ON A GENERALIZATION OF RAO'S U STATISTIC

D. G. KABE

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1. Introduction and summary

Let the joint density of a $(p+q)\times(p+q)$ positive definite symmetric matrix S and a $(p+q)$ component column vector y be

$$(1)\qquad g(S, y)=K\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}[S+N(y-\mu)(y-\mu)']\right\}|S|^{(N-p-q-1)/2}.$$

Let $y=(y_1', y_2')'$ be partitioned into two parts, y_1 of first p components and y_2 of next q components, and let the corresponding partitions of S, Σ, Σ^{-1}, μ be

$$(2)\qquad S=\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix},\quad \Sigma=\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{pmatrix},\quad \Sigma^{-1}=\begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22}\end{pmatrix},\quad \mu=\begin{pmatrix} \mu_1 \\ \mu_2\end{pmatrix}.$$

Then Rao's U statistic [5] is defined by the relation

$$(3)\qquad 1+U=(1+T^2_{p+q})/(1+T^2_q),$$

where

$$(4)\qquad T^2_{p+q}=Ny'S^{-1}y, \qquad T^2_q=Ny_2'S_{22}^{-1}y_2.$$

The U statistic is used for testing the hypothesis that $\eta=\mu_1-\Sigma_{12}\Sigma_{22}^{-1}\mu_2=0$, against $\eta\neq 0$, and under the null hypothesis U has the beta density

$$(5)\qquad g(U)=[B(p/2, (N-q-p+1)/2)]^{-1}U^{p/2-1}(1+U)^{-(N-q+1)/2}.$$

Khatri [4] considers a certain generalization of the above problem. Let a $(p+q)\times(p+q)$ symmetric positive definite S and a $(p+q)\times N$ matrix Y have the density

$$(6)\qquad g(S, Y)=K\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}[S+(Y-\beta Z)(Y-\beta Z)']\right\}|S|^{(N-p-q-1)/2}$$

where β is $(p+q)\times t$, Z is $t\times N$ and of rank $t<N$. The matrix β is partitioned as $\beta'=(\beta_1', \beta_2')'$, β_1 is $p\times t$, and β_2 is $q\times t$. Then Khatri tested the null hypothesis concerning double linear compounds of $\delta=\beta_1-\Sigma_{12}\Sigma_{22}^{-1}\beta_2$,

using Roy's maximum root test. If the hypothesis specifies that $L\delta C'$, where L is $r\times p$ known and of rank $r<p$, C is $g\times t$ known and of rank $g\leqq p<t$, has a known value, then our purpose in this paper is to prove that, 1) the exact test statistics is Wilks' likelihood ratio test statistic; 2) the confidence bounds based on this (Wilks') likelihood ratio statistic are narrower than those based on Roy's maximum root test; 3) the optimum properties of Wilks' likelihood ratio statistic are well known while the optimum properties of Roy's maximum root test are still to be investigated. For testing $\delta=0$ against $\delta\neq 0$ we find a statistic $|Q|$ where the matrix Q has a second kind of multivariate beta density. The matrix Q may be termed as a generalization of Rao's U statistic.

All densities appearing in this paper are written without their normalizing constants and K denotes these constants. Some results found useful in the sequel are stated in the next section. We assume that all integrals occurring in this paper are evaluated over the appropriate ranges of their variables of integration.

2. Some useful results

Let Y be a $p\times N$ matrix, $-\infty<Y<\infty$, A an $N\times N$ positive definite symmetric matrix, Δ a $p\times N$ matrix, and D a given $q\times N$ matrix of rank $q<N$, $N\geqq p+q$. Then Kabe [2] has proved that

$$(7)\quad \int_{(Y-\Delta)A(Y-\Delta)'=G,\, DY'=V'} f((Y-\Delta)A(Y-\Delta)')dY$$
$$=2^{-p}\prod_{i=1}^{p} C(N-p-q+i)|A|^{-p/2}|DA^{-1}D'|^{-p/2}f(G)$$
$$\cdot\, |G-(V-\Delta D')(DA^{-1}D')^{-1}(V-\Delta D')'|^{(N-p-q-1)/2},$$

Where G is $p\times p$ and V is $p\times q$. In case f is a suitable density function, then obviously the right-hand side of (7) represents the joint density of G and V.

If X is $p\times N$, $-\infty<X<\infty$, then we may show that

$$(8)\quad \int \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}[XAX'+(X-M)\Phi(X-M)']\right\}dx$$
$$=(2\pi)^{pN/2}|\Sigma|^{N/2}|A+\Phi|^{-p/2}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}M\Phi(A+\Phi)^{-1}AM'\right\}.$$

Now we proceed with the derivation of the test statistic. The method used here is the same as that used by Kabe [3].

3. Derivation of the test statistic

If $(ZZ')^{-1}ZY'=B'$ is the maximum likelihood estimator of β, then the joint density of S and B is

$$(9)\qquad g(S, B)=K\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}[S+(B-\beta)ZZ'(B-\beta)']\right\}|S|^{(N-p-q-1)/2},$$

and writing $B'=(B_1', B_2')'$ corresponding to the partition of β we find the joint density of S_{12}, S_{22}, $D_{11}=S_{11}-S_{12}S_{22}^{-1}S_{21}$, B_2, $\hat{\delta}=B_1-S_{12}S_{22}^{-1}B_2$ to be

$$\begin{aligned}(10)\quad & g(D_{11}, S_{12}, S_{22}, \hat{\delta}, B_2)\\ &=K\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{11}[(\hat{\delta}-\delta+S_{12}S_{22}^{-1}B_2-\Sigma_{12}\Sigma_{22}^{-1}B_2)\right.\\ &\quad\cdot ZZ'(\hat{\delta}-\delta+S_{12}S_{22}^{-1}B_2-\Sigma_{12}\Sigma_{22}^{-1}B_2)'+D_{11}+(S_{12}-\Sigma_{12}\Sigma_{22}^{-1}S_{22})\\ &\quad\left.\cdot S_{22}^{-1}(S_{12}-\Sigma_{12}\Sigma_{22}^{-1}S_{22})']\right\}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_{22}^{-1}[S_{22}+(B_2-\beta_2)\right.\\ &\quad\left.\cdot ZZ'(B_2-\beta_2)']\right\}|D_{11}|^{(N-p-q-1)/2}|S_{22}|^{(N-p-q-1)/2}.\end{aligned}$$

Now by setting

$$(11)\qquad \begin{aligned}&(S_{12}-\Sigma_{12}\Sigma_{22}^{-1}S_{22})S_{22}^{-1}(S_{12}-\Sigma_{12}\Sigma_{22}^{-1}S_{22})'=G,\\ &B_2'S_{22}^{-1}S_{12}'=V',\end{aligned}$$

and then integrating (10) with respect to (w.r.t.) S_{12} by using (7), over the region (11), we obtain the joint density of the variates G, V, D_{11}, S_{22}, $\hat{\delta}$, B_2. We integrate this joint density w.r.t. G, and we find that

$$\begin{aligned}(12)\quad & g(D_{11}, S_{22}, \hat{\delta}, B_2, V)\\ &=K|B_2'S_{22}B_2|^{-p/2}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{11}[D_{11}+(\hat{\delta}-\delta+V-\Sigma_{12}\Sigma_{22}^{-1}B_2)\right.\\ &\quad\cdot ZZ'(\hat{\delta}-\delta+V-\Sigma_{12}\Sigma_{22}^{-1}B_2)'+(V-\Sigma_{12}\Sigma_{22}^{-1}B_2)(B_2'S_{22}^{-1}B_2)^{-1}\\ &\quad\left.\cdot(V-\Sigma_{12}\Sigma_{22}^{-1}B_2')]\right\}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_{22}^{-1}[S_{22}+(B_2-\beta_2)\right.\\ &\quad\left.\cdot ZZ'(B_2-\beta_2)']\right\}|D_{11}|^{(N-p-q-1)/2}|S_{22}|^{(N-q-1)/2}.\end{aligned}$$

Further setting $V-\Sigma_{12}\Sigma_{22}^{-1}B_2=\psi$ and then integrating (12) w.r.t. ψ by using (8), we obtain that

$$\begin{aligned}(13)\qquad & g(D_{11}, S_{22}, \hat{\delta}, B_2)\\ &=K|I+ZZ'B_2'S_{22}^{-1}B_2|^{-p/2}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{11}[D_{11}+(\hat{\delta}-\delta)\right.\\ &\quad\left.\cdot(B_2'S_{22}^{-1}B_2+(ZZ')^{-1})^{-1}(\hat{\delta}-\delta)']\right\}\end{aligned}$$

$$\cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_{22}^{-1}[S_{22}+(B_2-\beta_2)ZZ'(B_2-\beta_2)']\right\}$$
$$\cdot |D_{11}|^{(N-p-q-1)/2}|S_{22}|^{(N-q-1)/2} .$$

Further we set

$$(\hat{\delta}-\delta)(B_2'S_{22}^{-1}B_2+(ZZ')^{-1})^{-1}(\hat{\delta}-\delta)'=G , \qquad C\hat{\delta}'=V' , \tag{14}$$

where C is a known $g\times t$ matrix of rank $g\leqq p<t$, and integrate (13), over the region (14), by using (7). This integration gives us the joint density of the variates G, V, D_{11}, S_{22}, B_2. We integrate this joint density w.r.t. G, and we find that

$$\begin{aligned}(15)\qquad & g(D_{11}, V, B_2, S_{22}) \\ &=K|\xi|^{-p/2}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{11}[D_{11}+(V-\delta C')\xi^{-1}(V-\delta C')']\right\} \\ &\cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_{22}^{-1}[S_{22}+(B_2-\beta_2)ZZ'(B_2-\beta_2)']\right\} \\ &\cdot |D_{11}|^{(N-p-q-1)/2}|S_{22}|^{(N-q-1)/2}\end{aligned}$$

where

$$\xi=CB_2S_{22}^{-1}B_2C'+C(ZZ')^{-1}C' . \tag{16}$$

In (15) we set $W=D_{11}+(V-\delta C')\xi^{-1}(V-\delta C')'$, and we find from (15) that

$$\begin{aligned}(17)\qquad & g(W, V, B_2, S_{22}) \\ &=K|\xi|^{-(N-q-1)/2}\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{11}W-\frac{1}{2}\operatorname{tr}\Sigma_{22}^{-1}[S_{22}+(B_2-\beta_2)\right. \\ &\left.\cdot ZZ'(B_2-\beta_2)']\right\}|S_{22}|^{(N-q-1)/2}|W|^{(N-p-q-1)/2} \\ &\cdot |\xi-(V-\delta C')'W^{-1}(V-\delta C')|^{(N-p-q-1)/2} .\end{aligned}$$

Now we integrate (17) w.r.t. V, by using (7), over the region

$$(V-\delta C')'W^{-1}(V-\delta C')=G , \qquad LV=R , \tag{18}$$

where L is a specified $r\times p$ matrix of $r<p$, $g+r<p$. This integral gives the joint density of G, R, W, S_{22}, B_2, and on integrating this density w.r.t. G, we find that

$$\begin{aligned}(19)\qquad & g(W, R, B_2, S_{22}) \\ &=K|\xi|^{-(N-q-1)/2}|LWL'|^{-g/2} \\ &\cdot \exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma^{11}W-\frac{1}{2}\operatorname{tr}\Sigma_{22}^{-1}[S_{22}+(B_2-\beta_2)\right.\end{aligned}$$

$$\cdot\, ZZ'(B_2-\beta_2)']\Big\}\,|S_{22}|^{(N-q-1)/2}\,|W|^{(N-q-1)/2}$$

$$\cdot\, |\xi-(R-L\delta C')'(LWL')^{-1}(R-L\delta C')|^{(N-q-r-1)/2}\,.$$

Further, by using the identity

$$B'(A+BB')^{-1}B=(B'A^{-1}B)(I+B'A^{-1}B)^{-1}\,, \tag{20}$$

we observe that

$$\begin{aligned}(21)\qquad &\xi^{-1/2}(R-L\delta C')'(LWL')^{-1}(R-L\delta C')\xi^{-1/2}\\ &=\xi^{-1/2}(R-L\delta C')'(LD_{11}L'+(R-L\delta C')'\xi^{-1}(R-L\delta C'))^{-1}\\ &\quad\cdot (R-L\delta C')\xi^{-1/2}\\ &=[\xi^{-1/2}(R-L\delta C')'(LD_{11}L')^{-1}(R-L\delta C')\xi^{-1/2}]\\ &\quad\cdot [I+\xi^{-1/2}(R-L\delta C')'(LD_{11}L')^{-1}(R-L\delta C')\xi^{-1/2}]\,.\end{aligned}$$

Thus our criterion for testing that $L\delta C'$ has a specified value is to based on $M^*=\xi^{-1/2}(R-L\delta C')'(LD_{11}L')^{-1}(R-L\delta C')\xi^{-1/2}$. This criterion is

$$|M^*|=|(R-L\delta c')'(LD_{11}L')^{-1}(R-L\delta c')|/|\xi|\,. \tag{22}$$

The density of the matrix M^* is found by using (19), and we have that

$$g(M^*)=K|M^*|^{(r-g-1)/2}|I+M^*|^{-(N-q+g)/2}\,. \tag{23}$$

Obviously the (Wilks') likelihood ratio criterion $|M^*|$ is distributed as a product of g independent beta variates. Further, we note that

$$|M^*|\leqq \operatorname*{Max}_{C,L}|M^*|\leqq \theta_1\theta_2\cdots\theta_g\,, \tag{24}$$

where $\theta_1>\theta_2>\cdots>\theta_g>\cdots>\theta_r>\cdots>\theta_p$ are the roots of the matrix Q

$$Q=D_{11}^{-1}(\hat{\delta}-\delta)(B_2'S_{22}^{-1}B_2+(ZZ')^{-1})^{-1}(\hat{\delta}-\delta)'\,. \tag{25}$$

It thus follows that confidence bounds based on $|M^*|$ are narrower than those based on Roy's maximum root statistic $\theta_1\theta_2\cdots\theta_g$, whose distribution, as yet, is unknown. Further, the optimum properties of tests based on $|M^*|$ are known while not much is known of tests based on Roy's maximum root test. In case $g=1$ and $r=1$, then M^* reduces to a constant multiple of Student's t^2 statistic.

For testing $\delta=0$ against $\delta\neq 0$, we have the test statistic $|Q|$, the matrix Q has a second kind of multivariate beta density and this density may be termed as a generalization of the density of U.

ST. MARY'S UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA

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