NON-NULL DISTRIBUTIONS OF THE LIKELIHOOD RATIO CRITERIA FOR INDEPENDENCE AND EQUALITY OF MEAN VECTORS AND COVARIANCE MATRICES

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1. Introduction and summary

The purpose of this paper is to give the asymptotic expansions of the likelihood ratio (=L.R.) criteria for testing independence of q-sets of variates and for testing that several distributions are identical. The test of independence is investigated by many authors. The unbiasedness of this test is shown by Narain [7]. Under the null hypothesis, Box [2] gave the asymptotic expansion of this test criterion in a more general formulation, the first term of which is χ^2-distribution. Recently Consul [4] obtained the exact distributions in some special cases by considering the inverse Mellin transformation. On the other hand, under the alternative hypothesis Siotani and Hayakawa [8] gave the limiting distribution of this criterion by using the differential operator due to Welch [11], which is a normal. Especially, with the help of the hypergeometric function of matrix argument due to Constantine [3], Sugiura and Fujikoshi [10] derived the asymptotic expansion of the criterion for testing independence of two sets. Also Sugiura [9] gave the asymptotic expansion under local alternatives. However, the criterion for testing the equality of means and covariance matrices is seldom researched. Only Fujikoshi [5] gave the asymptotic expansion under the assumption that covariance matrices are equal, the first term of which is non-central χ^2. The author [6] gave the asymptotic expansion in other problem by deriving a natural extension of the normal approximation to the χ^2 distribution to multivariate case. We shall give the asymptotic expansions by applying this method to these two test criteria, the first terms of which are normal distributions.

2. Some useful formulas

In this part we shall state some fundamental lemmas used in the latter sections.

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LEMMA 2.1. *Let S $(p\times p)$ be distributed according to the Wishart distribution $W(\Sigma, n)$. Then for any $p\times p$ matrix B, we have*

$$E[\operatorname{tr} BS]=n \operatorname{tr} B\Sigma \tag{2.1}$$

and

$$E[\operatorname{tr} S^2]=n \operatorname{tr} \Sigma^2+n(\operatorname{tr} \Sigma)^2+n^2 \operatorname{tr} \Sigma^2 . \tag{2.2}$$

The above proof is given by Nagao [6].

LEMMA 2.2. *Let a $p(p+1)/2\times 1$ vector $(y_{11}, y_{22},\cdots, y_{pp}, y_{12}, y_{13},\cdots, y_{p-1,p})'$ be distributed according to the normal distribution with mean vector $(\omega_{11}, \omega_{22},\cdots, \omega_{pp}, \omega_{12}, \omega_{13},\cdots, \omega_{p-1,p})'$ and covariance*

$$\begin{pmatrix} \overbrace{1 \quad \ddots \quad 1}^{p} & \overbrace{\qquad 0 \qquad}^{p(p-1)/2} \\ 0 & 1/2 \quad 1/2 \quad \ddots \quad 1/2 \end{pmatrix} . \tag{2.3}$$

Define symmetric matrices Y and Ω as

$$Y=\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1p} \\ y_{21} & y_{22} & \cdots & y_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ y_{p1} & y_{p2} & \cdots & y_{pp} \end{pmatrix} \quad \text{and} \quad \Omega=\begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1p} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{p1} & \omega_{p2} & \cdots & \omega_{pp} \end{pmatrix} . \tag{2.4}$$

Then for any $p\times p$ matrices A, B and symmetric C and diagonal matrices Λ, K, the following formulas hold.

$$E[\operatorname{tr} AY]=\operatorname{tr} A\Omega , \tag{2.5}$$

$$E[\operatorname{tr} AYBY]=\operatorname{tr} A\Omega B\Omega+\frac{1}{2} \operatorname{tr} AB' +\frac{1}{2} \operatorname{tr} A \operatorname{tr} B , \tag{2.6}$$

$$E[\operatorname{tr} AY \operatorname{tr} BY]=\operatorname{tr} A\Omega \operatorname{tr} B\Omega+\frac{1}{2} \operatorname{tr} AB+\frac{1}{2} \operatorname{tr} AB' , \tag{2.7}$$

$$\begin{aligned} E[\operatorname{tr} \Lambda YKY \operatorname{tr} AY]&=\operatorname{tr} (\Lambda\Omega K\Omega) \operatorname{tr} A\Omega +\frac{1}{2} \operatorname{tr} \Lambda \operatorname{tr} K \operatorname{tr} A\Omega \\ &\quad +\frac{1}{2} \operatorname{tr} \Lambda K \operatorname{tr} A\Omega+\operatorname{tr} \Lambda AK\Omega+\operatorname{tr} \Lambda A'K\Omega , \end{aligned} \tag{2.8}$$

$$E[\operatorname{tr} (\Lambda Y)^3]=\operatorname{tr} (\Lambda\Omega)^3 +\frac{3}{2} \operatorname{tr} \Lambda^3\Omega+\frac{3}{2} \operatorname{tr} \Lambda \operatorname{tr} \Lambda^2\Omega , \tag{2.9}$$

$$\begin{aligned} &E[\operatorname{tr} (\Lambda Y)^2 \operatorname{tr} (CY)^2] \\ &\quad = \operatorname{tr} (\Lambda\Omega)^2 \operatorname{tr} (C\Omega)^2+\frac{1}{2}\{\operatorname{tr} \Lambda^2+(\operatorname{tr} \Lambda)^2\} \operatorname{tr} (C\Omega)^2+4 \operatorname{tr} (\Lambda\Omega\Lambda C\Omega C) \end{aligned} \tag{2.10}$$

$$+\frac{1}{2}\{\operatorname{tr} C^2+(\operatorname{tr} C)^2\} \operatorname{tr} (\Lambda\Omega)^2+(\operatorname{tr} \Lambda C)^2+\operatorname{tr} (\Lambda C)^2$$

$$+\frac{1}{4}\{\operatorname{tr} \Lambda^2+(\operatorname{tr} \Lambda)^2\}\{\operatorname{tr} C^2+(\operatorname{tr} C)^2\} .$$

These formulas were used except for (2.10) in deriving the asymptotic expansion of multivariate Bartlett's test in Nagao [6].

LEMMA 2.3. *Let $p\times 1$ random vector V be distributed to $N(b, I)$. For any symmetric matrix A and $p\times 1$ vectors α, β and γ, the following formulas hold.*

(2.11) $E[\operatorname{tr} AVV']=\operatorname{tr} Abb'+\operatorname{tr} A$,

(2.12) $E[\alpha'V\beta'V]=\alpha'b\beta'b+\alpha'\beta$,

(2.13) $E[\operatorname{tr} AV\alpha'VV']=\operatorname{tr} Ab\alpha'bb'+2\alpha'Ab+\alpha'b \operatorname{tr} A$,

(2.14) $E[\alpha'V\beta'V\gamma'V]=\alpha'b\beta'b\gamma'b+\alpha'\beta\gamma'b+\alpha'\gamma\beta'b+\alpha'b\beta'\gamma$,

(2.15) $E[\operatorname{tr} AVV' \operatorname{tr} V\alpha']=\operatorname{tr} Abb' \operatorname{tr} b\alpha'+\operatorname{tr} A \operatorname{tr} \alpha b'+2 \operatorname{tr} Ab\alpha'$,

(2.16) $E[\{\operatorname{tr} AVV'\}^2]=(\operatorname{tr} Abb')^2+2 \operatorname{tr} A \operatorname{tr}Abb'+4 \operatorname{tr} A^2bb'$
$+(\operatorname{tr} A)^2+2 \operatorname{tr} A^2$,

(2.17) $E[(\alpha'V)^4]=(\alpha'b)^4+6\alpha'\alpha(\alpha'b)^2+3(\alpha'\alpha)^2$,

(2.18) $E[(\alpha'V)^2 \operatorname{tr} AVV']$
$=(\alpha'b)^2 \operatorname{tr} Abb'+\alpha'\alpha \operatorname{tr} Abb'+2\alpha'b \operatorname{tr} A\alpha b'+(\alpha'b)^2 \operatorname{tr} A$
$+2\alpha'b \operatorname{tr} Ab\alpha'+\alpha'\alpha \operatorname{tr} A+2 \operatorname{tr} A\alpha\alpha'$.

The proof of these formulas is easy.

LEMMA 2.4. *Let $f_n(t)$ be the characteristic function of a statistic λ_n. For $|t|<\sqrt{n}C$ $(C>0)$, if this function $f_n(t)$ is given by*

$$f_n(t)=\exp\left[\frac{(it)^2}{2}\right]\left[1+\frac{1}{\sqrt{n}}\{a_1it+a_3(it)^3\} +\frac{1}{n}\{b_2(it)^2+b_4(it)^4+b_6(it)^6\}\right]+O(n^{-3/2}) , \tag{2.19}$$

then the asymptotic expansion of the distribution λ_n is obtained by

$$P(\lambda_n\leqq x)=\Phi(x)-\frac{1}{\sqrt{n}}\{a_1\Phi^{(1)}(x)+a_3\Phi^{(3)}(x)\} +\frac{1}{n}\{b_2\Phi^{(2)}(x)+b_4\Phi^{(4)}(x)+b_6\Phi^{(6)}(x)\}+O(n^{-3/2}) , \tag{2.20}$$

where $\Phi^{(j)}(x)$ means the jth derivative of the standard normal distribution *function* $\Phi(x)$.

PROOF. The inversion formula of the first term in (2.19) is given by

$$(2.21)\quad \frac{1}{2\pi}\int_{|t|<\sqrt{n}C} e^{-itx}\exp\left[\frac{(it)^2}{2}\right]dt$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\exp\left[\frac{(it)^2}{2}\right]dt-\frac{1}{2\pi}\int_{|t|\geq\sqrt{n}C} e^{-itx}\exp\left[\frac{(it)^2}{2}\right]dt\,.$$

Then we must evaluate the integration $(2\pi)^{-1}\int_{|t|\geq\sqrt{n}C} e^{-itx}\exp\left[\frac{(it)^2}{2}\right]dt.$

$$(2.22)\quad \left|\frac{1}{2\pi}\int_{|t|\geq\sqrt{n}C} e^{-itx}\exp\left[-\frac{t^2}{2}\right]dt\right|\leq\frac{1}{\pi}\int_{\sqrt{n}C}^{\infty}\exp\left[-\frac{t^2}{2}\right]dt$$
$$\leq\frac{1}{\pi\sqrt{n}C}\exp\left[-\frac{nC^2}{2}\right].$$

Therefore the order of the function $n^{-1/2}\exp[-nC^2/2]$ is $O(n^{-l})$ for sufficient large l. The same results of the other terms can be shown by the analogous argument.

3. Non-null distribution of the L.R. criterion for independence

Let $p\times 1$ vector X, distributed according to $N(\mu, \Sigma)$, be partitioned into q sub-vectors with components $p_1, p_2, \cdots, p_q$, respectively. The vector of mean μ and covariance matrix Σ are also partitioned similarly; i.e.,

$$(3.1)\quad X=\begin{pmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(q)} \end{pmatrix},\qquad \mu=\begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \\ \vdots \\ \mu^{(q)} \end{pmatrix}$$

and

$$(3.2)\quad \Sigma=\begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1q} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ \Sigma_{q1} & \Sigma_{q2} & \cdots & \Sigma_{qq} \end{pmatrix}.$$

Given a sample $X_1, X_2\cdots, X_N$ of N observations on X, we wish to test the hypothesis $H:\Sigma_{ij}=0$ $(i\neq j)$ against the alternatives $K:\Sigma_{ij}\neq 0$ for some i, j $(i\neq j)$. Wilks [12] has defined the L.R. criterion V by

$$(3.3)\quad V=\frac{|S|^{N/2}}{\prod_{\alpha=1}^{q}|S_{\alpha\alpha}|^{N/2}},$$

where S is defined by

$$S=\sum_{\alpha=1}^{N}(X_\alpha-\bar{X})(X_\alpha-\bar{X})' , \tag{3.4}$$

and is partitioned in the same manner as Σ in (3.2). The corresponding matrix $S_{\alpha\alpha}$ is defined and partitioned similarly. Putting $m=\rho N$, we can express (3.3) as follows.

$$-2\rho \log V=m\sum_{\alpha=1}^{q}\log|S_{\alpha\alpha}|-m\log|S| , \tag{3.5}$$

where the correction factor ρ is given by (see Anderson [1])

$$\rho=1-\frac{2\left(p^3-\sum_{\alpha=1}^{q}p_\alpha^3\right)+9\left(p^2-\sum_{\alpha=1}^{q}p_\alpha^2\right)}{6N\left(p^2-\sum_{\alpha=1}^{q}p_\alpha^2\right)} . \tag{3.6}$$

Now we define the statistic $Y=(y_{\alpha\beta})_{\alpha,\beta=1,\cdots,p}$ as follows:

$$Y=\frac{\Sigma^{-1/2}S\Sigma^{-1/2}-mI}{\sqrt{2m}} . \tag{3.7}$$

Then the limiting distribution of this statistic can be easily seen to be $p(p+1)/2$ dimensional normal distribution with mean zero matrix and covariance (2.3). More precisely, the limiting distribution of a variable $(y_{11}, y_{22},\cdots, y_{pp}, y_{12},\cdots, y_{p-1,p})$ is given by

$$C\cdot \mathrm{etr}\left[-\frac{1}{2}Y^2\right] , \tag{3.8}$$

where $C=(2\pi)^{-p(p+1)/4}\cdot 2^{p(p-1)/4}$. Hence, using the statistic Y defined by (3.7), we can express the statistic $-2\rho\log V$ as follows.

$$\begin{aligned}-2\rho \log V=&\ m\left(\sum_{\alpha=1}^{q}\log|\Sigma_{\alpha\alpha}|-\log|\Sigma|\right)+\sqrt{m}q_0(Y)+q_1(Y)\\ &+m^{-1/2}q_2(Y)+O_p(m^{-1}) ,\end{aligned} \tag{3.9}$$

where the coefficients $q_0(Y)$, $q_1(Y)$ and $q_2(Y)$ are given by

$$\begin{aligned} q_0(Y)&=\sqrt{2}\left\{\sum_{\alpha=1}^{q}\mathrm{tr}\left(\sum_{\beta,\gamma}\Phi_{\alpha\beta}Y_{\beta\gamma}\Phi'_{\alpha\gamma}\right)-\mathrm{tr}\, Y\right\} ,\\ q_1(Y)&=\mathrm{tr}\, Y^2-\sum_{\alpha=1}^{q}\mathrm{tr}\left(\sum_{\beta,\gamma}\Phi_{\alpha\beta}Y_{\beta\gamma}\Phi'_{\alpha\gamma}\right)^2 ,\\ q_2(Y)&=\frac{2\sqrt{2}}{3}\left\{\sum_{\alpha=1}^{q}\mathrm{tr}\left(\sum_{\beta,\gamma}\Phi_{\alpha\beta}Y_{\beta\gamma}\Phi'_{\alpha\gamma}\right)^3-\mathrm{tr}\, Y^3\right\} .\end{aligned} \tag{3.10}$$

The above symbol $\sum_{\beta,\gamma}$ means the summation $\sum_{\beta=1}^{q}\sum_{\gamma=1}^{q}$ and $Y_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ are submatrices of Y and $\Phi=\Sigma_D^{-1/2}\Sigma^{1/2}$, respectively, partitioned in the same manner as Σ in (3.2), where

$$\Sigma_D=\begin{pmatrix} \Sigma_{11} & & & 0 \\ & \Sigma_{22} & & \\ & & \ddots & \\ 0 & & & \Sigma_{qq} \end{pmatrix}. \tag{3.11}$$

Now putting $V'=-2\rho\log V-m\left\{\sum_{\alpha=1}^{q}\log|\Sigma_{\alpha\alpha}|-\log|\Sigma|\right\}$ in (3.9), we can easily see that $V'/\sqrt{m}-\sqrt{2}\operatorname{tr}(\Phi'\Phi-I)Y=O_p(m^{-1/2})$. Hence the statistic $V'/\sqrt{m}$ converges in law to the normal distribution with mean zero and variance $\tau_V^2=2\operatorname{tr}(\Sigma\Sigma_D^{-1}-I)^2$, which was shown by Siotani and Hayakawa [8]. Further, the characteristic function of $V'/\sqrt{m}\tau_V$ $(\tau_V>0)$ can be expressed as

$$\begin{aligned} C_V(t)=E\Big[\exp(itq_0(Y)/\tau_V)\Big\{1&+m^{-1/2}itq_1(Y)/\tau_V \\ &+m^{-1}\Big[itq_2(Y)/\tau_V+\frac{1}{2}(it)^2q_1^2(Y)/\tau_V^2\Big]\Big\}\Big]+O(m^{-3/2}). \end{aligned} \tag{3.12}$$

Expressing $q_0(Y)$ and $q_1(Y)$ in (3.10) in terms of S, we have

$$q_0(Y)=\frac{1}{\sqrt{m}}\operatorname{tr}(\Sigma_D^{-1}-\Sigma^{-1})S \tag{3.13}$$

and

$$\begin{aligned} q_1(Y)=\frac{1}{2m}\Big\{\operatorname{tr}(\Sigma^{-1}S)^2&-\sum_{\alpha=1}^{q}\operatorname{tr}(\Sigma_{\alpha\alpha}^{-1}S_{\alpha\alpha})^2-2m\operatorname{tr}\Sigma^{-1}S \\ &+2m\sum_{\alpha=1}^{q}\operatorname{tr}(\Sigma_{\alpha\alpha}^{-1}S_{\alpha\alpha})\Big\}. \end{aligned} \tag{3.14}$$

Since the statistic S distributes according to Wishart distribution $W(\Sigma, n=N-1)$, we have

$$\begin{aligned} &E[\exp(itq_0(Y)/\tau_V)] \\ &=C_{p,n}\int|S|^{(n-p-1)/2}|\Sigma|^{-n/2}\operatorname{etr}\Big[-\frac{1}{2}\Big\{\Sigma^{-1}-\frac{2it}{\sqrt{m}\tau_V}(\Sigma_D^{-1}-\Sigma^{-1})\Big\}S\Big]dS \\ &=\Big|I-\frac{2it}{\sqrt{m}\tau_V}(\Sigma^{1/2}\Sigma_D^{-1}\Sigma^{1/2}-I)\Big|^{-m/2-\Delta}, \end{aligned} \tag{3.15}$$

where $2\Delta=N-m-1$. Applying the asymptotic formula $\log|I-A/\sqrt{m}|=-\sum_{\alpha=1}^{l}m^{-\alpha/2}\operatorname{tr}A^\alpha/\alpha+O(m^{-(l+1)/2})$, we can get the following asymptotic formula (3.16) for $|t|<\sqrt{m}C$ with $C=\tau_V/2\min_{j=1,\cdots,p}|ch_j(\Sigma^{1/2}\Sigma_D^{-1}\Sigma^{1/2}-I)|^{-1}$, where $ch_j(B)$ means the characteristic root of a matrix B.

(3.16) $E[\exp(itq_0(Y)/\tau_V)]$

$$=\exp\left[\frac{(it)^2}{2}\right]\left\{1+\frac{4(it)^3}{3\sqrt{m}\tau_V^3}\operatorname{tr}(\Sigma\Sigma_D^{-1}-I)^3+\frac{1}{m}\left[\Delta\tau_V^2(it/\tau_V)^2\right.\right.$$
$$\left.\left.+\frac{2(it)^4}{\tau_V^4}\operatorname{tr}(\Sigma\Sigma_D^{-1}-I)^4+\frac{8(it)^6}{9\tau_V^6}\{\operatorname{tr}(\Sigma\Sigma_D^{-1}-I)^3\}^2\right]\right\}+O(m^{-3/2})\,.$$

Moreover, applying Lemma 2.1 to $E[q_1(Y)\exp(itq_0(Y)/\tau_V)]$, we have

(3.17) $E[q_1(Y)\exp(itq_0(Y)/\tau_V)]$

$$=\exp\left[\frac{(it)^2}{2}\right]\left\{\frac{1}{2}\left(p^2-\sum_{\alpha=1}^q p_\alpha^2\right)+\left(\frac{it}{\tau_V}\right)^2\left[3\tau_V^2+2p\right.\right.$$
$$\left.-2\sum_{\alpha=1}^q\operatorname{tr}\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}^2\right]+\frac{1}{\sqrt{m}}\left[\frac{it}{\tau_V}\left\{2\sum_{\alpha=1}^q p_\alpha^2-(2\Delta+1)\tau_V^2\right.\right.$$
$$\left.-2\sum_{\alpha=1}^q p_\alpha\operatorname{tr}(\Sigma\Sigma_D^{-1}\Sigma)_{\alpha\alpha}\Sigma_{\alpha\alpha}^{-1}\right\}+\frac{2(it)^3}{3\tau_V^3}\left[\left(p^2-\sum_{\alpha=1}^q p_\alpha^2+24\right)\right.$$
$$\times\operatorname{tr}(\Sigma\Sigma_D^{-1}-I)^3-12\sum_{\alpha=1}^q\operatorname{tr}\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}\{(\Sigma\Sigma_D^{-1})^3\}_{\alpha\alpha}$$
$$\left.+24\sum_{\alpha=1}^q\operatorname{tr}\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}^2-12p\right]+\frac{4(it)^5}{3\tau_V^5}\left[3\tau_V^2+2p\right.$$
$$\left.\left.\left.-2\sum_{\alpha=1}^q\operatorname{tr}\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}^2\right]\operatorname{tr}(\Sigma\Sigma_D^{-1}-I)^3\right]\right\}+O(m^{-1})\,.$$

As the third term in (3.12) is of order m^{-1}, we can regard the variable Y as the random matrix having a normal distribution with mean 0 and covariance (2.3). Also putting $A_\alpha=(\Phi_{\alpha1},\Phi_{\alpha2},\cdots,\Phi_{\alpha q})$ and $B=\sqrt{2}(\Phi'\Phi-I)it/\tau_V$, then we can easily see the following relationship.

$$A_\alpha BA_\alpha'=\frac{\sqrt{2}\,it}{\tau_V}\{\Sigma_{\alpha\alpha}^{-1/2}(\Sigma\Sigma_D^{-1}\Sigma)_{\alpha\alpha}\Sigma_{\alpha\alpha}^{-1/2}-I_{\alpha\alpha}\}\tag{3.18}$$

and

$$A_\alpha A_\alpha'A_\alpha=A_\alpha\,.\tag{3.19}$$

Therefore making use of Lemma 2.2 after some modification, we have

(3.20) $E[q_2(Y)\exp(itq_0(Y)/\tau_V)]$

$$=\exp\left[\frac{(it)^2}{2}\right]\frac{2\sqrt{2}}{3}\left[\frac{3\sqrt{2}\,it}{2\tau_V}\left\{\frac{\tau_V^2}{2}+\sum_{\alpha=1}^q p_\alpha\operatorname{tr}(\Sigma\Sigma_D^{-1}\Sigma)_{\alpha\alpha}\Sigma_{\alpha\alpha}^{-1}-\sum_{\alpha=1}^q p_\alpha^2\right\}\right.$$
$$+\frac{2\sqrt{2}(it)^3}{\tau_V^3}\left\{\sum_{\alpha=1}^q\operatorname{tr}\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}^3-3\sum_{\alpha=1}^q\operatorname{tr}\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}^2\right.$$
$$\left.\left.-\operatorname{tr}(\Sigma\Sigma_D^{-1}-I)^3+\frac{3}{2}\tau_V^2+2p\right\}\right]+O(m^{-1/2})\,.$$

Similarly we have

$$(3.21)\quad E[q_1^2(Y)\exp(itq_0(Y)/\tau_V)]$$
$$=\exp\left[\frac{(it)^2}{2}\right]\left[\frac{1}{4}\left(p^2-\sum_{\alpha=1}^{q}p_\alpha^2\right)^2+p(p+1)+\sum_{\alpha,\beta}(\mathrm{tr}\,\Sigma_{\beta\alpha}\Sigma_{\alpha\alpha}^{-1}\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1})^2\right.$$
$$+\sum_{\alpha,\beta}\mathrm{tr}\,(\Sigma_{\beta\alpha}\Sigma_{\alpha\alpha}^{-1}\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1})^2-2\left(\sum_{\alpha=1}^{q}p_\alpha^2+p\right)$$
$$+\frac{(it)^2}{\tau_V^2}\left\{\left(\sum_{\alpha=1}^{q}p_\alpha^2-p^2\right)\left[2\sum_{\alpha=1}^{q}\mathrm{tr}\,\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}^2-3\tau_V^2-2p\right]+24\tau_V^2+24p\right.$$
$$-16\sum_{\alpha=1}^{q}\mathrm{tr}\,\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}^2+8\sum_{\alpha,\beta}\mathrm{tr}\,\Sigma_{\beta\alpha}\Sigma_{\alpha\alpha}^{-1}\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}$$
$$\left.\times\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\{(\Sigma\Sigma_D^{-1})^2\}_{\beta\beta}-16\sum_{\alpha,\beta}\mathrm{tr}\,\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\{(\Sigma\Sigma_D^{-1})^2\}_{\beta\beta}\Sigma_{\beta\alpha}\Sigma_{\alpha\alpha}^{-1}\right\}$$
$$\left.+\frac{(it)^4}{\tau_V^4}\left\{3\tau_V^2-2\sum_{\alpha=1}^{q}\mathrm{tr}\,\{(\Sigma\Sigma_D^{-1})^2\}_{\alpha\alpha}^2+2p\right\}^2\right]+O(m^{-1/2})\;.$$

Therefore for $|t|<\sqrt{m}C^*$ with some positive number C^*, the characteristic function of $V'/\sqrt{m}\tau_V$ is given by

$$(3.22)\quad C_V(t)=\exp\left[\frac{(it)^2}{2}\right]\left[1+\frac{1}{\sqrt{m}}\left\{\frac{1}{2}\left(p^2-\sum_{\alpha=1}^{q}p_\alpha^2\right)\frac{it}{\tau_V}+\left[\frac{4}{3}\,\mathrm{tr}\,(A-I)^3\right.\right.\right.$$
$$\left.\left.+3\tau_V^2+2p-2\sum_{\alpha=1}^{q}\mathrm{tr}\,\{A^2\}_{\alpha\alpha}^2\right]\left(\frac{it}{\tau_V}\right)^3\right\}$$
$$\left.+\frac{1}{m}\sum_{\alpha=1}^{3}h_{2\alpha}\left(\frac{it}{\tau_V}\right)^{2\alpha}+O(m^{-3/2})\right]\,,$$

where $\tau_V^2=2\,\mathrm{tr}\,(A-I)^2$ with $A=\Sigma\Sigma_D^{-1}$ and the coefficients h_2, h_4 and h_6 are given by

$$h_2=\frac{1}{8}\left(p^2-\sum_{\alpha=1}^{q}p_\alpha^2\right)^2+\frac{1}{2}\left(p^2-2\sum_{\alpha=1}^{q}p_\alpha^2-p\right)-\Delta\tau_V^2$$
$$+\frac{1}{2}\sum_{\alpha,\beta}(\mathrm{tr}\,\Sigma_{\beta\alpha}\Sigma_{\alpha\alpha}^{-1}\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1})^2+\frac{1}{2}\sum_{\alpha,\beta}\mathrm{tr}\,(\Sigma_{\beta\alpha}\Sigma_{\alpha\alpha}^{-1}\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1})^2\,,$$

$$(3.23)\quad h_4=\left(\sum_{\alpha=1}^{q}p_\alpha^2-p^2\right)\left[\sum_{\alpha=1}^{q}\mathrm{tr}\,\{A^2\}_{\alpha\alpha}^2-\frac{3}{2}\tau_V^2-p-\frac{2}{3}\,\mathrm{tr}\,(A-I)^3\right]$$
$$+2\,\mathrm{tr}\,(A-I)^4+\frac{40}{3}\,\mathrm{tr}\,(A-I)^3-8\sum_{\alpha=1}^{q}\mathrm{tr}\,\{A^2\}_{\alpha\alpha}\{A^3\}_{\alpha\alpha}$$
$$+\frac{8}{3}\sum_{\alpha=1}^{q}\mathrm{tr}\,\{A^2\}_{\alpha\alpha}^3+4\sum_{\alpha,\beta}\mathrm{tr}\,\Sigma_{\beta\alpha}\Sigma_{\alpha\alpha}^{-1}\{A^2\}_{\alpha\alpha}\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\{A^2\}_{\beta\beta}$$
$$-8\sum_{\alpha,\beta}\mathrm{tr}\,\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\{A^2\}_{\beta\beta}\Sigma_{\beta\alpha}\Sigma_{\alpha\alpha}^{-1}+16\tau_V^2+\frac{28}{3}p\,,$$

$$h_6=\frac{1}{2}\left[\frac{4}{3}\,\mathrm{tr}\,(A-I)^3+3\tau_V^2-2\sum_{\alpha=1}^{q}\mathrm{tr}\,\{A^2\}_{\alpha\alpha}^2+2p\right]^2.$$

By Lemma 2.4, we have the following theorem.

THEOREM 3.1. *Under the fixed alternative* $K: \Sigma_{ij} \neq 0$ *for some* i, j $(i \neq j)$, *the distribution of* $V' = -2\rho \log V + m \log |A|$, *where* V *is given by* (3.3) *with* $A = \Sigma\Sigma_D^{-1}$, *can be expanded asymptotically for large* m $(=\rho N)$ *as*

$$\begin{aligned}(3.24)\quad & P(V'/\sqrt{m}\tau_V \leqq x) \\ &= \Phi(x) - m^{-1/2}\Big\{\Phi^{(1)}(x)\frac{1}{2}\Big(p^2 - \sum_{\alpha=1}^{q} p_\alpha^2\Big)\Big/\tau_V \\ &\quad + \Phi^{(3)}(x)\Big[\frac{4}{3}\operatorname{tr}(A-I)^3 + 3\tau_V^2 + 2p - 2\sum_{\alpha=1}^{q}\operatorname{tr}\{A^2\}_{\alpha\alpha}^2\Big]\Big/\tau_V^3\Big\} \\ &\quad + m^{-1}\sum_{\alpha=1}^{3} h_{2\alpha}\Phi^{(2\alpha)}(x)\Big/\tau_V^{2\alpha} + O(m^{-3/2}),\end{aligned}$$

where $\tau_V^2 = 2\operatorname{tr}(A-I)^2$ *and* $\Phi^{(j)}(x)$ *means the jth derivative of the standard normal distribution function* $\Phi(x)$. *The coefficients* $h_{2\alpha}$ *with* $\Delta = (n-m)/2$ *are given by* (3.23).

This theorem was given by Sugiura and Fujikoshi [10] in case $q=2$.

4. Asymptotic expansion of the modified L.R. criterion for testing that several populations are equal

Let $p \times 1$ vectors $X_{i1}, X_{i2}, \cdots, X_{iN_i}$ be a random sample from $N(\mu_i, \Sigma_i)$ $(i=1,2,\cdots,k)$. For testing the hypothesis $H: \Sigma_1 = \Sigma_2 = \cdots = \Sigma_k$, $\mu_1 = \mu_2 = \cdots = \mu_k$ against all alternatives $K: \Sigma_i \neq \Sigma_j$ or $\mu_i \neq \mu_j$ for some i, j $(i \neq j)$, the modified L.R. criterion is given by (see Anderson [1])

$$\begin{aligned}(4.1)\quad -2\rho\log\Lambda &= m\log\Big|\Big\{S + \sum_{\alpha=1}^{k} N_\alpha(\bar{X}^{(\alpha)} - \bar{X})(\bar{X}^{(\alpha)} - \bar{X})'\Big\}\Big/m\Big| \\ &\quad - \sum_{\alpha=1}^{k} m_\alpha \log|S_\alpha/m_\alpha|,\end{aligned}$$

where $S = \sum_{\alpha=1}^{k} S_\alpha$, $S_\alpha = \sum_{\beta=1}^{N_\alpha}(X_{\alpha\beta} - \bar{X}^{(\alpha)})(X_{\alpha\beta} - \bar{X}^{(\alpha)})'$, $\bar{X}^{(\alpha)} = N_\alpha^{-1}\sum_{\beta=1}^{N_\alpha} X_{\alpha\beta}$, $\bar{X} = N^{-1}\sum_{\alpha=1}^{k} N_\alpha \bar{X}^{(\alpha)}$, $N = \sum_{\alpha=1}^{k} N_\alpha$, $n_\alpha = N_\alpha - 1$, $n = \sum_{\alpha=1}^{k} n_\alpha$, $m_\alpha = \rho n_\alpha$, $m = \sum_{\alpha=1}^{k} m_\alpha$, $\rho_\alpha = m_\alpha/m$ and correction factor ρ is (4.2).

$$(4.2)\qquad \rho = 1 - \Big(\sum_{\alpha=1}^{k}\frac{1}{n_\alpha} - \frac{1}{n}\Big)\frac{2p^2+3p-1}{6(k-1)(p+3)} - \frac{p-k+2}{n(p+3)}.$$

We define a random vector V_α and matrix Y_α as $\bar{X}^{(\alpha)} = m_\alpha^{-1/2}\Sigma_\alpha^{1/2}V_\alpha + \mu_\alpha$ and $S_\alpha = m_\alpha\Sigma_\alpha + \sqrt{2m_\alpha}\Sigma_\alpha^{1/2}Y_\alpha\Sigma_\alpha^{1/2}$, respectively. Also we define the abbreviated notations $\tilde{\Sigma} = \sum_{\alpha=1}^{k}\rho_\alpha\Sigma_\alpha + \sum_{\alpha=1}^{k}\rho_\alpha\eta_\alpha\eta_\alpha'$ with $\eta_\alpha = \mu_\alpha - \sum_{\alpha=1}^{k}\rho_\alpha\mu_\alpha$ and Ψ_α

$=\tilde{\Sigma}^{-1/2}\Sigma_\alpha^{1/2}$, $\xi_\alpha=\tilde{\Sigma}^{-1/2}\eta_\alpha$, $\bar{\xi}=k^{-1}\sum_{\alpha=1}^{k}\xi_\alpha$, $\tilde{\tilde{V}}=\sum_{\alpha=1}^{k}\Psi_\alpha V_\alpha V_\alpha'\Psi_\alpha'$, $\bar{V}=\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\Psi_\alpha V_\alpha\xi_\alpha'$, $\tilde{V}=\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\Psi_\alpha V_\alpha$ and $\tilde{Y}=\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\Psi_\alpha Y_\alpha\Psi_\alpha'$. Then the statistic $-2\rho\log\Lambda$ in (4.1) can be expressed in terms of V_α and Y_α $(\alpha=1, 2,\cdots, k)$ as

$$(4.3)\qquad -2\rho\log\Lambda=\mathrm{m}\left\{\log|\tilde{\Sigma}|-\sum_{\alpha=1}^{k}\rho_\alpha\log|\Sigma_\alpha|\right\}+\sqrt{m}q_0(Y, V)+q_1(Y, V)+m^{-1/2}q_2(Y, V)+O_p(m^{-1})\,.$$

The coefficients $q_0(Y, V)$, $q_1(Y, V)$ and $q_2(Y, V)$ are given by

$$q_0(Y, V)=\sqrt{2}\ \mathrm{tr}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}(\Psi_\alpha'\Psi_\alpha-I)Y+2\,\mathrm{tr}\,\bar{V}\,,$$

$$(4.4)\qquad q_1(Y, V)=\mathrm{tr}\,\tilde{\tilde{V}}-\mathrm{tr}\,\tilde{V}\tilde{V}'+\sum_{\alpha=1}^{k}\xi_\alpha'\xi_\alpha+2\Delta\sum_{\alpha=1}^{k}\rho_\alpha\xi_\alpha'\xi_\alpha-\mathrm{tr}\,\tilde{Y}^2-2\sqrt{2}\ \mathrm{tr}\,\tilde{Y}\bar{V}-\mathrm{tr}\,\bar{V}^2-\mathrm{tr}\,\bar{V}\bar{V}'+\sum_{\alpha=1}^{k}\mathrm{tr}\,Y_\alpha^2\,,$$

$$\begin{aligned}q_2(Y, V)=&4\Delta\,\mathrm{tr}\,\bar{V}+2\,\mathrm{tr}\sum_{\alpha=1}^{k}\frac{1}{\sqrt{\rho_\alpha}}\Psi_\alpha V_\alpha\xi_\alpha'-2k\,\mathrm{tr}\,\tilde{V}\bar{\xi}'-\sqrt{2}\ \mathrm{tr}\,\tilde{Y}\tilde{\tilde{V}}\\&+\sqrt{2}\ \mathrm{tr}\,\tilde{Y}\tilde{V}\tilde{V}'-\sqrt{2}\ \mathrm{tr}\,\tilde{Y}\sum_{\alpha=1}^{k}\xi_\alpha\xi_\alpha'-2\sqrt{2}\,\Delta\,\mathrm{tr}\,\tilde{Y}\sum_{\alpha=1}^{k}\rho_\alpha\xi_\alpha\xi_\alpha'\\&-2\,\mathrm{tr}\,\bar{V}\tilde{\tilde{V}}+2\,\mathrm{tr}\,\bar{V}\tilde{V}\tilde{V}'-2\,\mathrm{tr}\,\bar{V}\sum_{\alpha=1}^{k}\xi_\alpha\xi_\alpha'-4\Delta\,\mathrm{tr}\,\bar{V}\sum_{\alpha=1}^{k}\rho_\alpha\xi_\alpha\xi_\alpha'\\&+\frac{2\sqrt{2}}{3}\,\mathrm{tr}\,\tilde{Y}^3+4\,\mathrm{tr}\,\tilde{Y}^2\bar{V}+2\sqrt{2}\ \mathrm{tr}\,\tilde{Y}\bar{V}^2+\sqrt{2}\ \mathrm{tr}\,\tilde{Y}\bar{V}\bar{V}'\\&+\sqrt{2}\ \mathrm{tr}\,\tilde{Y}\bar{V}'\bar{V}+\frac{2}{3}\,\mathrm{tr}\,\bar{V}^3+2\,\mathrm{tr}\,\bar{V}^2\bar{V}'\\&-\frac{2\sqrt{2}}{3}\sum_{\alpha=1}^{k}\frac{1}{\sqrt{\rho_\alpha}}\,\mathrm{tr}\,Y_\alpha^3\,,\end{aligned}$$

where $\Delta=\frac{1}{2}\sum_{\alpha=1}^{k}(n_\alpha-m_\alpha)$.

From (4.3), we can easily see that $\Lambda'/\sqrt{m}=\left\{-2\rho\log\Lambda+m\sum_{\alpha=1}^{k}\rho_\alpha\cdot\log|A_\alpha|\right\}\Big/\sqrt{m}$ converges in law to a normal distribution with mean zero and variance $\tau_A^2=2\sum_{\alpha=1}^{k}\rho_\alpha\,\mathrm{tr}\,(A_\alpha-I)^2+4\sum_{\alpha=1}^{k}\rho_\alpha\,\mathrm{tr}\,A_\alpha\eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}$ with $A_\alpha=\Sigma_\alpha\tilde{\Sigma}^{-1}$. Thus the characteristic function of $\Lambda'/\sqrt{m}\tau_A$ $(\tau_A>0)$ is given by

$$(4.5)\qquad C_A(t)=E\left[\exp(itq_0(Y, V)/\tau_A)\left\{1+m^{-1/2}q_1(Y, V)it/\tau_A+m^{-1}\left[q_2(Y, V)it/\tau_A+\frac{1}{2}q_1^2(Y, V)(it/\tau_A)^2\right]\right\}\right]+O(m^{-3/2})\,.$$

By the lemmas in Part 2, we have the following theorem.

THEOREM 4.1. *Under the fixed alternative* $K: \Sigma_i \neq \Sigma_j$ *or* $\mu_i \neq \mu_j$ *for some* i, j, *the distribution* $\Lambda' = -2\rho \log \Lambda + m \sum_{\alpha=1}^{k} \rho_\alpha \log |A_\alpha|$ *with* $A_\alpha = \Sigma_\alpha \tilde{\Sigma}^{-1}$, *where* $-2\rho \log \Lambda$ *is given by* (4.1), *can be expanded asymptotically for large* m $(=\rho n)$ *as*

$$(4.6)\quad P(\Lambda'/\sqrt{m}\tau_\Lambda \leqq x)$$
$$= \Phi(x) - m^{-1/2}\Big[\Phi^{(1)}(x)\Big\{2\Delta a_1 + t_1 + s_1 - \frac{1}{2}t_2 - s_2 + \frac{k}{2}p(p+1)\Big\}\Big/\tau_\Lambda$$
$$+ \Phi^{(3)}(x)\Big\{\frac{4}{3}a_3 + 4\operatorname{tr}\tilde{\Psi} - 4\operatorname{tr} bb'\tilde{\Sigma}^{-1} - 2\operatorname{tr}\tilde{A}^2 - 8\operatorname{tr}\tilde{A}\Xi - 4\operatorname{tr}\Xi^2$$
$$-4\operatorname{tr}\Xi\Xi' + 2a_2\Big\}\Big/\tau_\Lambda^3\Big] + m^{-1}\sum_{\alpha=1}^{3} g_{2\alpha}\Phi^{(2\alpha)}(x)\Big/\tau_\Lambda^{2\alpha} + O(m^{-3/2}),$$

where the coefficients $g_{2\alpha}$ $(\alpha=1, 2, 3)$ *are given by*

$$g_2 = \Delta a_1(2\Delta a_1 + t_2 + 4) + 2\Delta a_2 + 8\sum_{\alpha=1}^{k}\eta'_\alpha\tilde{\Sigma}^{-1}A_\alpha\eta_\alpha$$
$$+\frac{1}{2}\{t_1 + s_1 - t_2 - s_2 + kp(p+1)\}\{t_1 - s_2 + s_1 + 4\Delta a_1\}$$
$$-2\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha^2(A_\alpha - I) - 2\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha \operatorname{tr} A_\alpha(A_\alpha - I)$$
$$+2\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha\tilde{A} + 4\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha\Xi - 4k\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha A_\alpha\eta_\alpha\bar{\eta}'\tilde{\Sigma}^{-1}$$
$$-2\operatorname{tr}\tilde{A}\sum_{\alpha=1}^{k}A_\alpha - 2\operatorname{tr}\tilde{A}\sum_{\alpha=1}^{k}\eta_\alpha\eta'_\alpha\tilde{\Sigma}^{-1} - 4\Delta\operatorname{tr}\tilde{A} - 4\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha^2\eta_\alpha\eta'_\alpha\tilde{\Sigma}^{-1}$$
$$-4\sum_{\alpha=1}^{k}\rho_\alpha\eta'_\alpha\tilde{\Sigma}^{-1}A_\alpha\eta_\alpha \operatorname{tr} A_\alpha - 4\operatorname{tr}\Xi\sum_{\alpha=1}^{k}A_\alpha + 4\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha A_\alpha\eta_\alpha b'\tilde{\Sigma}^{-1}$$
$$+4\sum_{\alpha=1}^{k}\rho_\alpha\eta'_\alpha\tilde{\Sigma}^{-1}b\operatorname{tr} A_\alpha - 4\operatorname{tr}\Xi\sum_{\alpha=1}^{k}\eta_\alpha\eta'_\alpha\tilde{\Sigma}^{-1} - 8\Delta\operatorname{tr}\Xi$$
$$+2\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha \operatorname{tr} A_\alpha\tilde{A} + 2\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha A_\alpha^2\tilde{A} + 4\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha^2\Xi$$
$$+4\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha \operatorname{tr} A_\alpha\Xi + 4\operatorname{tr}\Xi\tilde{A} + 2\sum_{\alpha=1}^{k}\eta'_\alpha\tilde{\Sigma}^{-1}\eta_\alpha \operatorname{tr} A_\alpha\tilde{A}$$
$$+4\operatorname{tr}\Xi^2 + 4\operatorname{tr}\Xi\Xi' + 4\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha \operatorname{tr}\eta_\alpha\eta'_\alpha\tilde{\Sigma}^{-1}\Xi$$
$$+4\sum_{\alpha=1}^{k}\rho_\alpha\eta'_\alpha\tilde{\Sigma}^{-1}\eta_\alpha \operatorname{tr} A_\alpha\Xi + \sum_{\alpha=1}^{k}\operatorname{tr} A_\alpha^2 + \operatorname{tr}\Big(\sum_{\alpha=1}^{k}\rho_\alpha A_\alpha\Big)^2$$
$$+2\sum_{\alpha,\beta=1}^{k}\rho_\alpha\rho_\beta \operatorname{tr} A_\alpha\eta_\alpha\eta'_\alpha\tilde{\Sigma}^{-1}A_\alpha A_\beta + 2\sum_{\alpha,\beta=1}^{k}\rho_\alpha\rho_\beta \operatorname{tr} A_\alpha\eta_\alpha\eta'_\beta\tilde{\Sigma}^{-1}\operatorname{tr} A_\alpha A_\beta$$
$$+\sum_{\alpha,\beta=1}^{k}\rho_\alpha\rho_\beta \operatorname{tr} A_\beta\eta_\alpha\eta'_\alpha\tilde{\Sigma}^{-1}\operatorname{tr} A_\alpha\eta_\beta\eta'_\beta\tilde{\Sigma}^{-1} + \sum_{\alpha,\beta=1}^{k}\rho_\alpha\rho_\beta(\eta'_\alpha\tilde{\Sigma}^{-1}\eta_\beta)^2 \operatorname{tr} A_\alpha A_\beta$$

$$-2\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha^2-2\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha^2\eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}-2\sum_{\alpha=1}^{k}\rho_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}\eta_\alpha \operatorname{tr} A_\alpha^2$$
$$+2\sum_{\alpha,\beta=1}^{k}\rho_\alpha\rho_\beta \operatorname{tr} A_\alpha A_\beta\eta_\alpha\eta_\beta'\tilde{\Sigma}^{-1}+2\sum_{\alpha=1}^{k}\rho_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}\eta_\alpha \operatorname{tr} A_\alpha\Big(\sum_{\alpha=1}^{k}\rho_\alpha A_\alpha\Big)$$
$$+2\sum_{\alpha,\beta=1}^{k}\rho_\alpha\rho_\beta\eta_\alpha'\tilde{\Sigma}^{-1}\eta_\beta \operatorname{tr} A_\beta A_\alpha\eta_\beta\eta_\alpha'\tilde{\Sigma}^{-1}+\frac{1}{8}t_2^2+\frac{1}{2}\sum_{\alpha,\beta=1}^{k}\rho_\alpha\rho_\beta(\operatorname{tr} A_\alpha A_\beta)^2$$
$$+\frac{1}{2}\sum_{\alpha,\beta=1}^{k}\rho_\alpha\rho_\beta \operatorname{tr}(A_\alpha A_\beta)^2+\frac{1}{8}(kp^2+kp+4)\{kp(p+1)-2t_2\}$$
$$+2\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha \operatorname{tr} \eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}\tilde{A}\ ,$$

(4.7)
$$g_4=2a_4+\frac{8}{3}\varDelta a_1a_3+\Big\{t_1+s_1-\frac{1}{2}t_2-s_2+\frac{1}{2}kp(p+1)\Big\}\Big\{\frac{4}{3}a_3+4\operatorname{tr}\tilde{\Psi}$$
$$-4\operatorname{tr} bb'\tilde{\Sigma}^{-1}-4\operatorname{tr}\Xi^2-4\operatorname{tr}\Xi\Xi'-2\operatorname{tr}\tilde{A}^2-8\operatorname{tr}\tilde{A}\Xi+2a_2\Big\}$$
$$+8a_3+2\varDelta a_1(4\operatorname{tr}\tilde{\Psi}-4\operatorname{tr} bb'\tilde{\Sigma}^{-1}-4\operatorname{tr}\Xi^2-4\operatorname{tr}\Xi\Xi'+2a_2$$
$$-2\operatorname{tr}\tilde{A}^2-8\operatorname{tr}\tilde{A}\Xi)-8\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha(A_\alpha-I)^2A_\alpha\tilde{A}$$
$$-16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}(A_\alpha-I)^2A_\alpha\Xi-8\operatorname{tr}\tilde{A}\tilde{\Psi}+8\operatorname{tr}\tilde{A}bb'\tilde{\Sigma}^{-1}-16\operatorname{tr}\Xi\tilde{\Psi}$$
$$+16\operatorname{tr}\Xi bb'\tilde{\Sigma}^{-1}+\frac{8}{3}\operatorname{tr}\tilde{A}^3+16\operatorname{tr}\tilde{A}^2\Xi+16\operatorname{tr}\tilde{A}\Xi^2+8\operatorname{tr}\tilde{A}\Xi\Xi'$$
$$+8\operatorname{tr}\tilde{A}\Xi'\Xi+\frac{16}{3}\operatorname{tr}\Xi^3+16\operatorname{tr}\Xi^2\Xi'-\frac{8}{3}a_3-8\operatorname{tr}\tilde{A}^2-16\operatorname{tr}\tilde{A}\Xi+4a_2$$
$$+8\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha^3\eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}+8\operatorname{tr}\Big(\sum_{\alpha=1}^{k}\rho_\alpha A_\alpha\Big)bb'\tilde{\Sigma}^{-1}+8\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha\Xi A_\alpha\Xi'$$
$$+8\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}(A_\alpha\Xi)^2+8\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}\tilde{A}A_\alpha\tilde{A}\eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}+8\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}\Xi A_\alpha\Xi\eta_\alpha$$
$$\times\eta_\alpha'\tilde{\Sigma}^{-1}+8\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}\Xi'A_\alpha\Xi\eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}-8\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha A_\alpha b\eta_\alpha'A_\alpha'\tilde{\Sigma}^{-1}$$
$$-16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha\tilde{A}\eta_\alpha\eta_\alpha'A_\alpha'\tilde{\Sigma}^{-1}+16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha\Xi'\eta_\alpha b'\tilde{\Sigma}^{-1}+16\sum_{\alpha=1}^{k}\rho_\alpha$$
$$\times\operatorname{tr} A_\alpha\Xi\eta_\alpha b'\tilde{\Sigma}^{-1}+16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha\tilde{A}A_\alpha\Xi+16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}\eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}\Xi A_\alpha\tilde{A}$$
$$+16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}\eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}\Xi'A_\alpha\tilde{A}+16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha\Xi\eta_\alpha\eta_\alpha'\tilde{\Sigma}^{-1}\Xi$$
$$+8\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}(A_\alpha\tilde{A})^2-16\operatorname{tr}\Xi'\sum_{\alpha=1}^{k}\rho_\alpha\tilde{\Sigma}^{-1}\eta_\alpha\eta_\alpha'A_\alpha'^2-16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr}\eta_\alpha\eta_\alpha'$$
$$\times A_\alpha'^2\tilde{\Sigma}^{-1}\Xi+16\sum_{\alpha=1}^{k}\rho_\alpha \operatorname{tr} A_\alpha\tilde{A}\eta_\alpha\sum_{\alpha=1}^{k}\rho_\alpha\eta_\alpha'A_\alpha'\tilde{\Sigma}^{-1}\ ,$$

$$g_6=\frac{2}{9}(2a_3+6\operatorname{tr}\tilde{\Psi}-6\operatorname{tr} bb'\tilde{\Sigma}^{-1}-3\operatorname{tr}\tilde{A}^2-6\operatorname{tr}\Xi^2-6\operatorname{tr}\Xi\Xi'$$
$$+3a_2-12\operatorname{tr}\tilde{A}\Xi)^2$$

and $\tau_A^2 = 2a_2 + 4\,\mathrm{tr}\,\Xi$. *The above notations mean that*

$$t_1 = \sum_{\alpha=1}^{k} \mathrm{tr}\,A_\alpha - \sum_{\alpha=1}^{k} \rho_\alpha\,\mathrm{tr}\,A_\alpha\,, \qquad t_2 = \sum_{\alpha=1}^{k} \rho_\alpha\,\mathrm{tr}\,A_\alpha^2 + \sum_{\alpha=1}^{k} \rho_\alpha(\mathrm{tr}\,A_\alpha)^2\,,$$

$$s_1 = \sum_{\alpha=1}^{k} \eta_\alpha' \tilde{\Sigma}^{-1} \eta_\alpha + 2\Delta \sum_{\alpha=1}^{k} \rho_\alpha \eta_\alpha' \tilde{\Sigma}^{-1} \eta_\alpha\,,$$

$$s_2 = \mathrm{tr}\,\Xi + \sum_{\alpha=1}^{k} \rho_\alpha \eta_\alpha' \tilde{\Sigma}^{-1} \eta_\alpha\,\mathrm{tr}\,A_\alpha\,, \tag{4.8}$$

$$a_\beta = \sum_{\alpha=1}^{k} \rho_\alpha\,\mathrm{tr}\,(A_\alpha - I)^\beta\,, \qquad (\beta = 1, 2, 3, 4)\,,$$

$$b = \sum_{\alpha=1}^{k} \rho_\alpha A_\alpha \eta_\alpha\,, \qquad \tilde{A} = \sum_{\alpha=1}^{k} \rho_\alpha (A_\alpha - I) A_\alpha\,, \qquad \Xi = \sum_{\alpha=1}^{k} \rho_\alpha A_\alpha \eta_\alpha \eta_\alpha' \tilde{\Sigma}^{-1}\,,$$

$$\tilde{\Psi} = \sum_{\alpha=1}^{k} \rho_\alpha A_\alpha \eta_\alpha \eta_\alpha' A_\alpha' \tilde{\Sigma}^{-1}\,.$$

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CORRECTIONS TO

"NON-NULL DISTRIBUTIONS OF THE LIKELIHOOD RATIO CRITERIA FOR INDEPENDENCE AND EQUALITY OF MEAN VECTORS AND COVARIANCE MATRICES"

HISAO NAGAO

The statement of Theorem 4.1 (Ann. Inst. Statist. Math. 24 (1972), 67–79) is in error. I would like to thank Dr. T. Hayakawa for pointing this out.

We should read as follows:

$$\sum_{\alpha,\beta} \rho_\alpha \rho_\beta \operatorname{tr} A_\alpha \eta_\alpha \eta'_\alpha \tilde{\Sigma}^{-1} A_\alpha A_\beta \longrightarrow \sum_{\alpha,\beta} \rho_\alpha \rho_\beta \operatorname{tr} A_\alpha \eta_\beta \eta'_\beta \tilde{\Sigma}^{-1} A_\alpha A_\beta$$

$$\sum_{\alpha=1}^{k} \rho_\alpha \eta'_\alpha \tilde{\Sigma}^{-1} \eta_\alpha \operatorname{tr} A_\alpha \left(\sum_{\alpha=1}^{k} \rho_\alpha A_\alpha \right) \longrightarrow \sum_{\alpha,\beta} \rho_\alpha \rho_\beta \eta'_\alpha \tilde{\Sigma}^{-1} \eta_\beta \operatorname{tr} A_\alpha A_\beta$$

$$\operatorname{tr} \Xi \Xi' \longrightarrow \operatorname{tr} \Xi \tilde{\Sigma} \Xi' \tilde{\Sigma}^{-1}$$

$$\operatorname{tr} \tilde{A} \Xi \Xi' \longrightarrow \operatorname{tr} \tilde{A} \Xi \tilde{\Sigma} \Xi' \tilde{\Sigma}^{-1}$$

$$\operatorname{tr} \tilde{A} \Xi' \Xi \longrightarrow \operatorname{tr} \tilde{A} \tilde{\Sigma} \Xi' \tilde{\Sigma}^{-1} \Xi$$

$$\operatorname{tr} \Xi^2 \Xi' \longrightarrow \operatorname{tr} \Xi^2 \tilde{\Sigma} \Xi' \tilde{\Sigma}^{-1}$$

$$\sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr} A_\alpha \Xi A_\alpha \Xi' \longrightarrow \sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr} A_\alpha \Xi A_\alpha \tilde{\Sigma} \Xi' \tilde{\Sigma}^{-1}$$

$$\sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr} \Xi' A_\alpha \Xi \eta_\alpha \eta'_\alpha \tilde{\Sigma}^{-1} \longrightarrow \sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr} \Xi' \tilde{\Sigma}^{-1} A_\alpha \Xi \eta_\alpha \eta'_\alpha$$

$$\sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr} A_\alpha \Xi' \eta_\alpha b' \tilde{\Sigma}^{-1} \longrightarrow \sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr} A_\alpha \tilde{\Sigma} \Xi' \tilde{\Sigma}^{-1} \eta_\alpha b' \tilde{\Sigma}^{-1}$$

$$\sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr} \eta_\alpha \eta'_\alpha \tilde{\Sigma}^{-1} \Xi' A_\alpha \tilde{A} \longrightarrow \sum_{\alpha=1}^{k} \rho_\alpha \operatorname{tr} \eta_\alpha \eta'_\alpha \Xi' \tilde{\Sigma}^{-1} A_\alpha \tilde{A}$$