DISTRIBUTIONS OF THE LARGEST LATENT ROOT OF THE MULTIVARIATE COMPLEX GAUSSIAN DISTRIBUTION[1]

T. SUGIYAMA[2]

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1. Summary

In this paper distributions of the largest latent root are studied for the following cases, based on the multivariate complex gaussian distribution, (i) that of the complex Wishart matrix; (ii) that of the complex Beta matrix, related with testing equality of two covariance matrices, with the MANOVA case, and with testing independence of two sets of variates; and (iii) that of the complex F matrix when two covariance matrices are not equal. Khatri [5] and [6] gave the result of the case (ii) in an analogous formula to Roy [8] for the real case, and also case (iii). In Sections 3 and 4 of this paper the results of the cases (ii) and (iii) are expressed by the hypergeometric functions with hermitian matrices as arguments. We also obtain the distributions of the smallest latent root. Hayakawa [3] gave the distribution of the largest latent root of the case (i). In Section 5 we obtain the approximation of the distribution of this case following the method in Sugiyama [11]. For the real case, Muirhead [7] gave an asymptotic distribution which is valid over some range.

2. Introduction

The discussion is based on the p-variate complex gaussian distribution studied by Wooding [12] and Goodman [2]

$$(1)\qquad f(z)=(\pi^p|\boldsymbol{\Sigma}|)^{-1}\exp\left(-(\bar{z}-\overline{E(z)})'\Sigma^{-1}(z-E(z))\right)$$

where $z=\boldsymbol{x}+i\boldsymbol{y}$ and $\boldsymbol{\Sigma}$ is $p\times p$ hermitian positive definite complex covariance matrix of which the jk elements σ_{jk} is given by

$$\sigma_{jk}=\begin{cases}\sigma_k^2 & \text{if } j=k\\ (\alpha_{jk}+i\beta_{jk})\sigma_j\sigma_k & \text{if } j\neq k\,.\end{cases}$$

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2) Now at the Kawasaki Medical University.

If the $p\times n$ matrix $\boldsymbol{Z}=(z_1, z_2, \cdots, z_n)$ is a sample of n complex valued vectors from the above distribution with $E(z)=\mathbf{0}$, then the sample Hermitian matrix $\boldsymbol{S}=n^{-1}\sum_{j=1}^{n} z_j\bar{z}_j'=n^{-1}\boldsymbol{Z}\bar{\boldsymbol{Z}}'$ is the maximum likelihood estimator for $\boldsymbol{\Sigma}$, and $\boldsymbol{A}=\boldsymbol{Z}\bar{\boldsymbol{Z}}'$ is distributed as

$$(2) \qquad (\tilde{\Gamma}_p(n)|\boldsymbol{\Sigma}|^n)^{-1}(\mathrm{etr}\ -\boldsymbol{\Sigma}^{-1}\boldsymbol{A})|\boldsymbol{A}|^{n-p}(d\boldsymbol{A})$$

where

$$\tilde{\Gamma}_p(n)=\pi^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(n-i+1)\ .$$

The above complex Wishart distribution $\tilde{W}(p, n, \boldsymbol{\Sigma})$ is derived by Goodman [2]. James ([4] and personal communication) develops the theory of the zonal polynomials based on hermitian matrices, and the hypergeometric functions of hermitian matrices. He also gives the results for the multivariate distribution problems derived from the complex Wishart distribution. The complex analogue for the distribution of the largest latent root discussed in Sugiyama [9] and [10] may be derived starting with James's results. In this paper the theory is developed along this direction in the cases (ii) and (iii). It seems very possible that the method of Constantine and Venables [1] can be used in the complex case also, and so leading to finite series of positive terms in these cases. Throughout the paper the same notation as James [4] is used for the complex multivariate gamma function

$$\tilde{\Gamma}_p(a)=\pi^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(a-i+1)\ ,$$

the complex multivariate hypergeometric coefficient

$$[a]_\kappa=\prod_{i=1}^{p}(a-i+1)_{k_i}$$

where $\kappa=(k_1, \cdots, k_p)$, $k_1\geqq\cdots\geqq k_p\geqq 0$, and $k_1+\cdots+k_p=k$, the zonal polynomials of an hermitian matrix $\tilde{C}_\kappa(A)$, and the hypergeometric functions

$${}_p\tilde{F}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; A)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_p]_\kappa\tilde{C}_\kappa(A)}{[b_1]_\kappa\cdots[b_q]_\kappa k!}$$

$${}_p\tilde{F}_q(a_1, \cdots, a_p, b_1, \cdots, b_q; \boldsymbol{A}, \boldsymbol{B})=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_p]_\kappa\tilde{C}_\kappa(\boldsymbol{A})\tilde{C}_\kappa(\boldsymbol{B})}{[b_1]_\kappa\cdots[b_q]_\kappa\tilde{C}_\kappa(I_p)k!}\ .$$

The inequality $A<B$ means that the hermitian matrix $B-A$ is positive definite. James (personal communication) gives simple recursive formula of $\tilde{C}_\kappa(A)$

$$\tilde{C}_{(k_1,\cdots,k_p)}(A)=a_{\hat{k}_1}\tilde{C}_{(\hat{k}_2,\cdots,\hat{k}_p)}(A)-a_{\hat{k}_2-1}\tilde{C}_{(\hat{k}_1+1,\hat{k}_3,\cdots,\hat{k}_p)}(A)$$
$$+a_{\hat{k}_3-2}\tilde{C}_{(\hat{k}_1+1,\hat{k}_2+1,\hat{k}_4,\cdots,\hat{k}_p)}(A)-\cdots$$

where a_i is the ith elementary symmetric function and $(\hat{k}_1, \hat{k}_2, \cdots, \hat{k}_p)$ is the conjugate partition to $\kappa=(k_1, k_2, \cdots, k_p)$. We know that $\tilde{C}_{(k)}(A)=a_k$. So from this we can calculate the zonal polynomials from the bottom upwards.

The following two equations are easily proved by the uniqueness of Laplace transforms, by the same way as the real case,

$$(3)\qquad \operatorname{etr}(-A)\,{}_1\tilde{F}_1(a; b; A)={}_1\tilde{F}_1(b-a; b; -A)\ ,$$

$$(4)\qquad {}_2\tilde{F}_1(a_1, a_2; b; A)=|I-A|^{-a_2}\,{}_2\tilde{F}_1(b-a_1, a_2; b; -A(I-A)^{-1})$$
$$=|I-A|^{b-a_1-a_2}\,{}_2\tilde{F}_1(b-a_1, b-a_2; b; A)\ .$$

Applying the reproductive property of zonal polynomial in James [4]

$$(5)\qquad \int_{\bar{A}'=A>0}\operatorname{etr}(-A)|A|^{a-p}(\tilde{C}_\kappa(AB)/\tilde{C}_\kappa(B))(dA)=\tilde{\Gamma}_p(a, \kappa)\ ,$$

we may obtain the following result of the beta integral

$$(6)\qquad \int_0^I |A|^{t-p}|I-A|^{u-p}\tilde{C}_\kappa(A)(dA)=[\tilde{\Gamma}_p(t, \kappa)\tilde{\Gamma}_p(u)/\tilde{\Gamma}_p(t+u, \kappa)]\tilde{C}_\kappa(I)\ ,$$

where $\tilde{\Gamma}_p(a, \kappa)=(a)_\kappa\tilde{\Gamma}_p(a)$. To show this

$$\tilde{\Gamma}_p(t, \kappa)\tilde{\Gamma}_p(u)$$
$$=\int_{\bar{T}'=T>0}\int_{\bar{S}'=S>0}\operatorname{etr}(-(S+T))|S|^{t-p}|T|^{u-p}(\tilde{C}_\kappa(S)/\tilde{C}_\kappa(I))(dS)(dT)$$
$$(\text{let } R=S+T)$$
$$=\int_{\bar{R}'=R>0}\int_{\bar{S}'=S>0}^{R}\operatorname{etr}(-R)|S|^{t-p}|R-S|^{u-p}(\tilde{C}_\kappa(S)/\tilde{C}_\kappa(I))(dS)(dR)$$
$$(\text{let } S=RS)$$
$$=\int_0^I\int_{\bar{R}'=R>0}[\operatorname{etr}(-R)|R|^{t+u-p}(\tilde{C}_\kappa(RS)/\tilde{C}_\kappa(S))]$$
$$\times|S|^{t-p}|I-S|^{u-p}(\tilde{C}_\kappa(S)/\tilde{C}_\kappa(I))(dS)(dR)$$
$$=\tilde{\Gamma}_p(t+u, \kappa)\int_0^I |S|^{t-p}|I-S|^{u-p}(\tilde{C}_\kappa(S)/\tilde{C}_\kappa(I))(dS)\ .$$

From the above result we have

$$(7)\qquad \int_0^I |A|^{t-p}|I-A|^{u-p}\tilde{C}_\kappa(AB)(dA)=(\tilde{\Gamma}_p(t, \kappa)\tilde{\Gamma}_p(u)/\tilde{\Gamma}_p(t+u, \kappa))\tilde{C}_\kappa(B)\ ,$$

where $u>p-1$, and $t+k_p>p-1$. Let $u=p$. Then with A replaced by xA in (7) we obtain the following lemma which is used to find the distributions of the largest latent root in Sections 3 and 4.

LEMMA.

$$\int_0^{xI} |A|^{t-p}\tilde{C}_\kappa(AB)(dA) = (\tilde{\Gamma}_p(t)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(t+p))([t]_\kappa/[t+p]_\kappa)\tilde{C}_\kappa(B)x^{pt+k} . \tag{8}$$

3. Distributions of the largest latent root and the smallest latent root of the complex Beta matrix

Let A_1 and A_2 be two independent matrices having the complex Wishart distribution $\tilde{W}(p, n_1, \Sigma)$ and $\tilde{W}(p, n_2, \Sigma)$ respectively, where n_1, $n_2 \geqq p$. Then the cdf of the largest latent root of the equation $|A_1-(A_1+A_2)\lambda|=0$ is written as

$$P(B<xI)=C\int_0^{xI} |B|^{n_1-p}|I-B|^{n_2-p}(dB) \tag{9}$$

where $B=(A_1+A_2)^{-1/2}A_1(A_1+A_2)^{-1/2}$, and $C=\tilde{\Gamma}_p(n_1+n_2)/[\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)]$. Expanding $|I-B|^{n_2-p}$ in a series of the zonal polynomials, and applying the lemma, we obtain for the cdf of the largest latent root

$$\begin{aligned} P(B<xI) &= C\sum_{k=0}^{\infty}\sum_{\kappa}([-n_2+p]_\kappa/k!)\int_0^{xI} |B|^{n_1-p}C_\kappa(B)(dB) \\ &= (\tilde{\Gamma}_p(n_1+n_2)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(n_1+p))x^{n_1p} \\ &\quad \times {}_2\tilde{F}_1(-n_2+p, n_1; n_1+p; xI) , \end{aligned} \tag{10}$$

where

$$\begin{aligned} &(\tilde{\Gamma}_p(n_1+n_2)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(n_1+p)) \\ &\quad = \prod_{i=1}^{p}(\Gamma(n_1+n_2-p+i)\Gamma(i)/\Gamma(n_2-p+i)\Gamma(n_1+i)) . \end{aligned}$$

This may be rewritten by (4) as

$$\begin{aligned} P(B<xI) &= (\tilde{\Gamma}_p(n_1+n_2)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(n_1+p))x^{n_1p}(1-x)^{n_2p} \\ &\quad \times {}_2\tilde{F}_1(n_1+n_2, p; n_1+p; xI) . \end{aligned} \tag{11}$$

The cdf of the smallest latent root of the equation $|A_1-(A_1+A_2)\lambda|=0$ is given by the same method as follows

$$\begin{aligned} P(B>xI) &= P(I-B<(1-x)I) \\ &= (\tilde{\Gamma}_p(n_1+n_2)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2+p))(1-x)^{n_2p} \\ &\quad \times {}_2\tilde{F}_1(-n_1+p, n_2; n_2+p; (1-x)I) . \end{aligned} \tag{12}$$

The hypergeometric functions in the (10) and (12) are terminated in the finite terms. The percentile points may be found by applying the Newton-Raphson method or other known procedures.

4. Distributions of the largest latent root and the smallest latent root with unequal covariance matrices $\boldsymbol{\Sigma}_1 \neq \boldsymbol{\Sigma}_2$

Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be independently distributed as the complex Wishart distribution $\tilde{W}(p, n_1, \boldsymbol{\Sigma}_1)$ and $\tilde{W}(p, n_2, \boldsymbol{\Sigma}_2)$ as (2), respectively. Then the distribution of $\boldsymbol{F}=\boldsymbol{A}_2^{-1/2}\boldsymbol{A}_1\boldsymbol{A}_2^{-1/2}$ is given by James [4] as follows

$$(13)\quad f(\boldsymbol{F})=|\boldsymbol{\Sigma}|^{-n_1}(\tilde{\Gamma}_p(n_1+n_2)/\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2))\,{}_1\tilde{F}_0(n_1+n_2;\,-\boldsymbol{\Sigma}^{-1},\,\boldsymbol{F})\,|\boldsymbol{F}|^{n_1-p}(d\boldsymbol{F})$$

where $\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$. Since

$$\begin{aligned}&{}_1F_0(n_1+n_2;\,-\boldsymbol{\Sigma}^{-1},\,\boldsymbol{F})\\ &\quad=\sum_{k=0}^{\infty}\sum_{\kappa}([n_1+n_2]_\kappa/k!)\int_{U(p)}\tilde{C}_\kappa(-\boldsymbol{\Sigma}^{-1}\boldsymbol{U}\boldsymbol{F}\bar{\boldsymbol{U}}')(d\boldsymbol{U})\end{aligned}$$

the cdf of the largest latent root is given by

$$(14)\quad \begin{aligned}P(\boldsymbol{F}<y\boldsymbol{I})&=|\boldsymbol{\Sigma}|^{-n_1}(\tilde{\Gamma}_p(n_1+n_2)/\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2))\sum_{k=0}^{\infty}\sum_{\kappa}([n_1+n_2]_\kappa/k!)\\ &\quad\times\int_{U(p)}\int_0^{y\boldsymbol{I}}|\boldsymbol{F}|^{n_1-p}\tilde{C}_\kappa(-\boldsymbol{\Sigma}^{-1}\boldsymbol{U}\boldsymbol{F}\bar{\boldsymbol{U}}')(d\boldsymbol{U})(d\boldsymbol{F})\end{aligned}$$

where $(d\boldsymbol{U})$ is the normalized invariant measure on the unitary group $U(p)$. By the lemma we have

$$(15)\quad \begin{aligned}P(\boldsymbol{F}<y\boldsymbol{I})&=|\boldsymbol{\Sigma}|^{-n_1}(\tilde{\Gamma}_p(n_1+n_2)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(n_1+p)\tilde{\Gamma}_p(n_2))y^{pn_1}\\ &\quad\times\sum_{k=0}^{\infty}\sum_{\kappa}([n_1+n_2]_\kappa[n_1]_\kappa/[n_1+p]_\kappa k!)\\ &\quad\times\int_{U(p)}C_\kappa(-y\bar{\boldsymbol{U}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{U})(d\boldsymbol{U})\,.\end{aligned}$$

Let the latent roots of $\Sigma=\Sigma_1\Sigma_2^{-1}$ be $\omega_1,\cdots,\omega_p$, and $\Omega=\mathrm{diag}\,(\omega_i)$. Then we obtain the cdf of the largest latent root as

$$(16)\quad \begin{aligned}P(\boldsymbol{F}<y\boldsymbol{I})&=(\tilde{\Gamma}_p(n_1+n_2)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(n_1+p)\tilde{\Gamma}_p(n_2))\\ &\quad\times|y\boldsymbol{\Omega}^{-1}|^{n_1}\,{}_2\tilde{F}_1(n_1+n_2,\,n_1;\,n_1+p;\,-y\boldsymbol{\Omega}^{-1})\,.\end{aligned}$$

Applying the formula (4) we may rewrite the cdf as

$$(17)\quad \begin{aligned}P(\boldsymbol{F}<y\boldsymbol{I})&=(\tilde{\Gamma}_p(n_1+n_2)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(n_1+p)\tilde{\Gamma}_p(n_2))\\ &\quad\times|y\boldsymbol{\Omega}^{-1}(\boldsymbol{I}+y\boldsymbol{\Omega}^{-1})^{-1}|^{n_1}\,{}_2\tilde{F}_1(p-n_2,\,n_1;\,n_1+p;\,y\boldsymbol{\Omega}^{-1}\\ &\quad\times(\boldsymbol{I}+y\boldsymbol{\Omega}^{-1})^{-1})\,.\end{aligned}$$

The distribution of the smallest latent root is given by

$$(18)\quad \begin{aligned}P(\boldsymbol{F}>y\boldsymbol{I})&=[\tilde{\Gamma}_p(n_1+n_2)\tilde{\Gamma}_p(p)/\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2+p)]\\ &\quad\times|y^{-1}\boldsymbol{\Omega}^{-1}(\boldsymbol{I}+y^{-1}\boldsymbol{\Omega}^{-1})^{-1}|^{n_2}\,{}_2\tilde{F}_1(p-n_1,\,n_2;\,n_2+p;\,y^{-1}\boldsymbol{\Omega}^{-1}\\ &\quad\times(\boldsymbol{I}+y^{-1}\boldsymbol{\Omega}^{-1})^{-1})\,.\end{aligned}$$

Let $\boldsymbol{\Omega}=\boldsymbol{I}$. Then we have the formulae (10) and (12) with $y(1+y)^{-1}$ in (17) and (18) replaced by x.

5. Approximation for the distribution of the largest latent root of the complex Wishart matrix, and the non-central complex Wishart matrix with known covariance

Let $\boldsymbol{A}=n\boldsymbol{S}$ in (2). Then the cdf of the largest latent root of the sample hermitian covariance matrix $\boldsymbol{S}$ is given by

$$(19)\qquad P(\boldsymbol{S}<z\boldsymbol{I})=P(\boldsymbol{A}<nz\boldsymbol{I})=(\tilde{\Gamma}_p(n))^{-1}\int_{0<\boldsymbol{B}<nz}\boldsymbol{\Sigma}^{-1}\operatorname{etr}(-\boldsymbol{B})|\boldsymbol{B}|^{n-p}(d\boldsymbol{B})\,.$$

Without loss of generality we may assume $\boldsymbol{\Sigma}^{-1}=\operatorname{diag}(\sigma_1^{-1},\cdots,\sigma_p^{-1})$, where $\sigma_1,\cdots,\sigma_p$ are the latent roots of $\boldsymbol{\Sigma}$. Let $\boldsymbol{T}$ be an upper triangular matrix with diagonal elements t_{ii} and off-diagonal elements $t_{ij}=t_{ijR}+it_{ijI}$. The Jacobian for the transformation $\boldsymbol{B}=\bar{\boldsymbol{T}}'\boldsymbol{T}$ is $2^p t_{11}^{2p-1}t_{22}^{2p-3}\cdots t_{pp}$ given by Goodmann [2]. Then (19) is rewritten as

$$\begin{aligned}(20)\qquad P(\boldsymbol{S}<z\boldsymbol{I})&=(2^p/\Gamma_p(n))\int_{0<\bar{T}'T<nz\Sigma^{-1}}\exp\Big(-\sum_{i<j}(t_{ijR}^2+t_{ijI}^2)\Big)\\&\quad\times\exp\Big(-\sum_{i=1}^p t_{ii}^2\Big)\prod_{i=1}^p t_{ii}^{2n-(2i-1)}\prod_{i<j}dt_{ijR}dt_{ijI}\prod_{i=1}^p dt_{ii}\\&\leqq(2^p/\tilde{\Gamma}_p(n))\int_{-\infty}^{\infty}\exp\Big(-\sum_{i<j}(t_{ijR}^2+t_{ijI}^2)\Big)\prod_{i<j}dt_{ijR}dt_{ijI}\\&\quad\times\int_{\{t_{ii}^2\}}\exp\Big(-\sum_{i=1}^p t_{ii}^2\Big)\prod_{i=1}^p t_{ii}^{2n-(2i-1)}\prod_{i=1}^p dt_{ii}\,.\end{aligned}$$

For the ranges of t_{ii}, $i=1,\cdots,p$, we give the following necessary conditions, based on $0<\boldsymbol{B}<nz\boldsymbol{\Sigma}^{-1}$,

$$0<t_{ii}^2<nz\sigma_i^{-1}\,,\qquad i=1,\cdots,p\,.$$

And further, let $s_i=t_{ii}^2$, $i=1,\cdots,p$. Then we obtain

$$(21)\qquad P(\boldsymbol{A}<z\boldsymbol{I})\leqq\prod_{i=1}^p\Big(\int_{0<s_i<nz\sigma_i^{-1}}\exp(-s_i)s_i^{n-i}ds_i/\Gamma(n-i+1)\Big)\,.$$

So for large n we have the approximation of the cdf of the largest latent root as

$$P(\boldsymbol{A}<z\boldsymbol{I})\doteqdot\prod_{i=1}^p P(\chi^2_{2(n-i+1)}\leqq 2nz\sigma_i^{-1})$$

where χ_j^2 denotes a random variable having a χ^2-distribution with j degrees of freedom.

Let the matrix $\boldsymbol{A}=n\boldsymbol{S}$ be distributed as a non-central Wishart (see James [4]). We may prove that

$$(22) \qquad \lim_{n\to\infty} {}_0\tilde{F}_1(n; \boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A}) = \exp \operatorname{tr} (\boldsymbol{\Sigma}^{-1}\boldsymbol{M}\bar{\boldsymbol{M}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{S}) ,$$

and the right-hand side exp tr ( ) has larger value than the ${}_0\tilde{F}_1$ for sufficiently large n. Suppose that $\boldsymbol{\Sigma}$ is known. Then the distribution of $\boldsymbol{W}=\boldsymbol{\Sigma}^{-1}\boldsymbol{S}$ is approximated by

$$(23) \qquad f(\boldsymbol{W}) = \exp \operatorname{tr} (-\boldsymbol{\Omega}) \exp \operatorname{tr} (\boldsymbol{\Omega}\boldsymbol{W}) \times [n^{pn}/\tilde{\Gamma}_p(n)] \exp \operatorname{tr} (-n\boldsymbol{W}) |\boldsymbol{W}|^{(n-p)} ,$$

where $\boldsymbol{\Omega}=\boldsymbol{M}\bar{\boldsymbol{M}}'\boldsymbol{\Sigma}^{-1}$. Applying the same method as before, we may obtain the following approximation for the cdf of the largest latent root of the matrix $\boldsymbol{W}$

$$(24) \qquad P(\boldsymbol{W}<z\boldsymbol{I}) = \exp (-\operatorname{tr} \boldsymbol{\Omega}) |\boldsymbol{I}-n^{-1}\boldsymbol{\Omega}|^{-n} \prod_{i=1}^{p} P(\chi^2_{2(n-i+1)} \leqq 2nz(1-n^{-1}\omega_i)) ,$$

where $\omega_1, \cdots, \omega_p$ are the latent roots of non-centrality parameter $\boldsymbol{\Omega}$. This approximate cdf (24) is greater than the exact cdf for sufficiently large n.

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