THE EXACT NON-CENTRAL DISTRIBUTION OF THE GENERALIZED VARIANCE

A. M. MATHAI

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1. Summary

This article gives the exact non-central distribution of Wilks' generalized variance in the most general case, in terms of computable functions involving Zonal polynomials, Psi and Zeta functions. The exact distribution is obtained by using inverse Mellin transform, properties of Meijer's G-function and Calculus of residues. The cumulative distribution function is also available from the representation of the exact density.

2. Introduction

Let $S=XX'$, where the $m\times n$ matrix X has the normal density,

$$(2.1)\qquad (\det 2\pi\Sigma)^{-n/2}\exp\left[\mathrm{tr}\ -\frac{1}{2}\Sigma^{-1}(X-M)(X-M)'\right],$$

where $\det(\cdot)$ means the determinant of the square matrix $(\cdot)$ and $\mathrm{tr}(\cdot)$ means the trace$(\cdot)$=sum of the diagonal elements of the matrix $(\cdot)$. Here S is called a non-central Wishart matrix with n degrees of freedom with non-centrality parameters $\Omega=MM'\Sigma^{-1}/2$. Constantine [7] gives the density of S as,

$$(2.2)\qquad \left[\Gamma_m\left(\frac{n}{2}\right)\right]^{-1}(\det 2\Sigma)^{-n/2}\exp(\mathrm{tr}\ -\Omega)\exp\left(\mathrm{tr}\ -\frac{1}{2}\Sigma^{-1}S\right)$$
$$\times(\det S)^{(n-m-1)/2}{}_0F_1\left(\frac{n}{2};\frac{1}{2}\Sigma^{-1}\Omega S\right),$$

where, in general,

$$(2.3)\qquad \Gamma_m(u)=\pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma[u-(i-1)/2],$$

and ${}_0F_1(\cdot)$ is a hypergeometric function of a matrix argument which is defined as a certain series involving Zonal polynomials. For convenience the definition of a generalized hypergeometric function ${}_pF_q(\cdot)$, with matrix arguments, will be given in Section 3. Wilks [18] defined the determinant of the sample dispersion matrix as the sample generalized variance. Hence the aim of this article is to give the exact distribution of the determinant of the non-central Wishart matrix S. Constantine [7] gives the tth moment, about the origin, of $(\det S)$ as follows.

$$\begin{aligned}(2.4)\qquad E[(\det S)^t]=&[\Gamma_m(t+n/2)/\Gamma_m(n/2)](\det 2\Sigma)^t\\ &\times\exp(\mathrm{tr}\,-\Omega)_1F_1(t+n/2;\,n/2;\,\Omega)\ .\end{aligned}$$

In a recent paper, Sugiura and Fujikoshi [17], it is reported that Fujikoshi [11] obtained an asymptotic distribution of $(\det S)$. In the non-central linear case, that is, when all the eigen values of the determinental equation,

$$(2.5)\qquad \det(\Omega-\lambda^2\Sigma)=0$$

are zeros except one of them, (2.4) reduces to a very simple form. The hth moment in the non-central linear case was given by Anderson [1]. Bagai [3] used some complicated integrals, arising from some convolutions, to obtain the exact density of $(\det S)$ in the linear case for $m=2, 3$ and 4. Consul [9] used inverse Mellin transform technique and obtained the distribution for $m=2, 3, 4, 5, 6$ and 7. Bagai [4] and Consul [8] also considered some limiting distributions in the non-central linear case. Mathai and Rathie [15] gave the exact distribution, for the general value of m, in the non-central linear case. Here we will obtain the exact distribution of $(\det S)$ for the most general case.

3. Some definitions

(i) Braaksma's H-function: This is the most generalized Special Function and is defined as,

$$(3.1)\qquad H(z)=H_{p,q}^{m,n}\left(z\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right)=(2\pi i)^{-1}\int_C h(s)z^{-s}ds\ ,$$

where $i=(-1)^{1/2}$, z is not equal to zero and

$$(3.2)\qquad z^s=\exp\{s(\log|z|+i\arg z)\}\ ,$$

in which $\log|z|$ denotes the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value.

$$(3.3)\qquad h(s)=\left\{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\right\}\Big/\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)\right\}$$

where p, q, m, n are integers such that,

$$0 \leqq n \leqq p , \qquad 1 \leqq m \leqq q , \tag{3.4}$$

α_j $(j=1, 2, \cdots, p)$, β_j $(j=1, 2, \cdots, q)$ are positive numbers and a_j $(j=1, 2, \cdots, p)$, b_j $(j=1, 2, \cdots, q)$ are complex numbers such that,

$$\begin{aligned} &\alpha_j(b_h+v) \neq \beta_h(a_j-1-r) , \\ &\qquad \text{for } v, r=0, 1, \cdots ; \ h=1, \cdots, m ; \ j=1, \cdots, n . \end{aligned} \tag{3.5}$$

C is a contour in the complex s-plane separating the points,

$$-s=(b_j+v)/\beta_j , \qquad j=1, \cdots, m ; \ v=0, 1, \cdots \tag{3.6}$$

and

$$-s=(a_j-1-v)/\alpha_j , \qquad j=1, \cdots, n ; \ v=0, 1, \cdots . \tag{3.7}$$

When $\alpha_1=\alpha_2=\cdots=\alpha_p=1=\beta_1=\cdots=\beta_q$, the H-function reduces to a Meijer's G-function,

$$G(z)=G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right) . \tag{3.8}$$

A definition of a G-function may also be found in Erdélyi ([10], p. 207). The H-function exists for every $z \neq 0$ if $\mu=\sum_{j=1}^{q} \beta_j-\sum_{j=1}^{p} \alpha_j>0$ and for $0<|z|<\beta^{-1}$ if $\mu=0$ where,

$$\beta=\prod_{j=1}^{p} \alpha_j^{\alpha_j} \prod_{j=1}^{q} \beta_j^{-\beta_j} . \tag{3.9}$$

A detailed discussion of the H-function is given in Braaksma [6]. In our discussion a special case of the G-function occurs and it is easy to see that the G-function exists. In the evaluation of the exact density of $(\det S)$ we will use a property of the H-function which is proved in Braaksma ([6], p. 278, (6.1)) and which in effect says that $H(z)$ is available as the sum of the residues of $h(s)z^{-s}$ in the points (3.6). The definition of the H-function is slightly modified in (3.1) to present it as an inverse Mellin transform. This modification does not affect the result of Braaksma ([6], p. 278, (6.1)) which is mentioned above.

Since the non-central moments of the generalized variance are given in terms of a hypergeometric function with a matrix argument a definition of the generalized hypergeometric function will be given here. These functions are defined in terms of Zonal polynomials. A discussion of Zonal polynomials can be found in James ([13], [14]) and Constantine [7]. The notation $C_K(Z)$ for the Zonal polynomial can be found in Constantine [7] and the conditions for the existence of the generalized hypergeometric function can be found in Herz [12] and Constantine [7] and hence these won't be given here.

(ii) Hypergeometric functions with matrix arguments:

$$(3.10) \quad {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;Z) = \sum_{k=0}^{\infty} \Sigma_K\{[(a_1)_K(a_2)_K\cdots(a_p)_K C_K(Z)]/[(b_1)_K\cdots(b_q)_K k!]\}$$

where

$$(3.11) \quad (a)_K=\prod_{i=1}^{m}(a-(i-1)/2)_{k_i}\,, \quad K=(k_1,\cdots,k_m),\ k_1\geqq\cdots\geqq k_m\geqq 0,\ k_1+k_2+\cdots+k_m=k\,,$$

$$(3.12) \quad (a)_n=(a)(a+1)\cdots(a+n-1)$$

and $C_K(Z)$ is the Zonal polynomial. Hence (3.10) is a generalization to matrix arguments of the generalized hypergeometric function with scalar arguments. A detailed discussion of the Bessel function with matrix arguments may be found in Herz [12]. The generalized hypergeometric function with scalar arguments is also available as a special case from the H-function in (3.1).

(iii) The ψ-function: This is the logarithmic derivative of the Gamma function and is defined as,

$$(3.13) \quad \psi(z)=\frac{d}{dz}\log\Gamma(z)=-\gamma+(z-1)\sum_{m=0}^{\infty}[(m+1)(m+z)]^{-1}$$

where γ is the Euler's constant; $\gamma=0.577\cdots$.

(iv) The generalized Riemann Zeta function $\zeta(s,v)$:

$$(3.14) \quad \zeta(s,v)=\sum_{m=0}^{\infty}(v+m)^{-s}\,, \qquad R(s)>1,\ v\neq 0,\ -1,\cdots$$

where $R(\cdot)$ denotes the real part of $(\cdot)$.

4. The exact density

The exact density is given in (4.2), (4.7) and (4.8). For convenience, we will consider the density function of $(\det S)/(\det 2\Sigma)$. From (2.4), we have,

$$\begin{aligned}(4.1) \quad & E\{[(\det S)/(\det 2\Sigma)]^t\} \\ &=[\Gamma_m(t+n/2)/\Gamma_m(n/2)]\exp(\operatorname{tr}-\Omega)\,{}_1F_1(t+n/2;n/2;\Omega) \\ &=[\exp(\operatorname{tr}-\Omega)/\Gamma_m(n/2)]\sum_{k=0}^{\infty}\Sigma_K\{C_K(\Omega)\Gamma_m(t+n/2) \\ &\quad\times(t+n/2)_K/[k!(n/2)_K]\}\,.\end{aligned}$$

Since (4.1) exists and is unique for all complex t such that $R(t)>-n/2$

$+(m-1)/2$ the density function of $(\det S)/(\det 2\Sigma)$, denoted by $f(x)$, is uniquely determined by the uniqueness of inverse Mellin transform of (4.1). Further, since the series corresponding to ${}_1F_1(t+n/2; n/2; \Omega)$ is absolutely convergent, see Herz [12], $f(x)$ can be obtained as follows:

$$f(x)=\frac{\exp(\operatorname{tr}-\Omega)}{\Gamma_m(n/2)}\sum_{k=0}^{\infty}\Sigma_K\frac{C_K(\Omega)}{k!(n/2)_K}(2\pi i)^{-1} \times\int_{c-i\infty}^{c+i\infty}\Gamma_m(t+n/2)(t+n/2)_K x^{-t-1}dt\,, \tag{4.2}$$

where $i=(-1)^{1/2}$ and c is chosen properly. But,

$$\Gamma_m(t+n/2)=\pi^{m(m-1)/4}\Gamma(t+n/2)\Gamma(t+n/2-1/2)\cdots \Gamma[t+n/2-(m-1)/2] \tag{4.3}$$

and

$$(t+n/2)_K=(t+n/2)_{k_1}(t+n/2-1/2)_{k_2}\cdots(t+n/2-(m-1)/2)_{k_m}\,. \tag{4.4}$$

Therefore,

$$\Gamma_m(t+n/2)(t+n/2)_K=\pi^{m(m-1)/4}\Gamma(\alpha+k_1)\Gamma(\alpha-1/2+k_2)\cdots \Gamma[\alpha-(m-1)/2+k_m] \tag{4.5}$$

where

$$\alpha=t+n/2\,. \tag{4.6}$$

Now by using (3.8) we can write (4.2) as,

$$f(x)=[\pi^{m(m-1)/4}/\Gamma_m(n/2)]\exp(\operatorname{tr}-\Omega) \times\sum_{k=0}^{\infty}\Sigma_K\frac{C_K(\Omega)}{k!(n/2)_K}x^{-1}G^{m,0}_{0,m}(x\,|\,n/2+k_1,\, n/2-1/2+k_2,\cdots, n/2-(m-1)/2+k_m)\,,\qquad 0<x<\infty\,. \tag{4.7}$$

The representation in (4.7) is not in a computable form due to the fact that the parameters in $G^{m,0}_{0,m}(\cdot)$ in (4.7) differ by integers and hence a representation in computable forms is not available in the literature for the G-function appearing in (4.7). We will give a representation of the above G-function into computable forms by using Calculus of residues.

THEOREM 4.1.

$$G^{m,0}_{0,m}(x\,|\,n/2+k_1,\, n/2-1/2+k_2,\cdots,\, n/2-(m-1)/2+k_m) =\sum_{j=1}^{\infty}\left\{\frac{x^{n/2-(m+1)/2+k_m+j}}{(a_j-1)!}\sum_{v=0}^{a_j-1}\binom{a_j-1}{v}(-\log x)^{a_j-1-v}A'_v +\frac{x^{n/2-m/2+k_{m-1}+j}}{(b_j-1)!}\sum_{v=0}^{b_j-1}\binom{b_j-1}{v}(-\log x)^{b_j-1-v}C'_v\right\} \tag{4.8}$$

where

$$(4.9)\qquad A'_v=\left[\sum_{v_1=0}^{v-1}\binom{v-1}{v_1}A_0^{(v-1-v_1)}\sum_{v_2=0}^{v_1-1}\binom{v_1-1}{v_2}A_0^{(v_1-1-v_2)}\cdots\right]B_0\ ,$$

and

$$(4.10)\qquad C'_v=\left[\sum_{v_1=0}^{v-1}\binom{v-1}{v_1}C_0^{(v-1-v_1)}\sum_{v_2=0}^{v_1-1}\binom{v_1-1}{v_2}C_0^{(v_1-1-v_2)}\cdots\right]D_0\ ,$$

where a_j, b_j, A_0, $A_0^{(r)}$, B_0, C_0, $C_0^{(r)}$, D_0 *are available from* (4.16), (4.17), (4.40), (4.41), (4.42), (4.43) *for the case m-odd and from* (4.46), (4.47), (4.56), (4.57) *for the case m-even. It may be noticed that* (4.9) *and* (4.10) *are finite sums and* (4.8) *can be programmed and computed. In order to prove Theorem* 4.1 *we will make the following observations.*

4.1. *Case* I. *m-odd.*

Consider the following sets of Gammas,

$$(4.11)\qquad \Gamma(\alpha+k_1),\ \Gamma(\alpha-1+k_3),\cdots,\ \Gamma[\alpha-(m-1)/2+k_m]\ ,$$

and

$$(4.12)\qquad \Gamma(\alpha-1/2+k_2),\ \Gamma(\alpha-3/2+k_4),\cdots,\ \Gamma[\alpha-(m-2)/2+k_{m-1}]\ .$$

The poles of the Gammas within the sets (4.11) and (4.12) overlap whereas the poles of the Gammas between the sets do not overlap. In general, the poles of $\Gamma(z)$ are available from the equation,

$$(4.13)\qquad z=-v\ ,\qquad v=0,1,\cdots\ .$$

Further, in (4.11) and (4.12), $k_1\geqq k_2\geqq\cdots\geqq k_m\geqq 0$ and hence by a little simplification, it can be easily seen that the poles of

$$(4.14)\qquad \Delta=\Gamma(\alpha+k_1)\Gamma(\alpha-1/2+k_2)\cdots\Gamma[\alpha-(m-1)/2+k_m]$$

are available by equating to zero, the various factors in (4.15) where the exponents denote the orders of the poles.

$$(4.15)\qquad [\alpha-(m+1)/2+k_m+j]^{a_j}(\alpha-m/2+k_{m-1}+j)^{b_j}\ ,$$

where

$$(4.16)\qquad a_j=\begin{cases}1\ , & j=1,2,\cdots,1+k_{m-2}-k_m\ ,\\ 2\ , & j=2+k_{m-2}-k_m,\cdots,2+k_{m-4}-k_m\ ,\\ \cdots & \cdots\\ (m-1)/2\ , & j=(m-1)/2+k_3-k_m,\cdots,(m-1)/2+k_1-k_m\ ,\\ (m+1)/2\ , & j\geqq(m+1)/2+k_1-k_m\ ,\end{cases}$$

and

$$(4.17)\qquad b_j=\begin{cases} 1\,, & j=1,2,\cdots,1+k_{m-3}-k_{m-1}\,,\\ 2\,, & j=2+k_{m-3}-k_{m-1},\cdots,2+k_{m-5}-k_{m-1}\,,\\ \cdots & \cdots\\ (m-3)/2\,, & j=(m-3)/2+k_4-k_{m-1},\cdots,\\ & \qquad (m-3)/2+k_2-k_{m-1}\,,\\ (m-1)/2\,, & j\geqq(m-1)/2+k_2-k_{m-1}\,. \end{cases}$$

In general, if $G(\alpha)$ is a Gamma product with a pole of order s at $\alpha=a$ then the residue R of $G(\alpha)x^{-\alpha}$ at $\alpha=a$ is available from Calculus of residues as,

$$(4.18)\qquad R=\frac{1}{(s-1)!}\frac{\partial^{s-1}}{\partial\alpha^{s-1}}[(\alpha-a)^sG(\alpha)x^{-\alpha}]\,,\qquad \text{at } \alpha=a\,.$$

Also (4.18) can be simplified to the form,

$$(4.19)\qquad R=\frac{x^{-a}}{(s-1)!}\sum_{v=0}^{s-1}\binom{s-1}{v}(-\log x)^{s-1-v}\frac{\partial^v}{\partial\alpha^v}[(\alpha-a)^sG(\alpha)]\,,\qquad \text{at } \alpha=a\,.$$

But,

$$(4.20)\qquad \frac{\partial^v}{\partial\alpha^v}[(\alpha-a)^sG(\alpha)]=\frac{\partial^{v-1}}{\partial\alpha^{v-1}}[(\alpha-a)^sG(\alpha)H]\,,$$

where

$$(4.21)\qquad H=\frac{\partial}{\partial\alpha}\log[(\alpha-a)^sG(\alpha)]\,.$$

Now extending the result in (4.21) we can write,

$$(4.22)\qquad \frac{\partial^v}{\partial\alpha^v}[(\alpha-a)^sG(\alpha)]=\left[\sum_{v_1=0}^{v-1}\binom{v-1}{v_1}H^{(v-1-v_1)}\times\sum_{v_2=0}^{v_1-1}\binom{v_1-1}{v_2}H^{(v_1-1-v_2)}\cdots\right][(\alpha-a)^sG(\alpha)]\,,$$

where

$$(4.23)\qquad H^{(j)}=\frac{\partial^j}{\partial\alpha^j}H\,.$$

According to Braaksma ([6], p. 278, (6.1)), the G-function in (4.7) is the sum of the residues at the poles of (4.14). But the poles are available from (4.15) when m is odd. Hence from the result (4.18) to (4.23) it is easily seen that the expansion of the G-function is as given in (4.8) where, (4.24) $B_0=[\alpha-(m+1)/2+k_m+j]^{a_j}\Delta$,

at $\alpha=(m+1)/2-k_m-j$; $\alpha=t+n/2$,

(4.25) $A_0=\dfrac{\partial}{\partial t}\log[\alpha-(m+1)/2+k_m+j]^{a_j}\Delta$,

at $\alpha=(m+1)/2-k_m-j$,

(4.26) $A_0^{(r)}=\dfrac{\partial^{r+1}}{\partial t^{r+1}}\log[\alpha-(m+1)/2+k_m+j]^{a_j}\Delta$,

at $\alpha=(m+1)/2-k_m-j$, $r\geqq 1$,

(4.27) $D_0=(\alpha-m/2+k_{m-1}+j)^{b_j}\Delta$, at $\alpha=m/2-k_{m-1}-j$,

(4.28) $C_0=\dfrac{\partial}{\partial t}\log(\alpha-m/2+k_{m-1}+j)^{b_j}\Delta$, at $\alpha=m/2-k_{m-1}-j$,

(4.29) $C_0^{(r)}=\dfrac{\partial^{r+1}}{\partial t^{r+1}}\log(\alpha-m/2+k_{m-1}+j)^{b_j}\Delta$,

at $\alpha=m/2-k_{m-1}-j$, $r\geqq 1$,

where Δ is given in (4.14). Now (4.24) to (4.29) will be evaluated with the help of the following lemmas. In order to see the steps clearly, no simplification is done in (4.31) to (4.40). From (4.41) onwards, the expressions are simplified by using $\sum$ and $\prod$ notations.

LEMMA 4.1. *For,*

(4.30) $i+k_{m-2(i-1)}-k_m\leqq j\leqq i+k_{m-2i}-k_m$, $i=1,2,\cdots,(m-1)/2$,

(4.31)
$$\begin{aligned}&[\alpha-(m+1)/2+k_m+j]^{a_j}\{\Gamma(\alpha+k_1)\Gamma(\alpha-1+k_3)\cdots\Gamma[\alpha-(m-1)/2+k_m]\}\\&=\Gamma(\alpha+k_1)\Gamma(\alpha-1+k_3)\cdots\Gamma[\alpha-(m+1)/2+i+1+k_{m-2i}]\\&\quad\times\Gamma^i[\alpha-(m+1)/2+k_m+j+1]/\{[\alpha-(m+1)/2+k_m+j-1]^i\\&\quad\times[\alpha-(m+1)/2+k_m+j-2]^i\cdots[\alpha-(m+1)/2+i+k_{m-2(i-1)}]^i\\&\quad\times[\alpha-(m+1)/2+i+k_{m-2(i-1)}-1]^{i-1}\cdots[\alpha-(m+1)/2\\&\quad+i-1+k_{m-2(i-2)}]^{i-1}\cdots[\alpha-(m+1)/2+k_m+1]^1\}\ .\end{aligned}$$

The proof follows from the observation that when j satisfies (4.30), $a_j=i$. Now (4.31) at $\alpha=(m+1)/2-k_m-j$ becomes,

(4.32)
$$\begin{aligned}&\Gamma[(m+1)/2-k_m-j+k_1]\Gamma[(m+1)/2-k_m-j-1+k_3]\cdots\Gamma[i+1-j\\&-k_m+k_{m-2i}]\Gamma^i(1)/\{[(-1)(-2)\cdots(-k_m-j+i+k_{m-2(i-1)})]^i\\&\times[(-k_m-j+i-1+k_{m-2(i-1)})\cdots(-k_m-j+i-1\\&+k_{m-2(i-2)})]^{i-1}\cdots(-j+1)^1\}\ .\end{aligned}$$

In a similar way it can be seen that, when $j\geqq(m+1)/2+k_1-k_m$, $a_j=(m+1)/2$ and (4.31) at $\alpha=(m+1)/2-k_m-j$ becomes,

$$(4.33)\quad \Gamma^{(m+1)/2}(1)/\{[(-1)(-2)\cdots((m+1)/2-k_m-j+k_1)]^{(m+1)/2} \times[((m+1)/2-k_m-j+k_1-1)\cdots((m+1)/2-k_m-j-1 +k_3)]^{(m-1)/2}\cdots(-j+1)^1\}\ .$$

It is easy to see that (4.32) and (4.33) can be combined by using the convention that,

$$(4.34)\quad \Gamma[(m+1)/2-k_m-j+k_1]\Gamma[(m+1)/2-k_m-j-1+k_3]\cdots \Gamma[i+1-j-k_m+k_{m-2i}]=1$$

if $m-2i\leqq 0$. In this case (4.31) at $\alpha=(m+1)/2-k_m-j$ is given by (4.32) for $i=1, 2, \cdots, (m+1)/2$. (4.34) is consistent with the convention that an empty product is interpreted as unity.

LEMMA 4.2. *For,*

$$(4.35)\quad i+k_{m-(2i-1)}-k_{m-1}\leqq j\leqq i+k_{m-(2i+1)}-k_{m-1}\ ,\qquad i=1, 2,\cdots,(m-1)/2\ ,$$

$$(4.36)\quad (\alpha-m/2+k_{m-1}+j)^{b_j}\{\Gamma(\alpha-1/2+k_2)\Gamma(\alpha-3/2+k_4)\cdots \Gamma[\alpha-(m-2)/2+k_{m-1}]\}$$

at $\alpha=m/2-k_{m-1}-j$,

$$=\Gamma[(m-1)/2-j+k_2-k_{m-1}]\Gamma[(m-3)/2-j+k_4-k_{m-1}]\cdots \Gamma[i+1-j+k_{m-(2i+1)}-k_{m-1}]\Gamma^i(1)/\{[(-1)(-2)\cdots(i-j +k_{m-(2i-1)}-k_{m-1})]^i[(i-1-j+k_{m-(2i-1)}-k_{m-1})\cdots(i-1-j +k_{m-(2i-3)}-k_{m-1})]^{i-1}\cdots(-j+1)^1\}\ ,$$

with the convention that,

$$(4.37)\quad \Gamma[(m-1)/2-j+k_2-k_{m-1}]\Gamma[(m-3)/2-j+k_4-k_{m-1}]\cdots \Gamma(i+1-j+k_{m-(2i+1)}-k_{m-1})=1 \qquad \textit{if } m-(2i+1)\leqq 0\ .$$

LEMMA 4.3.

$$(4.38)\qquad \frac{\partial^{r+1}}{\delta z^{r+1}}\log\prod_{j=1}^{p}\Gamma(c_j+z)=\sum_{j=1}^{p}\psi(c_j+z)\ ,\qquad \textit{if } r=0\ ,$$

$$(4.39)\qquad =(-1)^{r+1}r!\sum_{j=1}^{p}\zeta(r+1, c_j+z)\ ,\qquad \textit{if } r\geqq 1\ ,$$

where the ψ *and* ζ*-functions are defined in* (3.17) *and* (3.18) *respectively.*

Now by using the Lemmas 4.1, 4.2 and 4.3 we can write down B_0, A_0, $A_0^{(r)}$, D_0, C_0, $C_0^{(r)}$ for the case m-odd. These are given in the following equations.

$$(4.40)\quad B_0=\{\Gamma[(m+1)/2-k_m-j+k_1]\Gamma[(m+1)/2-k_m-j-1+k_3]\cdots \Gamma[i+1-j-k_m+k_{m-2i}]\Gamma^i(1)\}\{\Gamma(m/2-k_m-j+k_2)$$

$$\times\Gamma(m/2-1-k_m-j+k_4)\cdots\Gamma(3/2-k_m-j+k_{m-1})\}/$$
$$\{[(-1)(-2)\cdots(-k_m-j+i+k_{m-2(i-1)})]^i$$
$$\times[(-k_m-j+i+k_{m-2(i-1)}-1)\cdots(-k_m-j+i-1$$
$$+k_{m-2(i-2)})]^{i-1}\cdots(-j+1)^1\},$$

for $i=1, 2,\cdots,(m+1)/2$ under (4.34).

$$(4.41)\quad A_0=\sum_{t=0}^{(m-1)/2-i}\psi[(m+1)/2-k_m-j-t+k_{2t+1}]+i\psi(1)$$
$$+\sum_{t=0}^{(m-3)/2}\psi(m/2-k_m-j+k_{2t+2}-t)-i\sum_{t=0}^{a}(-1-t)^{-1}$$
$$-(i-1)\sum_{t=0}^{b}(-k_m-j+i+k_{m-2(i-1)}-1-t)^{-1}\cdots-(-j+1)^{-1},$$

where $a=-1+k_m+j-i-k_{m-2(i-1)}$ and $b=k_{m-2(i-1)}-k_{m-2(i-2)}$, for $i=1, 2, \cdots,(m+1)/2$ under (4.34).

$$(4.42)\quad A_0^{(r)}=(-1)^{r+1}r!\left\{\sum_{t=0}^{(m-1)/2-i}\zeta(r+1,(m+1)/2-k_m-j-t+k_{2t+1})\right.$$
$$+i\zeta(r+1, 1)+\sum_{t=0}^{(m-3)/2}\zeta(r+1, m/2-k_m-j-t+k_{2t+2})$$
$$+i\sum_{t=0}^{a}(-1-t)^{-(r+1)}+(i-1)\sum_{t=0}^{b}(-k_m-j+i$$
$$\left.+k_{m-2(i-1)}-1-t)^{-(r+1)}+\cdots+(-j+1)^{-(r+1)}\right\},\qquad r\geqq 1,$$

where a and b are given in (4.41) and $i=1, 2,\cdots,(m+1)/2$ under (4.34). It may be noticed that (4.41) and (4.42) can be easily obtained from (4.40), by using the following procedure. Introduce a dummy variable, say z, in every factor of (4.40). Then evaluate the logarithmic derivative of B_0 with respect to z, at $z=0$ to obtain (4.41). Multiply (4.41) by $(-1)^{r+1}r!$, replace $\psi(\cdot)$ by $\zeta(r+1,\cdot)$, multiply the terms not containing $\psi(\cdot)$ by (-1) and raise the denominators of the terms not containing $\psi(\cdot)$ to the power $(r+1)$, to obtain (4.42). These are also seen from Lemma 4.3. Hence we will give only D_0 for the case m-odd and B_0 and D_0 for the case m-even.

$$(4.43)\quad D_0=\left\{\prod_{t=0}^{(m-3)/2-i}\Gamma[(m-1)/2-j-k_{m-1}-t+k_{2t+2}]\Gamma^i(1)\right.$$
$$\left.\times\prod_{t=0}^{(m-1)/2}\Gamma(m/2-k_{m-1}-j+k_{2t+1}-t)\right\}\Big/\left\{\prod_{t=0}^{c}(-1-t)^i\right.$$
$$\left.\times\prod_{t=0}^{d}(i-1-j+k_{m-(2i-1)}k_{m-1}-t)^{i-1}\cdots(-j+1)^1\right\},$$

where

$$(4.44)\quad c=-1+j-i+k_{m-1}-k_{m-(2i-1)}\,,\qquad d=k_{m-(2i-1)}-k_{m-(2i-3)}\,,$$
$$i=1,2,\cdots,(m-1)/2 \text{ under } (4.37)\,.$$

C_0 and $C_0^{(r)}$ are available from D_0 by using the procedure discussed after (4.42).

4.2. *Case* II. *m-even.*

In this case the poles of the Gammas in (4.14) are available by equating to zero the various factors of

$$(4.45)\quad [\alpha-(m+1)/2+k_m+j]^{a_j}[\alpha-m/2+k_{m-1}+j]^{b_j}\,,\qquad \alpha=n/2+t\,,$$

where the exponents denote the orders of the poles and

$$(4.46)\quad a_j=\begin{cases} 1\,, & j=1,2,\cdots,1+k_{m-2}-k_m\,,\\ 2\,, & j=2+k_{m-2}-k_m,\cdots,2+k_{m-4}-k_m\,,\\ \cdots & \cdots\\ (m-2)/2\,, & j=(m-2)/2+k_4-k_m,\cdots,(m-2)/2+k_2-k_m\,,\\ (m/2)\,, & j\geqq m/2+k_2-k_m\,,\end{cases}$$

and

$$(4.47)\quad b_j=\begin{cases} 1\,, & j=1,2,\cdots,1+k_{m-3}-k_{m-1}\,,\\ 2\,, & j=2+k_{m-3}-k_{m-1},\cdots,2+k_{m-5}-k_{m-1}\,,\\ \cdots & \cdots\\ (m-2)/2\,, & j=(m-2)/2+k_3-k_{m-1},\cdots,\\ & \qquad (m-2)/2+k_1-k_{m-1}\,,\\ m/2\,, & j\geqq m/2+k_1-k_{m-1}\,.\end{cases}$$

Again the density function is given by (4.7), (4.8), (4.9) and (4.10) where the quantities A_0 to D_0 are calculated in a similar fashion as in the case, m-odd. The procedure of getting A_0, $A_0^{(r)}$ from B_0 and C_0, $C_0^{(r)}$ from D_0 is given in the discussion after (4.42). Therefore we will give only B_0 and D_0 here. These are calculated with the help of the following lemmas.

LEMMA 4.4. *For,*

$$(4.48)\quad i+k_{m-2(i-1)}-k_m\leqq j\leqq i+k_{m-2i}-k_m\,,\qquad i=1,2,\cdots,m/2\,,$$

$$(4.49)\quad [\alpha-(m+1)/2+k_m+j]^{a_j}\Gamma(\alpha-1/2+k_2)\Gamma(\alpha-3/2+k_4)\cdots$$
$$\Gamma[\alpha-(m-1)/2+k_m]\,,\qquad at\ \alpha=(m+1)/2-k_m-j\,,$$
$$=\Delta_1=\prod_{t=0}^{(m-2)/2-i}\Gamma(m/2-j-k_m-t+k_{2t+2})\Gamma^i(1)\Big/\Big\{\prod_{t=0}^{a}(-1-t)^i$$
$$\times\prod_{t=0}^{b}(-k_m-j+i+k_{m-2(i-1)}-1-t)^{i-1}\cdots(-j+1)^1\Big\}\,,$$

where

$$(4.50)\qquad a=-1+k_m+j-i-k_{m-2(i-1)}\ ,\qquad b=k_{m-2(i-1)}-k_{m-2(i-2)}$$

with the convention that

$$(4.51)\qquad \prod_{t=0}^{(m-2)/2-i}\Gamma(m/2-j-k_m-t+k_{2t+2})=1\ ,\qquad \text{if } m-2i\leqq 0\ .$$

LEMMA 4.5. *For,*

$$(4.52)\qquad i+k_{m-(2i-1)}-k_{m-1}\leqq j\leqq i+k_{m-(2i+1)}-k_{m-1}\ ,\qquad i=1,2,\cdots,m/2\ ,$$

$$\begin{aligned}(4.53)\qquad &(\alpha-m/2+k_{m-1}+j)^{b_j}\Gamma(\alpha+k_1)\Gamma(\alpha-1+k_3)\cdots\\ &\qquad\Gamma[\alpha-(m-2)/2+k_{m-1}]\ ,\qquad \text{at } \alpha=m/2-k_{m-1}-j\ ,\\ &=\Delta_2=\prod_{t=0}^{(m-2)/2-i}\Gamma(m/2-j-k_{m-1}+k_{2t+1}-t)\Gamma^i(1)\Big/\Big\{\prod_{t=0}^{c}(-1-t)^i\\ &\qquad\times\prod_{t=0}^{d}(i-j-1+k_{m-(2i-1)}-k_{m-1}-t)^{i-1}\cdots(-j+1)^1\Big\}\ ,\end{aligned}$$

where

$$(4.54)\qquad c=-1+j-i+k_{m-1}-k_{m-(2i-1)}\ ,\qquad d=k_{m-(2i-1)}-k_{m-(2i-3)}\ ,$$

with the convention that

$$(4.55)\qquad \prod_{t=0}^{(m-2)/2-i}\Gamma(m/2-j-k_{m-1}+k_{2t+1}-t)=1\ ,\qquad \text{if } m-(2i+1)\leqq 0\ .$$

From Lemma 4.4 we get B_0 and from Lemma 4.5 we get D_0. These are given below.

$$(4.56)\qquad B_0=\Delta_1\left\{\prod_{t=0}^{(m-2)/2}\Gamma[(m+1)/2-k_m-j+k_{2t+1}-t]\right\}\ ,$$

where Δ_1 is given in (4.49).

$$(4.57)\qquad D_0=\Delta_2\left\{\prod_{t=0}^{(m-2)/2}\Gamma[(m-1)/2-k_{m-1}-j+k_{2t+2}-t]\right\}\ ,$$

where Δ_2 is given in (4.53).

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DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, MONTREAL, CANADA

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