'\Sigma^{-1/2})^h] = (2p)^{mh}(\Gamma_m((n/2)+h)/\Gamma_m(n/2)) \cdot p^{mn/2}(1/(\det AB)^{m/2})\,\mathrm{etr}(-(1/2)\Sigma^{-1}MB^{-1}M') \cdot \sum_{k=0}^{\infty}(p^k/(k!))\sum_{\kappa}(((n/2)+h)_\kappa/(n/2)_\kappa)P_\kappa((1/\sqrt{2})\Sigma^{-1/2}MB^{-1} \cdot A^{-1/2}(C^{-1}-p^{-1}I)^{-1/2}, C^{-1}-p^{-1}I) , \qquad p/2 < \overline{\overline{AB}} < \|AB\| < p .$$

If we set $\Sigma = I_m$ *and* $M = 0$, *we have, by* (3),

$$E[(\det XAX')^h] = (2p)^{mh}p^{mn/2}(\Gamma_m((n/2)+h)/\Gamma_m(n/2))(1/(\det AB)^{m/2}) \cdot \sum_{k=0}^{\infty}((-p)^k/(k!))\sum_{\kappa}(((n/2)+h)_\kappa C_\kappa(C^{-1}-p^{-1}I)C_\kappa(I_m))/C_\kappa(I_n) , \qquad p/2 < \overline{\overline{AB}} < \|AB\| < p ,$$

which agrees with (46) *of Khatri* [8] *with* $\Sigma = I_m$. $\overline{\overline{AB}}$ *denotes the minimum characteristic root of* AB.

COROLLARY 2. *If we set* $A = I_n$ *and* $B = I_n$, *then the p.d.f. of the latent roots of a non-central Wishart matrix with known covariance is represented as follows.*

$$(24) \quad ((\pi^{m^2/2}\,\mathrm{etr}(-(1/2)\Sigma^{-1}MM'))/(2^{mn/2}\Gamma_m(n/2)\Gamma_m(m/2))) \cdot \mathrm{etr}(-(1/(2p))\Lambda)(\det\Lambda)^{(n-m-1)/2}\prod_{i<j}(\lambda_i-\lambda_j) \cdot \sum_{k=0}^{\infty}\sum_{\kappa}(((p-1)/p)^k(H_\kappa(\sqrt{p/(2(p-1))}\,\Sigma^{-1/2}M) \cdot C_\kappa((1/2)\Lambda))/(k!(n/2)_\kappa C_\kappa(I_m)) , \qquad \textit{for } p > 1 ,$$

since $P_\kappa(T, A)$ *is a homogeneous function with respect to* A.

This representation (24) is a different form compared with one of James [6]. (Brian G. Leach [13] has obtained

$$\sum_{k=0}^{\infty}\sum_{\kappa}(L_\kappa^a(S)C_\kappa(Z))/((a+(m+1)/2)_\kappa k!C_\kappa(I))$$
$$=\operatorname{etr}(Z)_0F_1^{(m)}(a+(m+1)/2;S,-Z).$$

Using this formula and (10) of [3], we can check that (24) is the same form as one of James [6]. The author wishes to thank the referee who has pointed out the above formula.). If we set $p\to\infty$, then (24) approaches the same form as (26) of [3].

Note. Since

$$\int_{\lambda_1>\lambda_2>\cdots>\lambda_m>0}\operatorname{etr}(-(1/(2p))\Lambda)(\det\Lambda)^{(n-m-1)/2}\prod_{i<j}(\lambda_i-\lambda_j)$$
$$\cdot C_\kappa((1/2)\Lambda)d\Lambda$$
$$=((\Gamma_m(n/2)\Gamma_m(m/2))/\pi^{m^2/2})p^{(mn/2)+k}2^{mn/2}(n/2)_\kappa C_\kappa(I_m),$$

we have a following relation from (23)

$$\operatorname{etr}(-(1/2)\Sigma^{-1}MB^{-1}M')/(\det AB/p)^{m/2}\tag{25}$$
$$=\sum_{k=0}^{\infty}\sum_{\kappa}(p^k/(k!))P_\kappa((1/\sqrt{2}\,\Sigma^{-1/2}MB^{-1}A^{-1/2}(C^{-1}-p^{-1}I)^{-1/2}$$
$$\cdot C^{-1}-p^{-1}I).$$

This formula can be obtained also from (12) directly if we replace x with $-p$, T with $(1/\sqrt{2})\Sigma^{-1/2}MB^{-1}A^{-1/2}(C^{-1}-p^{-1}I)^{-1/2}$, and A with $C^{-1}-p^{-1}I$, respectively.

THEOREM 3 (*Mixture type representation*). *Let X be an $m\times n$ ($m\leqq n$) matrix whose p.d.f. is given by*

$$(1/(\pi^{mn/2}(\det 2\Sigma)^{n/2}))\operatorname{etr}(-(1/2)\Sigma^{-1}XX'),\tag{26}$$

and let A be an $n\times n$ positive definite symmetric matrix. Then the mixture type representation of the p.d.f. of the latent roots of $\Sigma^{-1/2}(X-M)\cdot A(X-M)'\Sigma^{-1/2}$ is given by

$$\sum_{k=0}^{\infty}\sum_{\kappa}R_\kappa f_\kappa(\Lambda),\qquad for\ \bar{\bar{A}}>p>0,\tag{27}$$

where

$$f_\kappa(\Lambda)=(\pi^{m^2/2})/((2p)^{mn/2}\Gamma_m(n/2;\kappa)\Gamma_m(m/2)C_\kappa(I_m))\tag{28}$$
$$\cdot\operatorname{etr}(-(1/(2p))\Lambda)(\det\Lambda)^{(n-m-1)/2}C_\kappa((1/(2p))\Lambda)\prod_{i<j}(\lambda_i-\lambda_j)$$

and

$$(29)\qquad k!R_\kappa=(1/(2\pi)^{mn/2})\int_Y \operatorname{etr}(-(1/2)YY')\operatorname{etr}(-\Omega(Y))C_\kappa(\Omega(Y))dY\,,$$

and

$$(30)\qquad 2\Omega(Y)=[Y(A/p-I)^{1/2}+\Sigma^{-1/2}M(A/p)^{1/2}]\cdot[Y(A/p-I)^{1/2}+\Sigma^{-1/2}M(A/p)^{1/2}]'\,.$$

$\bar{\bar{A}}$ *denotes the minimum characteristic root of* A. *The* R_κ*'s satisfy the following conditions.*

$$(31)\qquad R_\kappa>0\,,\qquad \textit{for all partition } \kappa\,.$$

$$(32)\qquad \sum_{k=0}^{\infty}\sum_{\kappa}R_\kappa=1\,.$$

(27) *converges absolutely for all* $\Lambda>0$.

PROOF. Let Z and Y be $m\times n$ $(m\leqq n)$ independent matrices whose p.d.f.'s are the same as (26) with $\Sigma=I$.

On setting

$$(33)\qquad X=\sqrt{p}\,\Sigma^{1/2}ZA^{-1/2}-\Sigma^{1/2}Y(I-pA^{-1})^{1/2}\,,\qquad \text{for } 0<p<\bar{\bar{A}}\,,$$

we have

$$(34)\qquad \begin{aligned} S/p&=\Sigma^{-1/2}(X-M)A(X-M)'\Sigma^{-1/2}/p\\ &=\{Z-Y(A/p-I)^{1/2}-\Sigma^{-1/2}M(A/p)^{1/2}\}\\ &\quad\cdot\{Z-Y(A/p-I)^{1/2}-\Sigma^{-1/2}M(A/p)^{1/2}\}'\,.\end{aligned}$$

For fixed Y, the variate of the right hand side of (34) is non-central Wishart matrix with n degrees of freedom and non-central parameter matrix $\Omega(Y)$, where

$$(35)\qquad 2\Omega(Y)=\{Y(A/p-I)^{1/2}+\Sigma^{-1/2}M(A/p)^{1/2}\}\cdot\{Y(A/p-I)^{1/2}+\Sigma^{-1/2}M(A/p)^{1/2}\}'\,.$$

Hence the p.d.f. of the latent roots Λ of S/p under the condition Y fixed is given by

$$(36)\qquad \begin{aligned} &((\pi^{m^2/2}\operatorname{etr}(-\Omega(Y)))/((2p)^{mn/2}\Gamma_m(n/2)\Gamma_m(m/2)))\\ &\quad\cdot\operatorname{etr}(-(1/(2p))\Lambda)(\det\Lambda)^{(n-m-1)/2}\prod_{i<j}(\lambda_i-\lambda_j)\\ &\quad\cdot\sum_{k=0}^{\infty}\sum_{\kappa}(C_\kappa((1/(2p))\Lambda)C_\kappa(\Omega(Y)))/(k!(n/2)_\kappa C_\kappa(I_m))\\ &=\sum_{k=0}^{\infty}\sum_{\kappa}D_\kappa(Y)f_\kappa(\Lambda)\,,\end{aligned}$$

where

(37) $$f_\kappa(\Lambda)=\pi^{m^2/2}/((2p)^{mn/2}\Gamma_m(n/2;\kappa)\Gamma_m(m/2)C_\kappa(I_m)) \cdot \operatorname{etr}(-(1/(2p))\Lambda)(\det\Lambda)^{(n-m-1)/2}C_\kappa((1/(2p))\Lambda)\prod_{i<j}(\lambda_i-\lambda_j)\ ,$$

and

(38) $$D_\kappa(Y)=\operatorname{etr}(-\Omega(Y))C_\kappa(\Omega(Y))/k!\ .$$

Therefore the p.d.f. of the latent roots Λ of the multivariate non-central quadratic form $\Sigma^{-1/2}(X-M)A(X-M)'\Sigma^{-1/2}$ is given by

(39) $$\sum_{k=0}^{\infty}\sum_{\kappa}R_\kappa f_\kappa(\Lambda)\ ,$$

where

(40) $$k!R_\kappa=E_Y(D_\kappa(Y)) =(1/(2\pi)^{mn/2})\int_Y \operatorname{etr}(-(1/2)YY')\operatorname{etr}(-\Omega(Y))C_\kappa(\Omega(Y))dY\ .$$

Since $\Omega(Y)>0$, $C_\kappa(\Omega(Y))>0$. Hence $R_\kappa>0$ for all partition κ of k without measure 0 with respect to the normal distribution.

It is also obvious that

(41) $$\sum_{k=0}^{\infty}\sum_{\kappa}R_\kappa=1\ ,$$

since $\operatorname{etr}(\Omega(S))=\sum_{k=0}^{\infty}\sum_{\kappa}C_\kappa(\Omega(Y))/k!$. Hence (27) is the mixture type representation.

COROLLARY 3. *The relation betwein R_κ and $P_\kappa(T, A)$ is given by*

(42) $$k!R_\kappa=(\det A/p)^{-m/2}\operatorname{etr}(-(1/2)\Sigma^{-1}MM') \cdot(-1)^k P_\kappa((i/\sqrt{2})\Sigma^{-1/2}M(A/p-I)^{-1/2}, I-(A/p)^{-1})\ .$$

PROOF. This relation can be obtained very easily. We will omit the proof.

COROLLARY 4. *The generating functions of R_κ and $\sum_\kappa R_\kappa$ are given by the following way, respectively.*

(43) $$(\operatorname{etr}(-(1/2)\Sigma^{-1}MM')\operatorname{etr}((1/2)SS'))/(\det A/p)^{m/2} \cdot\int_{O(m)}\int_{O(n)}\operatorname{etr}\{-(1/2)SH_2(pA^{-1})H_2'S' -H_1SH_2(pA^{-1})^{1/2}M'\Sigma^{-1/2}\}d(H_1)d(H_2) =\sum_{k=0}^{\infty}\sum_{\kappa}(R_\kappa C_\kappa((1/2)SS'))/((n/2)_\kappa C_\kappa(I_m))\ ,$$

(44) $$(1/((\det A/p)^{m/2}\det\{I+x(I-(pA^{-1}))\}^{m/2}))$$

$$\cdot\ \mathrm{etr}\,\{-(1/2)(1+x)\Sigma^{-1/2}M\{I+x(I-(pA^{-1}))\}^{-1}M'\Sigma^{-1/2}\}$$
$$=\sum_{k=0}^{\infty}\sum_{\kappa}(-x)^k R_\kappa\,, \qquad |x|<\overline{\overline{(I-pA^{-1})^{-1}}}\,.$$

PROOF. (43) can be obtained by using (6) and (42), and (44) also can be obtained by using (12) and (42).

4. The p.d.f. of $\mathrm{tr}\,XAX'$ and the relations with the univariate quadratic form

In this section we discuss the representation of the p.d.f. of $\mathrm{tr}\,XAX'$ (and of $\mathrm{tr}\,\Sigma^{-1/2}XAX'\Sigma^{-1/2}$ as a special case). When $m=1$, the p.d.f. of the univariate quadratic form are treated by many authors. We derive three representations of it by the use of $P_\kappa(T, A)$, and give the relations between our results and the one of Kotz et al. [9] and of Ruben [10], [11].

Before discussing the representation, we give a simple lemma.

LEMMA 1.

(45) $$\sum_{\kappa} P_k(T, A)=P_k(\boldsymbol{t}, I_m\otimes A)\,,$$

(46) $$\left|\sum_{\kappa} P_\kappa(T, A)\right|\leqq \mathrm{etr}\,(TT')(mn/2)_k C_{(k)}(I_m\otimes A)/C_{(k)}(I_{mn})\,,$$

where $\boldsymbol{t}=(\boldsymbol{t}_1, \boldsymbol{t}_2,\cdots, \boldsymbol{t}_m)$ *and* $T'=[\boldsymbol{t}_1', \boldsymbol{t}_2',\cdots, \boldsymbol{t}_m']$, *and* $\otimes$ *denotes a Kronecker product of* I_m *and* A, *and* $P_k(\cdot, \cdot)$ *is a polynomial for* $m=1$ *in the definition* (1) *of* $P_\kappa(T, A)$.

PROOF. Let $U'=[\boldsymbol{u}_1', \boldsymbol{u}_2',\cdots, \boldsymbol{u}_m']$ and $T'=[\boldsymbol{t}_1', \boldsymbol{t}_2',\cdots, \boldsymbol{t}_m']$, then $\mathrm{tr}\,UAU' =\boldsymbol{u}(I_m\otimes A)\boldsymbol{u}'$, where $\boldsymbol{u}=[\boldsymbol{u}_1, \boldsymbol{u}_2,\cdots, \boldsymbol{u}_m]$. Hence

$$\begin{aligned}\sum_{\kappa} P_\kappa(T, A)&=\mathrm{etr}\,(TT')((-1)^k/\pi^{mn/2})\\&\quad\cdot\int_U \mathrm{etr}\,(-2iTU')\,\mathrm{etr}\,(-UU')(\mathrm{tr}\,UAU')^k dU\\&=\exp\,(\boldsymbol{tt}')((-1)^k/\pi^{mn/2})\\&\quad\cdot\int_U \exp\,(-2i\boldsymbol{tu}'-\boldsymbol{uu}')\{\boldsymbol{u}(I_m\otimes A)\boldsymbol{u}'\}^k dU\\&=P_k(\boldsymbol{t}, I_m\otimes A)\,.\end{aligned}$$

$$\begin{aligned}\left|\sum_{\kappa} P_\kappa(T, A)\right|&\leqq \exp\,(\boldsymbol{tt}')(1/\pi^{mn/2})\int_U \exp\,(-\boldsymbol{uu}')\{\boldsymbol{u}(I\otimes A)\boldsymbol{u}'\}^k dU\\&=\mathrm{etr}\,(TT')(1/\pi^{mn/2})\int_{\boldsymbol{u}} \exp\,(-\boldsymbol{uu}')C_{(k)}(\boldsymbol{u}'\boldsymbol{u}(I\otimes A))d\boldsymbol{u}\,,\end{aligned}$$

since rank of $\boldsymbol{u}(I_m\otimes A)\boldsymbol{u}'$ is 1 and $\{\boldsymbol{u}(I\otimes A)\boldsymbol{u}'\}^k=\{\mathrm{tr}\,\boldsymbol{u}'\boldsymbol{u}(I\otimes A)\}^k=$

$C_{(k)}(\boldsymbol{u}'\boldsymbol{u}(I\otimes A))$. Transforming $\boldsymbol{u}\to\boldsymbol{u}H$, $H\in 0(mn)$ and integrating H over $0(mn)$, the R.H.S. becomes

$$=\operatorname{etr}(TT')(1/\pi^{mn/2})\int_{\boldsymbol{u}}\exp(-\boldsymbol{u}\boldsymbol{u}')((C_{(k)}(I\otimes A)C_{(k)}(\boldsymbol{u}'\boldsymbol{u}))/(C_{(k)}(I_{mn}))d\boldsymbol{u}\ .$$

Hence by using the Hsu's lemma, we have the R.H.S. of (46).

Let X be an $m\times n$ $(m\leqq n)$ matrix whose p.d.f. is (20). We denote X and M as

$$X'=[\boldsymbol{x}_1',\boldsymbol{x}_2',\cdots,\boldsymbol{x}_m']'\ ,\qquad M'=[\boldsymbol{\mu}_1',\boldsymbol{\mu}_2',\cdots,\boldsymbol{\mu}_m']'\ ,$$

where $\boldsymbol{x}_\alpha=(x_{\alpha1},x_{\alpha2},\cdots,x_{\alpha n})$ and $\boldsymbol{\mu}_\alpha=(\mu_{\alpha1},\mu_{\alpha2},\cdots,\mu_{\alpha n})$, $\alpha=1,2,\cdots,m$. Let $\boldsymbol{x}=(\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_m)$ and $\boldsymbol{\mu}=(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\cdots,\boldsymbol{\mu}_m)$, then $\boldsymbol{x}$ is mn dimensional normal random vector with mean $\boldsymbol{\mu}$ and covariance matrix $\Sigma\otimes B$. On the other hand,

$$\text{(47)}\qquad \operatorname{tr}XAX'=\sum_{\alpha=1}^{m}\boldsymbol{x}_\alpha A\boldsymbol{x}_\alpha'=\boldsymbol{x}(I_m\otimes A)\boldsymbol{x}'\ .$$

Hence the problem is reduced to the one of the univariate non-central quadratic form. To compare with the results of Kotz et al. [9] and of Ruben [10], [11], we assume without loss of generality that A is a diagonal matrix where diagonal elements are $a_1,a_2,\cdots,a_n$ and $a_1\geqq a_2\geqq\cdots\geqq a_n>0$.

The p.d.f. of $T=\operatorname{tr}XAX'$ is derived by the following multiple integral.

$$\text{(48)}\qquad (1/(\pi^{mn/2}(\det 2\Sigma)^{n/2}(\det B)^{m/2}))$$
$$\cdot\int_{T=\boldsymbol{x}(I_m\otimes A)\boldsymbol{x}'}\exp[-(1/2)(\boldsymbol{x}-\boldsymbol{\mu})(\Sigma^{-1}\otimes B^{-1})(\boldsymbol{x}-\boldsymbol{\mu})']d\boldsymbol{x}$$
$$=((\exp\{-(1/2)\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1})\boldsymbol{\mu}'\})/(\pi^{mn/2}(\det 2\Sigma)^{n/2}(\det AB)^{m/2}))$$
$$\cdot\int_{T=\boldsymbol{x}\boldsymbol{x}'}\exp[-(1/2)\boldsymbol{x}(\Sigma^{-1}\otimes C^{-1})\boldsymbol{x}'+\boldsymbol{x}(\Sigma^{-1}\otimes A^{-1/2}B^{-1})\boldsymbol{\mu}']d\boldsymbol{x}\ ,$$

where $C=A^{1/2}BA^{1/2}$.

THEOREM 4 (*Power series type*). *Let X be distributed with p.d.f.* (20), *then the p.d.f. of $T=\operatorname{tr}XAX'$ is given by*

$$\text{(49)}\qquad ((\exp(-(1/2)\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1})\boldsymbol{\mu}'))/(\Gamma(mn/2)(\det 2\Sigma)^{n/2}(\det AB)^{m/2}))$$
$$\cdot\ T^{mn/2-1}\sum_{k=0}^{\infty}(1/(k!(mn/2)_k))(T/2)^kP_k((1/\sqrt{2})\boldsymbol{\mu}$$
$$\cdot(\Sigma^{-1/2}\otimes B^{-1}A^{-1/2}C^{1/2}),\ \Sigma^{-1}\otimes C^{-1})\ ,$$

where $C=A^{1/2}BA^{1/2}$. The power series converges absolutely for $T>0$.

PROOF. By using (48), the p.d.f. $g(T)$ of $T=\operatorname{tr}XAX'$ is expressed by

$$g(T)=C^*(1/\pi^{mn/2})\int_{T=\boldsymbol{x}\boldsymbol{x}'}\int_{O(mn)}\exp[-(1/2)\boldsymbol{x}H(\Sigma^{-1}\otimes C^{-1})H'\boldsymbol{x}'$$
$$+\boldsymbol{x}H(\Sigma^{-1/2}\otimes C^{-1/2})(\Sigma^{-1/2}\otimes C^{1/2}A^{1/2}B^{-1})\boldsymbol{\mu}']d(H)d\boldsymbol{x}$$
$$=C^*(1/\pi^{mn/2})\int_{T=\boldsymbol{x}\boldsymbol{x}'}\sum_{k=0}^{\infty}(1/(k!))(\boldsymbol{x}\boldsymbol{x}'/2)^k P_k((1/\sqrt{2})\boldsymbol{\mu}$$
$$\cdot(\Sigma^{-1/2}\otimes B^{-1}A^{1/2}C^{1/2}),\ \Sigma^{-1}\otimes C^{-1})d\boldsymbol{x}$$
$$=((C^*T^{mn/2-1})/\Gamma(mn/2))\sum_{k=0}^{\infty}((T/2)^k/(k!(mn/2)_k))P_k((1/\sqrt{2})\boldsymbol{\mu}$$
$$\cdot(\Sigma^{-1/2}\otimes B^{-1}A^{-1/2}C^{1/2}),\ \Sigma^{-1}\otimes C^{-1}),$$

where

$$C^*=(\exp(-(1/2)\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1})\boldsymbol{\mu}'))/((\det 2\Sigma)^{n/2}(\det AB)^{m/2}).$$

The third equality is shown by using Hsu's lemma.

COROLLARY 5. *The p.d.f. of* $T=\operatorname{tr}\Sigma^{-1/2}XAX'\Sigma^{-1/2}$ *is given by*

(50) $$((\exp(-(1/2)\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1})\boldsymbol{\mu}'))/(\Gamma(mn/2)(\det 2AB)^{m/2}))T^{mn/2-1}$$
$$\cdot\sum_{k=0}^{\infty}(1/(k!(mn/2)_k))(T/2)^k P_k(1/\sqrt{2}\,\boldsymbol{\mu}(\Sigma^{-1/2}\otimes B^{-1}A^{-1/2}C^{1/2}),$$
$$I_m\otimes C^{-1}).$$

Note. (50) can be obtained by using the Laplace transform of $\operatorname{tr}\Lambda$ with respect to the p.d.f. (21), see Hayakawa [3]. Here we compare with the results of Kotz et al. [9]. Kotz et al. showed the following lemma.

LEMMA (*Kotz et al.*) *Let* $\boldsymbol{x}=(x_1,\cdots,x_n)$ *be normally distributed with mean* $\mathbf{0}$ *and covariance matrix* I_n *and* A *be a diagonal matrix, i.e.* $\operatorname{diag}(a_1,\cdots,a_n)$, $a_1\geqq a_2\geqq\cdots\geqq a_n>0$ *and* $b=(b_1,\cdots,b_n)$, *then the p.d.f. of* $T=(\boldsymbol{x}+\boldsymbol{b})A(\boldsymbol{x}+\boldsymbol{b})'$ *is given by*

(51) $$\sum_{k=0}^{\infty}(\alpha_k^p(-1)^k(T/2)^{n/2+k-1})/(2\Gamma((n/2)+k)).$$

The α_k^{p}*'s are determined by*

(52) $$\sum_{k=0}^{\infty}\alpha_k^p\theta^k=(\det A)^{-1/2}\prod_{i=1}^{n}(1-\theta/a_i)^{-1/2}.$$
$$\cdot\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(b_i^2/(1-\theta/a_i)\right],\qquad \textit{for } \bar{\bar{A}}>\theta,$$

where the recurrence relation

(53) $$\alpha_0^p=(\det A)^{-1/2}\exp\left(-(1/2)\sum_{i=1}^{n}b_i^2\right),$$

$$\alpha_k^p=\sum_{r=0}^{k-1} b_{k-r}^p\alpha_r^p\ , \qquad k\geqq 1$$

with $b_k^p=(1/2)\sum_{i=1}^{n}(1-kb_i^2)/a_i^k$ *can be obtained.* (α_0^p *and* b_k^p *of Kotz et al. should be changed to the above form.*)

To compare with Theorem 4 and lemma (Kotz et al.); we set $\Sigma=I_m$ and $B=I_n$ in (49) and we have

$$(49)' \qquad ((\operatorname{etr}(-(1/2)\boldsymbol{\mu}\boldsymbol{\mu}'))/(2^{mn/2}\Gamma(mn/2)(\det A)^{m/2}))T^{mn/2-1} \cdot\sum_{k=0}^{\infty}(1/(k!(mn/2)_k))(T/2)^k P_k((1/\sqrt{2})\boldsymbol{\mu}, I\otimes A^{-1})\ .$$

Now replacing A by $I_m\otimes A$, $\boldsymbol{b}$ by $\boldsymbol{\mu}$ and n by mn in Lemma (Kotz et al.), we have the following form.

$$(51)' \qquad \sum_{k=0}^{\infty}(\alpha_k^p(-1)^k(T/2)^{mn/2+k-1})/(2\Gamma((mn/2)+k))\ ,$$

$$(52)' \qquad \sum_{k=0}^{\infty}\alpha_k^p\theta^k=(\det A)^{-m/2}\exp\left[-(1/2)\sum_{j=1}^{n}(1/(1-\theta/a_j))\sum_{i=1}^{m}\mu_{ij}^2\right] \cdot\prod_{j=1}^{n}(1-\theta/a_j)^{-m/2}\ ,$$

$$(53)' \qquad \alpha_0^p=(\det A)^{-m/2}\operatorname{etr}(-(1/2)MM')$$

and

$$b_k^p=(1/2)\sum_{j=1}^{n}(1/a_j)^k\sum_{i=1}^{m}(1-k\mu_{ij}^2)\ .$$

Therefore, by comparing (49)′ with (51)′ we have the following relation by (45),

$$(54) \qquad \begin{aligned}\alpha_k^p&=((-1)^k/(k!))((\operatorname{etr}(-(1/2)MM'))/(\det A)^{m/2})\\ &\quad\cdot P_k((1/\sqrt{2})\boldsymbol{\mu}, I_m\otimes A^{-1})\\ &=((-1)^k/(k!))((\operatorname{etr}(-(1/2)MM'))/(\det A)^{m/2})\\ &\quad\cdot\sum_{\kappa}P_\kappa((1/\sqrt{2})M, A^{-1})\ ,\end{aligned}$$

where $\sum_{\kappa}P_\kappa((1/\sqrt{2})M, A^{-1})$ is given by Proposition 2. Hence (54) gives an explicit form of α_k^p not involving a recurrence relation. We can also easily check by using (12) that if we insert (54) into the left hand side of (52)′, we have the right hand side of (52)′.

Next we discuss the Γ-type representation.

THEOREM 5 (*Γ-type representation*). *Let X be distributed with the p.d.f.* (20), *then the p.d.f. of* $T=\operatorname{tr}XAX'$ *is expressed as*

$$(55)\qquad ((\exp(-(1/2)\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1})\boldsymbol{\mu}'))/(\Gamma(mn/2)(\det 2\Sigma)^{n/2}(\det AB)^{m/2}))$$
$$\cdot \exp(-(1/(2p))T)T^{(mn/2)-1}$$
$$\cdot \sum_{k=0}^{\infty}(1/(k!(mn/2)_k))(T/2)^k P_k((1/\sqrt{2})\boldsymbol{\mu}$$
$$\cdot (\Sigma^{-1}\otimes B^{-1}A^{-1/2})D^{-1/2}, D)\ , \qquad \|\Sigma\otimes C\|<p$$

where $D=\Sigma^{-1}\otimes C^{-1}-I_m\otimes I_n/p,\ C=A^{1/2}BA^{1/2}$.

PROOF. From (48) and the proof of Theorem 4, we can easily show (55).

COROLLARY 6. *The p.d.f. of* $\operatorname{tr}\Sigma^{-1/2}XAX'\Sigma^{-1/2}$ *is given by*

$$(56)\qquad ((\exp(-(1/2)\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1})\boldsymbol{\mu}'))/(2^{mn/2}\Gamma(mn/2)(\det AB)^{m/2}))$$
$$\cdot \exp(-(1/(2p))T)T^{(mn/2)-1}$$
$$\cdot \sum_{k=0}^{\infty}(1/(k!(mn/2)_k))(T/2)^k P_k((1/\sqrt{2})\boldsymbol{\mu}(\Sigma^{-1/2}\otimes B^{-1}A^{-1/2})$$
$$\cdot (I\otimes(C^{-1}-I/p)^{-1/2}), I\otimes(C^{-1}-I/p))\ , \qquad \|C\|<p\ ,$$

where $C=A^{1/2}BA^{1/2}$.

Note. (56) can be also obtained by the use of Laplace transform of $\operatorname{tr}\Lambda$ with respect to the p.d.f. (22).

Ruben [10] gave a Γ-type representation of the p.d.f. of a quadratic form which is obtained in the following way.

LEMMA (*Ruben*). *Under the same condition of lemma* (*Kotz et al.*), *the p.d.f.* of $T=(\boldsymbol{x}+\boldsymbol{b})A(\boldsymbol{x}+\boldsymbol{b})'$ *is given by*

$$(57)\qquad \sum_{k=0}^{\infty}\alpha_k^c((\exp(-(1/(2p))T)T^{(n/2)+k-1})/(2^{(n/2)+k-1}\Gamma((n/2)+k)))(1/p)^{(n/2)+k}\ .$$

The α_k^c*'s are determined by*

$$(58)\qquad \sum_{k=0}^{\infty}\alpha_k^c\theta^k=(\det A/p)^{-1/2}\prod_{j=1}^{n}(1/\{1-(1-p/a_j)\theta\}^{1/2})$$
$$\cdot \exp\left[-(1/2)\sum_{k=1}^{n}b_k^2((1-\theta)/(1-(1-p/a_k)\theta))\right]\ .$$

Hence the recurrence relation

$$(59)\qquad \alpha_0^c=(\det A/p)^{-1/2}\exp[-(1/2)\boldsymbol{b}\boldsymbol{b}']$$

$$(60)\qquad \alpha_k^c=\sum_{r=0}^{k-1}b_{k-r}^c\alpha_r^c/2k\ , \qquad k\geqq 1\ ,$$

with

$$b_k^c=\sum_{j=1}^{n}(1-(p/a_j))^k+kp\sum_{j=1}^{n}(b_j^2/a_j)(1-(p/a_j))^{k-1}$$

can be obtained.

To compare Theorem 5 with lemma (Ruben), we set $\Sigma=I_m$ and $B=I_n$ in (55) and we replace A with $I_m\otimes A$, $\boldsymbol{b}$ with $\boldsymbol{\mu}$ and n with mn in (57). Then we have

$$(55)' \quad ((\exp(-(1/2)\boldsymbol{\mu\mu}'))/(2^{mn/2}\Gamma(mn/2)(\det A)^{m/2})) \cdot \exp(-(1/(2p))T)T^{mn/2-1} \cdot \sum_{k=0}^{\infty}(1/(k!(mn/2)_k))(T/2)^k \sum_{\kappa} P_{\kappa}((1/\sqrt{2})MA^{-1/2} \cdot (A^{-1}-I/p)^{-1/2}, A^{-1}-I/p)\ ,$$

$$(57)' \quad \sum_{k=0}^{\infty}\alpha_k^c((\exp(-(1/(2p))T)T^{(mn/2)+k-1})/ (2^{(mn/2)+k-1}\Gamma((mn/2)+k)))(1/p)^{(mn/2)+k}\ ,$$

$$(58)' \quad \sum_{k=0}^{\infty}\alpha_k^c\theta^k=(\det A)^{-m/2}\prod_{j=1}^{n}\{1-(1-p/a_j)\theta\}^{-m/2} \cdot \exp\left[-(1/2)\sum_{j=1}^{n}((1-\theta)/(1-(1-p/a_j)\theta))\sum_{i=1}^{m}\mu_{ij}^2\right].$$

Therefore, by comparing (55)′ with (57)′, we have the following relation.

$$(61) \quad \alpha_k^c=(p^k/(k!))(\det A/p)^{-m/2}\operatorname{etr}(-(1/2)MM') \cdot \sum_{\kappa}P_{\kappa}((1/\sqrt{2})MA^{-1/2}(A^{-1}-I/p)^{-1/2}, A^{-1}-I/p)$$

and $\sum_{\kappa}P_{\kappa}((1/\sqrt{2})MA^{-1/2}(A^{-1}-I/p)^{-1/2}, A^{-1}-I/p)$ is give by Proposition 2. Hence (61) gives an explicit expression of α_k^p not involving a recurrence relation. We can also check easily that if we insert (61) into the L.H.S. of (58)′, then we obtain R.H.S. of (58)′ by using (12) with appropriate change of T and A in (12).

Next we consider the mixture representation.

THEOREM 6 (*Mixture representation*). *Let X be distributed with p.d.f.* (26), *then the p.d.f. of* $\operatorname{tr}\Sigma^{-1/2}(X-M)A(X-M)'\Sigma^{-1/2}$ *is expressed as*

$$(62) \quad (1/((2p)^{mn/2}\Gamma(mn/2)))\exp(-(1/(2p))T)T^{mn/2-1} \cdot \sum_{k=0}^{\infty}(1/(mn/2)_k)(T/(2p))^k\sum_{\kappa}R_k\ , \qquad \bar{\bar{A}}>p>0\ ,$$

where R_κ is given by (29) *and* (62) *converges absolutely for* $T>0$.

PROOF. By taking the Fourier transform of $\operatorname{tr}\Lambda$ with respect to the p.d.f. (27) and inverting it, we have (62).

Note. We can also compare with the results of Ruben [11] by the similar way. We will omit.

5. The complex multivariate quadratic form

In this section, we shall state the above results for the complex Gaussian distribution studied by Goodman [1], James [6], and Khatri [7].

Let T and U be $m\times n$ $(m\leqq n)$ complex arbitrary matrices whose ranks are m, respectively, and A be an $n\times n$ positive definite Hermitian matrix. We define $\tilde{P}_\kappa(T, A)$ as follows:

(63) $$\operatorname{etr}(-T\bar{T}')\tilde{P}_\kappa(T, A) = ((-1)^k/\pi^{mn})\int_U \operatorname{etr}\{-i(T\bar{U}'+U\bar{T}')\}\operatorname{etr}(-U\bar{U}')\tilde{C}_\kappa(UA\bar{U}')dU$$

where $\tilde{C}_\kappa(UA\bar{U}')$ is a zonal polynomial of an Hermitian matrix $UA\bar{U}'$ and is expressed as

$$\tilde{C}_\kappa(UA\bar{U}')=\chi_{[\kappa]}(1)\chi_{\{\kappa\}}(UA\bar{U}') ,$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation corresponding to a partition κ of the symmetric group of k symbole and $\chi_{\{\kappa\}}(UA\bar{U}')$ is a character of the representation of the general linear group, (James [6]). As $U=U_1+iU_2$, $T=T_1+iT_2$ and $A=A_1+iA_2$, where A_1 is a symmetric matrix and A_2 is a skew symmetric matrix, we will set

$$U^*=(U_1 \quad U_2) , \qquad T^*=(T_1 \quad T_2) ,$$

$$B=\begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}=B' , \qquad C=\begin{bmatrix} A_2 & -A_1 \\ A_2 & A_1 \end{bmatrix}=-C'$$

and (63) becomes then

(64) $$\tilde{P}_\kappa(T, A)=((-1)^k/\pi^{mn})\int_U \operatorname{etr}(-U^*U^{*\prime})\tilde{C}_\kappa((U^*-iT^*)(B+iC) \cdot (U^*-iT^*)')dU^* .$$

The exact expressions for $\tilde{P}_\kappa(T, A)$ up to $k=3$ are given in Appendix. The fundamental properties of $\tilde{P}_\kappa(T, A)$ are similar to the real variate $P_\kappa(T, A)$ which are shown below.

(65) $$\tilde{P}_\kappa(0, A)=(-1)^k[n]_\kappa\tilde{C}_\kappa(A)\tilde{C}_\kappa(I_m)/\tilde{C}_\kappa(I_n) ,$$

(66) $$\tilde{P}_\kappa(T, I_n)=\tilde{H}_\kappa(T) ,$$

(67) $$|P_\kappa(T, A)|\leqq \operatorname{etr}(T\bar{T}')[n]_\kappa\tilde{C}_\kappa(A)\tilde{C}_\kappa(I_m)/\tilde{C}_\kappa(I_n) ,$$

where

$$[a]_\kappa=\prod_{\alpha=1}^{m}(a-\alpha+1)_{k_\alpha}\ ,$$

and $\tilde{H}_\kappa(T)$ is a complex generalized Hermite polynomial of a matrix argument T, (essentially $\tilde{H}_\kappa(T)$ is a real valued function). The detail of $\tilde{H}_\kappa(T)$ may be found in Hayakawa [4].

The generating function of $\tilde{P}_\kappa(T, A)$'s is given by

$$(68)\qquad \int_{U(m)}\int_{U(n)}\operatorname{etr}(-SU_2A\bar{U}_2'\bar{S}'+U_1SU_2A^{1/2}\bar{T}' + TA^{1/2}\bar{U}_2'\bar{S}'\bar{U}_1')d(U_1)d(U_2) =\sum_{k=0}^{\infty}\sum_{\kappa}(\tilde{P}_\kappa(T, A)\tilde{C}_\kappa(SS'))/(k![n]_\kappa\tilde{C}_\kappa(I_m))\ .$$

The right hand-side of (68) converges absolutely with respect to S, where S is an $m\times n$ $(m\leqq x)$ complex arbitrary matrix, U_1 and U_2 are unitary matrices of order m and n, respectively, and $d(U_1)$ and $d(U_2)$ are the normalized unitary invariant measures over the unitary groups $U(m)$ and $U(n)$, respectively.

PROPOSITION 3. Let T and S be $m\times n$ $(m\leqq n)$ complex matrices and let A and B be positive definite Hermitian matrices, then

$$(69)\qquad \int_{U(m)}\int_{U(n)}(1/\det(I-u^2U_2A\bar{U}_2'B)^m)\operatorname{etr}\{uU_1TA^{1/2}\bar{U}_2'B^{1/2}\bar{S}' +uSB^{1/2}U_2A^{1/2}\bar{T}'\bar{U}_1'-u^2TA^{1/2}\bar{U}_2'BU_2A^{1/2}\bar{T}' +u^2(S-uU_1TA^{1/2}\bar{U}_2'B^{1/2})B^{1/2}U_2(u^2\bar{U}_2'BU_2-A^{-1})^{-1} \cdot\bar{U}_2'B^{1/2}\overline{(S-uU_1TA^{1/2}\bar{U}_2'B^{1/2})'}\}d(U_1)d(U_2) =\sum_{k=0}^{\infty}\sum_{\kappa}((\tilde{P}_\kappa(T, A)\tilde{P}_\kappa(S, B))/(k![n]_\kappa\tilde{C}_\kappa(I_m)))u^{2k}\ , \quad \text{for } \max\{\|uA\|, \|uB\|\}<1\ .$$

PROOF. Similar to Proposition 1.

COROLLARY 3. *If we set* $A=B=I_n$ *in* (69), *we have*

$$(70)\qquad (1/(1-u^2)^{mn})\int_{U(m)}\int_{U(n)}\operatorname{etr}\{-(u^2/(1-u^2))(S\bar{S}'+T\bar{T}') +(u/(1-u^2))(U_1S\bar{U}_2'\bar{T}'+TU_2\bar{S}'\bar{U}_1')\}d(U_1)d(U_2) =\sum_{k=0}^{\infty}\sum_{\kappa}((\tilde{H}_\kappa(T)\tilde{H}_\kappa(S))/(k![n]_\kappa\tilde{C}_\kappa(I_m)))u^{2k}\ ,\qquad |u|<1\ .$$

This is the generalized Mehler's formulas of the generalized complex Hermite polynomials, Hayakawa [4].

PROPOSITION 4.

$$\sum_{\kappa} \tilde{P}_{\kappa}(T, A)=(-1)^k k!\Big[\tilde{A}_k+(1/2)\sum_{l_1+l_2=k}\tilde{A}_{l_1}\tilde{A}_{l_2} + (1/3!)\sum_{l_1+l_2+l_3=k}\tilde{A}_{l_1}\tilde{A}_{l_2}\tilde{A}_{l_3}+\cdots+(\tilde{A}_1^k/(k!))\Big] , \tag{71}$$

where l_α's are positive integers greater than or equal to 1, and

$$\tilde{A}_l=(m/l)\operatorname{tr} A^l-\operatorname{tr} TA^l\bar{T}' , \qquad l=1, 2, 3,\cdots, k$$

and

$$\tilde{A}_0\equiv 0 , \qquad \text{for convenience} .$$

PROOF. Similar to Proposition 2.

Remark. If we set $A=I_n$, then we have immediately,

$$\sum_{\kappa} \tilde{P}_{\kappa}(T, I_n)=(-1)^k L_k^{mn-1}(\operatorname{tr} T\bar{T}') , \tag{72}$$

where $L_k^{\alpha}(z)$ is a univariate Laguerre polynomial.

Let X be an $m\times n$ $(m\leqq n)$ complex matrix whose density function is given by

$$(1/\pi^{mn}(\det \Sigma)^n(\det B)^m)) \operatorname{etr}(-\Sigma^{-1}(X-M)B^{-1}(\overline{X-M})') , \tag{73}$$

where Σ is an $m\times m$ p.d. Hermitian matrix, M is an $m\times n$ complex matrix whose rank is m, and B is an $n\times n$ p.d. Hermitian matrix. Let A be an $n\times n$ p.d. Hermitian matrix. Then we have three types of representation of the p.d.f. of the latent roots of the multivariate quadratic form $XA\bar{X}'$.

THEOREM 6 (*Power series representation*). *Let X be distributed with the p.d.f.* (73). *Then the p.d.f. of the latent roots $\Lambda=\operatorname{diag}(\lambda_1, \lambda_2,\cdots, \lambda_m)$ of $\Sigma^{-1/2}XA\bar{X}'\Sigma^{-1/2}$ is expressed as*

$$((\pi^{m^2} \operatorname{etr}(-\Sigma^{-1}MB^{-1}\bar{M}'))/(\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)(\det AB)^m))(\det \Lambda)^{n-m} \cdot \prod_{i\neq j}(\lambda_i-\lambda_j)^2 \cdot \sum_{k=0}^{\infty}\sum_{\kappa}(\tilde{P}_{\kappa}(\Sigma^{-1/2}MB^{-1}A^{-1/2}C^{1/2}, C^{-1})\tilde{C}_{\kappa}(\Lambda))/(k![n]_{\kappa}\tilde{C}_{\kappa}(I_m)) , \tag{74}$$

where $C=A^{1/2}BA^{1/2}$, and $\tilde{\Gamma}_m(a)=\prod_{\alpha=1}^{m}\Gamma(a-\alpha+1)$.

PROOF. Similar to Theorem 1.

THEOREM 7 (*Γ-type representation*). *Under the same condition of Theorem* 6, *the p.d.f. of Λ is expressed as*

$$(75)\qquad ((\pi^{m^2}\operatorname{etr}(-\Sigma^{-1}MB^{-1}\bar{M}'))/(\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)(\det AB)^m)) \cdot \operatorname{etr}(-(1/p)\Lambda)(\det\Lambda)^{n-m}\prod_{i\neq j}(\lambda_i-\lambda_j)^2 \cdot \sum_{k=0}^{\infty}\sum_{\kappa}(\tilde{P}_\kappa(\Sigma^{-1/2}MA^{-1/2}(C^{-1}-p^{-1}I)^{-1/2}, C^{-1}-p^{-1}I)\tilde{C}_\kappa(\Lambda))/ (k![n]_\kappa\tilde{C}_\kappa(I_m)) ,$$

where $C=A^{1/2}BA^{1/2}$ *and* $\|AB\|<p$.

PROOF. Similar to Theorem 2.

THEOREM 8 (*Mixture type representation*). *Under the same condition of Theorem* 6 *with* $B=I$, *the p.d.f. of* Λ *is expressed as*

$$(76)\qquad \sum_{k=0}^{\infty}\sum_{\kappa}\tilde{R}_\kappa\tilde{f}_\kappa(\Lambda) ,\qquad for\ \bar{\bar{A}}>p ,$$

where

$$(77)\qquad \tilde{f}_\kappa(\Lambda)=(\pi^{m^2}/(p^{mn}\tilde{\Gamma}_m(n;\kappa)\tilde{\Gamma}_m(m)\tilde{C}_\kappa(I_m)))\operatorname{etr}(-(1/p)\Lambda)(\det\Lambda)^{n-m} \cdot\prod_{i\neq j}(\lambda_i-\lambda_j)^2\tilde{C}_\kappa(\Lambda/p) ,$$

and

$$(78)\qquad k!\tilde{R}_\kappa=(1/\pi^{mn})\int_Y \operatorname{etr}(-Y\bar{Y}')\operatorname{etr}(-\tilde{\Omega}(Y))\tilde{C}_\kappa(\tilde{\Omega}(Y))dY ,$$

and

$$(79)\qquad \tilde{\Omega}(Y)=[Y(A/p-I)^{1/2}+\Sigma^{-1/2}M(A/p)^{1/2}] \cdot\overline{[Y(A/p-I)^{1/2}+\Sigma^{-1/2}M(A/p)^{1/2}]}' .$$

The $\tilde{R}_\kappa$'s *satisfy the following conditions,*

$$(80)\qquad \tilde{R}_\kappa>0 ,\qquad for\ all\ \kappa ,$$

$$(81)\qquad \sum_{k=0}^{\infty}\sum_{\kappa}\tilde{R}_\kappa=1 ,$$

$$(82)\qquad k!\tilde{R}_\kappa=(\det A/p)^{-m}\operatorname{etr}(-\Sigma^{-1}M\bar{M}')(-1)^k\tilde{P}_\kappa(iT^*, I-pA^{-1})$$

where $T^*=[T_1\ \ T_2]_{m\times 2n}$ *and* $T=T_1+iT_2=\Sigma^{-1/2}M(A/p-I)^{-1/2}$.

PROOF. Similar to Theorem 3.

We can also obtain the representations of the p.d.f. of $\operatorname{tr}XA\bar{X}'$ and $\operatorname{tr}\Sigma^{-1/2}XA\bar{X}'\Sigma^{-1/2}$. We will give them as Theorem 9.

THEOREM 9. *Let* X *be distributed with the p.d.f. of* (73), *then the p.d.f.'s of* $T=\operatorname{tr}XA\bar{X}'$ *are given by*

Power series type:

$$(83)\qquad ((\exp(-\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1})\bar{\boldsymbol{\mu}}'))/(\Gamma(mn)(\det\Sigma)^{n}(\det AB)^{m}))T^{mn-1} \cdot\sum_{k=0}^{\infty}(T^{k}/(k!(mn)_{k}))\tilde{P}_{k}(\boldsymbol{\mu}(\Sigma^{-1/2}\otimes B^{-1}A^{-1/2}C^{1/2}),\ \Sigma^{-1}\otimes C^{-1}) .$$

Γ-type:

$$(84)\qquad ((\exp(-\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1})\bar{\boldsymbol{\mu}}'))/(\Gamma(mn)(\det\Sigma)^{n}(\det AB)^{m})) \cdot\exp(-(1/p)T)T^{mn-1} \cdot\sum_{k=0}^{\infty}(T^{k}/(k!(mn)_{k}))\tilde{P}_{k}(\boldsymbol{\mu}(\Sigma^{-1}\otimes B^{-1}A^{-1/2})D^{-1/2};\ D) ,\quad \text{for } \|AB\|<p ,$$

where

$$D=\Sigma^{-1}\otimes C^{-1}-I_{mn}/p ,\qquad C=A^{1/2}BA^{1/2} .$$

Mixture type:

$$(85)\qquad (1/(p^{mn}\Gamma(mn)))\exp(-(1/p)T)T^{mn-1}\sum_{k=0}^{\infty}(T^{k}/(k!(mn)_{k}))\sum_{\kappa}\tilde{R}_{\kappa} ,\quad \bar{\bar{A}}>p>0 ,$$

where $\tilde{R}_{\kappa}$*'s are given by* (78).

PROOF. Similar to Theorems 4, 5, 6.

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Appendix

In this appendix, we list the exact forms of $P_{\kappa}(T, A)$ up to $k=4$ and $\tilde{P}_{\kappa}(T, A)$ up to $k=3$.

$$(\text{A }1)\qquad P_{(1)}(T, A)=-(m/2)\operatorname{tr}A+\operatorname{tr}TAT' .$$

$$(\text{A }2)\qquad P_{(2)}(T, A)=(1/3)[(m(m+2)/4)\{2\operatorname{tr}A^{2}+(\operatorname{tr}A)^{2}\} -(m+2)\{2\operatorname{tr}TA^{2}T'+\operatorname{tr}A\operatorname{tr}TAT'\} +2\operatorname{tr}(TAT')^{2}+(\operatorname{tr}TAT')^{2}] .$$

$$P_{(1^2)}(T, A)=(1/3)[(m(m-1)/2)\{(\mathrm{tr}\,A)^2-\mathrm{tr}\,A^2\}+2(m-1)\{\mathrm{tr}\,TA^2T'-\mathrm{tr}\,A\,\mathrm{tr}\,TAT'\}+2\{(\mathrm{tr}\,TAT')^2-\mathrm{tr}\,(TAT')^2\}]\,. \tag{A 3}$$

$$P_{(3)}(T, A)=-(1/15)[(m(m+2)(m+4)/8)\cdot\{(\mathrm{tr}\,A)^3+6\,\mathrm{tr}\,A\,\mathrm{tr}\,A^2+8\,\mathrm{tr}\,A^3\}-(3/4)(m+2)(m+4)\{(\mathrm{tr}\,A)^2+2\,\mathrm{tr}\,A^2\}\,\mathrm{tr}\,TAT'+(3/2)(m+4)\,\mathrm{tr}\,A\{(\mathrm{tr}\,TAT')^2+2\,\mathrm{tr}\,(TAT')^2\}-3(m+2)(m+4)\{\mathrm{tr}\,A\,\mathrm{tr}\,TA^2T'+2\,\mathrm{tr}\,TA^3T'\}+6(m+4)\{\mathrm{tr}\,TAT'\,\mathrm{tr}\,TA^2T'+2\,\mathrm{tr}\,TAT'TA^2T'\}-(\mathrm{tr}\,TA'T)^3-6\,\mathrm{tr}\,TAT'\,\mathrm{tr}\,(TAT')^2-8\,\mathrm{tr}\,(TAT')^3]\,. \tag{A 4}$$

$$P_{(21)}(T, A)=-(3/5)[(m(m-1)(m+2)/8)\{(\mathrm{tr}\,A)^3+\mathrm{tr}\,A\,\mathrm{tr}\,A^2-2\,\mathrm{tr}\,A^3\}-(1/4)(m-1)(m+2)\{3(\mathrm{tr}\,A)^2+\mathrm{tr}\,A^2\}\,\mathrm{tr}\,TAT'+((3m+2)/2)\{\mathrm{tr}\,A(\mathrm{tr}\,TAT')^2-2\,\mathrm{tr}\,TAT'TA^2T'\}-(1/2)(m-1)(m+2)\cdot\{\mathrm{tr}\,A\,\mathrm{tr}\,TA^2T'-3\,\mathrm{tr}\,TA^3T'\}+(1/2)(m-6)\,\mathrm{tr}\,A\,\mathrm{tr}\,(TAT')^2+(m+4)\cdot\mathrm{tr}\,TAT'\,\mathrm{tr}\,TA^2T'-(\mathrm{tr}\,TAT')^3-\mathrm{tr}\,TAT'\,\mathrm{tr}\,(TAT')^2+2\,\mathrm{tr}\,(TAT')^3]\,. \tag{A 5}$$

$$P_{(1^3)}(T, A)=-(1/3)[(m(m-1)(m-2)/8)\{(\mathrm{tr}\,A)^3-3\,\mathrm{tr}\,A\,\mathrm{tr}\,A^2+2\,\mathrm{tr}\,A^3\}-(3/4)(m-1)(m-2)\{(\mathrm{tr}\,A)^2-\mathrm{tr}\,A^2\}\,\mathrm{tr}\,TAT'+(3/2)(m-2)\{(\mathrm{tr}\,TAT')^2-\mathrm{tr}\,(TAT')^2\}\,\mathrm{tr}\,A+(3/2)(m-1)(m-2)\cdot\{\mathrm{tr}\,A\,\mathrm{tr}\,TA^2T'-\mathrm{tr}\,TA^3T'\}-3(m-2)\{\mathrm{tr}\,TAT'\cdot\mathrm{tr}\,TA^2T'-\mathrm{tr}\,TAT'TA^2T'\}-(\mathrm{tr}\,TAT')^3+3\,\mathrm{tr}\,TAT'\,\mathrm{tr}\,(TAT')^2-2\,\mathrm{tr}\,(TAT')^3]\,. \tag{A 6}$$

$$105P_{(4)}(T, A)=(m/2)_{(4)}Z_{(4)}(A)-(1/2)(m+2)(m+4)(m+6)[Z_{(3)}(A)\cdot\mathrm{tr}\,TAT'+6Z_{(2)}(A)\,\mathrm{tr}\,TA^2T'+24\,\mathrm{tr}\,A\,\mathrm{tr}\,TA^3T'+48\,\mathrm{tr}\,TA^4T']+(3/2)(m+4)(m+6)\cdot[Z_{(2)}(A)Z_{(2)}(TAT')+8Z_{(2)}(TA^2T')+8\,\mathrm{tr}\,A\{\mathrm{tr}\,TAT'\,\mathrm{tr}\,TA^2T'+2\,\mathrm{tr}\,TAT'TA^2T'\}+16\{\mathrm{tr}\,TAT'\,\mathrm{tr}\,TA^3T'+2\,\mathrm{tr}\,TAT'TA^3T'\}]-2(m+6)[\mathrm{tr}\,AZ_{(3)}(TAT')+6\{\mathrm{tr}\,TA^2T'Z_{(2)}(TAT')+4\,\mathrm{tr}\,TAT'\,\mathrm{tr}\,TAT'TA^2T'+8\,\mathrm{tr}\,TA^2T'(TAT')^2\}]+Z_{(4)}(TAT')\,. \tag{A 7}$$

$$
\begin{aligned}
\text{(A 8)}\quad 105P_{(31)}(T, A) = {} & 20(m/2)_{(31)}Z_{(31)}(A) - 5(m+2)(m+4)(m-1) \\
& \cdot [\{2(\operatorname{tr} A)^3 + 5 \operatorname{tr} A \operatorname{tr} A^2 + 2 \operatorname{tr} A^3\} \operatorname{tr} TAT' \\
& + \{5(\operatorname{tr} A)^2 - 4 \operatorname{tr} A^2\} \operatorname{tr} TA^2T' + 6 \operatorname{tr} A \operatorname{tr} TA^3T' \\
& - 16 \operatorname{tr} TA^4T'] + 5(m+4)[\{(6m+1)(\operatorname{tr} A)^2 \\
& + (5m+9) \operatorname{tr} A^2\}(\operatorname{tr} TAT')^2 + \{(5m-12)(\operatorname{tr} A)^2 \\
& - 2(2m+5) \operatorname{tr} A^2\} \operatorname{tr} (TAT')^2 + 4 \operatorname{tr} A \{(5m+2) \\
& \cdot \operatorname{tr} TAT' \operatorname{tr} TA^2T' + (3m-10) \operatorname{tr} TAT'TA^2T'\} \\
& + 4\{(3m+11) \operatorname{tr} TAT' \operatorname{tr} TA^3T' \\
& - 2(4m+3) \operatorname{tr} TAT'TA^3T'\} - 8(m-1)Z_{(2)}(TAT')] \\
& - 20[\operatorname{tr} A \{(2m+5)(\operatorname{tr} TAT')^3 + (5m+2) \operatorname{tr} TAT' \\
& \cdot \operatorname{tr} (TAT')^2 + 2(m-8) \operatorname{tr} (TAT')^3\} \\
& + \{(5m+23)(\operatorname{tr} TAT')^2 - 2(2m+5) \operatorname{tr} (TAT')^2\} \\
& \cdot \operatorname{tr} TA^2T' + 6(m+6) \operatorname{tr} TAT' \operatorname{tr} TAT'TA^2T' \\
& - 8(2m+5) \operatorname{tr} TA^2T'(TAT')^2] + 20Z_{(31)}(TAT') .
\end{aligned}
$$

$$
\begin{aligned}
\text{(A 9)}\quad 105P_{(2^2)}(T, A) = {} & 14(m/2)_{(2^2)}Z_{(2^2)}(A) - 7(m+2)(m-1)(m+1) \\
& \cdot [\{(\operatorname{tr} A)^3 + \operatorname{tr} A \operatorname{tr} A^2 - 2 \operatorname{tr} A^3\} \operatorname{tr} TAT' + \{(\operatorname{tr} A)^2 \\
& + 7 \operatorname{tr} A^2\} \operatorname{tr} TA^2T' - 6 \operatorname{tr} A \operatorname{tr} TA^3T' - 2 \operatorname{tr} TA^4T'] \\
& + 7(m+1)[\{(3m+2)(\operatorname{tr} A)^2 + (m-6) \operatorname{tr} A^2\} \\
& \cdot (\operatorname{tr} TAT')^2 + \{(m-6)(\operatorname{tr} A)^2 + (7m-2) \operatorname{tr} A^2\} \\
& \cdot \operatorname{tr} (TAT')^2 + 4\{(m+4) \operatorname{tr} TAT' \operatorname{tr} TA^2T' \\
& - (3m+2) \operatorname{tr} TAT'TA^2T'\} \operatorname{tr} A - 4\{(3m+2) \\
& \cdot \operatorname{tr} TAT' \operatorname{tr} TA^3T' + (m-6) \operatorname{tr} TAT'TA^3T'\} \\
& + 2\{(7m+8)(\operatorname{tr} TA^2T')^2 - (m+14) \operatorname{tr} (TA^2T')^2\}] \\
& - 28(m+1)[\operatorname{tr} AZ_{(21)}(TAT') + \operatorname{tr} TA^2T'(\operatorname{tr} TAT')^2 \\
& + 7 \operatorname{tr} TA^2T' \operatorname{tr} (TAT')^2 - 2 \operatorname{tr} TA^2T'(TAT')^2 \\
& - 6 \operatorname{tr} TAT' \operatorname{tr} TAT'TA^2T'] + 14Z_{(2^2)}(TAT') .
\end{aligned}
$$

$$
\begin{aligned}
\text{(A10)}\quad 105P_{(21^2)}(T, A) = {} & 56(m/2)_{(21^2)}Z_{(21^2)}(A) - 14(m+2)(m-2)(m-1) \\
& \cdot [\{2(\operatorname{tr} A)^3 - \operatorname{tr} A \operatorname{tr} A^2 - \operatorname{tr} A^3\} \operatorname{tr} TAT' \\
& - \{(\operatorname{tr} A)^2 + 4 \operatorname{tr} A^2\} \operatorname{tr} TA^2T' - 3 \operatorname{tr} A \operatorname{tr} TA^3T' \\
& + 8 \operatorname{tr} TA^4T'] + 14(m-2)[\{(6m+7)(\operatorname{tr} A)^2 \\
& - (m-3) \operatorname{tr} A^2\}(\operatorname{tr} TAT')^2 - \{(m+12)(\operatorname{tr} A)^2 \\
& + 2(2m-1) \operatorname{tr} A^2\} \operatorname{tr} (TAT')^2 - 2\{(3m-4) \\
& \cdot \operatorname{tr} TAT'TA^2T' + (2m-1) \operatorname{tr} TAT' \operatorname{tr} TA^2T'\} \operatorname{tr} A \\
& - 2\{(3m+11) \operatorname{tr} TAT' \operatorname{tr} TA^3T' - 2(4m+3) \\
& \cdot \operatorname{tr} TAT'TA^3T'\} - 8(m+2)Z_{(1^2)}(TA^2T')]
\end{aligned}
$$

$$-56[(2m-1)(\mathrm{tr}\, TAT')^3-(m+7)\,\mathrm{tr}\, TAT'\,\mathrm{tr}\,(TAT')^2$$
$$-(m-8)\,\mathrm{tr}\,(TAT')^3]\,\mathrm{tr}\, A-(m-8)\,\mathrm{tr}\, TA^2T'$$
$$\cdot\,(\mathrm{tr}\, TAT')^2-2(2m-1)\,\mathrm{tr}\, TA^2T'\,\mathrm{tr}\,(TAT')^2$$
$$+4(2m-1)\,\mathrm{tr}\, TA^2T'(TAT')^2-3(m+2)\,\mathrm{tr}\, TAT'$$
$$\cdot\,\mathrm{tr}\, TAT'TA^2T']+56Z_{(21^2)}(TAT')\ .$$

(A11) $$105P_{(1^4)}(T, A)=14(m/2)_{(1^4)}Z_{(1^4)}(A)-7(m-1)(m-2)(m-3)$$
$$\cdot\,[Z_{(1^3)}(A)\,\mathrm{tr}\, TAT'-3Z_{(1^2)}(A)\,\mathrm{tr}\, TA^2T'$$
$$+6\,\mathrm{tr}\, A\,\mathrm{tr}\, TA^3T'-6\,\mathrm{tr}\, TA^4T']+21(m-2)(m-3)$$
$$\cdot\,[Z_{(1^2)}(A)(\mathrm{tr}\, TAT')^2-Z_{(1^2)}(A)\,\mathrm{tr}\,(TAT')^2$$
$$-4\{\mathrm{tr}\, TAT'\,\mathrm{tr}\, TA^2T'-\mathrm{tr}\, TAT'TA^2T'\}\,\mathrm{tr}\, A$$
$$+4\{\mathrm{tr}\, TAT'\,\mathrm{tr}\, TA^3T'-\mathrm{tr}\, TAT'TA^3T'\}$$
$$+2Z_{(1^2)}(TA^2T')]-28(m-3)[Z_{(1^3)}(TAT')\,\mathrm{tr}\, A$$
$$-3Z_{(1^2)}(TAT')\,\mathrm{tr}\, TA^2T'-6\,\mathrm{tr}\, TA^2T'(TAT')^2$$
$$+6\,\mathrm{tr}\, TAT'\,\mathrm{tr}\, TAT'TA^2T']+14Z_{(1^4)}(TAT')\ ,$$

where $C_\kappa(S)=(\chi_\kappa(1)/(2k-1)!!)Z_\kappa(S)$ and $Z_\kappa(S)$'s are given in James [6]. Here we only sketch the derivation of these polynomials. For example,

$$P_{(2)}(T, A)=E_U[C_{(2)}\{(U-iT)A(U-iT)'\}]$$
$$=(1/3)E_U[\{\mathrm{tr}\,(U-iT)A(U-iT)'\}^2$$
$$-2\,\mathrm{tr}\,\{(U-iT)A(U-iT)'\}^2]\ .$$

Then it is not so hand to check

$$E_U[\{\mathrm{tr}\,(U-iT)A(U-iT)'\}^2]$$
$$=(m^2/4)(\mathrm{tr}\, A)^2+(m/2)\,\mathrm{tr}\, A^2-m\,\mathrm{tr}\, A\,\mathrm{tr}\, TAT'$$
$$-2\,\mathrm{tr}\, TA^2T'+(\mathrm{tr}\, TAT')^2$$
$$=\sum_\kappa P_\kappa(T, A)\ , \qquad \text{for } k=2$$

$$E_U[\mathrm{tr}\,\{(U-iT)A(U-iT)'\}^2]$$
$$=(m/4)(\mathrm{tr}\, A)^2+(m(m+1)/4)\,\mathrm{tr}\, A^2-\mathrm{tr}\, A\,\mathrm{tr}\, TAT'$$
$$-(m+1)\,\mathrm{tr}\, TA^2T'+\mathrm{tr}\,(TAT')^2\ .$$

Therefore, we have (A2).

We have $\tilde{P}_\kappa(T, A)$'s by the similar way as the case of $P_\kappa(T, A)$'s.

(A12) $$\tilde{P}_{(1)}(T, A)=-m\,\mathrm{tr}\, A+\mathrm{tr}\, TA\bar{T}'$$

(A13) $$2\tilde{P}_{(2)}(T, A)=m(m+1)[(\mathrm{tr}\, A)^2+\mathrm{tr}\, A^2]-2(m+1)$$
$$\cdot\,[\mathrm{tr}\, A\,\mathrm{tr}\, TA\bar{T}'+\mathrm{tr}\, TA^2\bar{T}']+(\mathrm{tr}\, TA\bar{T}')^2+\mathrm{tr}\,(TA\bar{T}')^2$$

(A14) $$2\tilde{P}_{(1^2)}(T, A)=m(m-1)[(\mathrm{tr}\,A)^2-\mathrm{tr}\,A^2]-2(m-1)[\mathrm{tr}\,A\,\mathrm{tr}\,TA\bar{T}' -\mathrm{tr}\,TA^2\bar{T}']+(\mathrm{tr}\,TA\bar{T}')^2-\mathrm{tr}\,(TA\bar{T}')^2$$

(A15) $$\begin{aligned}6\tilde{P}_{(3)}(T, A)=-[&m(m+1)(m+2)\{(\mathrm{tr}\,A)^3+3\,\mathrm{tr}\,A\,\mathrm{tr}\,A^2+2\,\mathrm{tr}\,A^3\}\\ &-3(m+1)(m+2)\{(\mathrm{tr}\,A)^2+\mathrm{tr}\,A^2\}\,\mathrm{tr}\,TA\bar{T}'\\ &-6(m+1)(m+2)\{\mathrm{tr}\,A\,\mathrm{tr}\,TA^2\bar{T}'+\mathrm{tr}\,TA^3\bar{T}'\}\\ &+3(m+2)\{\mathrm{tr}\,A(\mathrm{tr}\,TA\bar{T}')^2+2\,\mathrm{tr}\,TA\bar{T}'\,\mathrm{tr}\,TA^2\bar{T}'\\ &+\mathrm{tr}\,A\,\mathrm{tr}\,(TA\bar{T}')^2+2\,\mathrm{tr}\,TA\bar{T}'TA^2\bar{T}'\}-(\mathrm{tr}\,TA\bar{T}')^3\\ &-3\,\mathrm{tr}\,TA\bar{T}'\,\mathrm{tr}\,(TA\bar{T}')^2-2\,\mathrm{tr}\,(TA\bar{T}')^3]\end{aligned}$$

(A16) $$\begin{aligned}6\tilde{P}_{(21)}(T, A)=-[&4m(m-1)(m+1)\{(\mathrm{tr}\,A)^3-\mathrm{tr}\,A^3\}\\ &-12(m+1)(m-1)\{(\mathrm{tr}\,A)^2\,\mathrm{tr}\,TA\bar{T}'-\mathrm{tr}\,TA^3\bar{T}'\}\\ &+12m\{\mathrm{tr}\,A(\mathrm{tr}\,TA\bar{T}')^2-\mathrm{tr}\,TA\bar{T}'TA^2\bar{T}'\}\\ &-12\{\mathrm{tr}\,A\,\mathrm{tr}\,(TA\bar{T}')^2-\mathrm{tr}\,TA\bar{T}'\,\mathrm{tr}\,TA^2\bar{T}'\}\\ &-4\{(\mathrm{tr}\,TA\bar{T}')^3-\mathrm{tr}\,(TA\bar{T}')^3\}]\end{aligned}$$

(A17) $$\begin{aligned}6\tilde{P}_{(1^3)}(T, A)=-[&m(m-1)(m-2)\{(\mathrm{tr}\,A)^3-3\,\mathrm{tr}\,A\,\mathrm{tr}\,A^2+2\,\mathrm{tr}\,A^3\}\\ &-3(m-1)(m-2)\{(\mathrm{tr}\,A)^2-\mathrm{tr}\,A^2\}\,\mathrm{tr}\,TA\bar{T}'\\ &+6(m-1)(m-2)\{\mathrm{tr}\,A\,\mathrm{tr}\,TA^2\bar{T}'-\mathrm{tr}\,TA^3\bar{T}'\}\\ &+3(m-2)\{\mathrm{tr}\,A(\mathrm{tr}\,TA\bar{T}')^2-\mathrm{tr}\,A\,\mathrm{tr}\,(TA\bar{T}')^2\}\\ &-6(m-2)\{\mathrm{tr}\,TA\bar{T}'\,\mathrm{tr}\,TA^2\bar{T}'-\mathrm{tr}\,TA\bar{T}'TA^2\bar{T}'\}\\ &-(\mathrm{tr}\,TA\bar{T}')^3+3\,\mathrm{tr}\,TA\bar{T}'\,\mathrm{tr}\,(TA\bar{T}')^2-2\,\mathrm{tr}\,(TA\bar{T}')^3]\,.\end{aligned}$$

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