ON THE UNIFORM ASYMPTOTIC JOINT NORMALITY OF SAMPLE QUANTILES

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Summary

Uniform (or type $(\boldsymbol{B})_d$) asymptotic normality of the joint distribution of an increasing number of sample quantiles as the sample size increases is investigated in both cases where the basic distributions are equal and are unequal. Under fairly general assumptions, sufficient conditions are derived for the asymptotic normality of sample quantiles.

Type $(\boldsymbol{B})_d$ asymptotic normality is a strictly stronger notion than the usual one which is based on the convergence in law, and the results obtained in this article will be helpful to widen the applicability of results on asymptotic normality of sample quantiles to related statistical inferences.

1. Introduction

Let, for each positive integer n, $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics of a random sample of size n drawn from a continuous distribution on the real line, whose pdf and cdf being given by $f_n(x)$ and $F_n(x)$, respectively. If $f_n(x)=f(x)$ and hence $F_n(x)=F(x)$ for all n, we shall call it *the case of equal basic distributions*, and *the case of unequal basic distributions* otherwise.

Most of the works in literature on asymptotic normality of sample quantiles have treated the case of equal basic distributions. Let $0<\lambda_1<\cdots<\lambda_k<1$ be any given set of positive numbers, and put $F(\mu_i)=\lambda_i$ and $\sigma_{ij}^2=\lambda_i(1-\lambda_i)/f(\mu_i)f(\mu_j)$, $i\leqq j$; $i, j=1,\cdots,k$. Let further $X_{nn_1}<X_{nn_2}<\cdots<X_{nn_k}$ be the corresponding sample quantiles with $n_i=[n\lambda_i]+1$ (sometimes $n_i=[n\lambda_i]$), $i=1,\cdots,k$. Under this situation, Mosteller [5] showed that the joint distribution of k variables, $\sqrt{n}(X_{nn_i}-\mu_i)$, $i=1,\cdots,k$, converges in law to a k-dimensional normal distribution $N(0, \Sigma_{(k)})$ as $n\to\infty$, where $\Sigma_{(k)}=\|\sigma_{ij}^2\|$, provided that $f(\mu_i)>0$, $i=1,\cdots,k$. A mathematically regorous treatment of this result has been given by Walker [7]. From this result, we can say, in our terminology [2], that the joint distribution of X_{nn_i}, $i=1,\cdots,k$, is asymptotically equivalent $(\boldsymbol{M})_d$ to a k-dimensional normal distribution $N(\mu_{(k)}, (1/n)\Sigma_{(k)})$ as $n\to\infty$, if $f(\mu_i)>0$, $i=1,\cdots,k$, where $\mu_{(k)}=(\mu_1, \mu_2,\cdots,\mu_k)'$.

Type $(\boldsymbol{M})_d$ asymptotic equivalence sometimes appears to be not strong enough for practical applications, for it only assures us the uniform coincidence of corresponding quantiles of both distributions under consideration. Indeed, there are some cases where type $(\boldsymbol{M})_d$ asymptotic equivalence does not guarantee the coincidence in the limit of the Shannon-Wiener information measures of both distributions. It should also be noted that Mosteller's result requires k and λ_i's to be fixed independently of n.

Recently, Weiss [8] considered the asymptotic joint normality of an increasing number of sample quantiles in a special case of unequal basic distributions, where the basic distributions are all defined over the interval $[0, 1]$ such that $F_n(1)-F_n(0)=1$, $0<D_1\leqq f_n(x)\leqq D_2<\infty$, $|f_n''(x)|\leqq D_3<\infty$ for all x in $(0, 1)$ for some positive numbers D_1, D_2 and D_3 independent of n, and $f_n(x)$, $f_n'(x)$ and $f_n''(x)$ are all right-continuous at $x=0$ and left-continuous at $x=1$. Let δ be any given number such that $3/4<\delta<1$, and put $k=n^{1-\delta}-1$ and $n_i=in^{\delta}$, $i=1,\cdots,k$. Further, let $l_{ni}^0=n_i/n$ and $s_{ni}^0=F_n^{-1}(l_{ni}^0)$, $i=1,\cdots,k$. Weiss [8] then showed that the joint distribution of k sample quantiles, X_{nn_i}, $i=1,\cdots,k$, is asymptotically equivalent to a k-dimensional normal distribution $N(s_{n(k)}^0, S_{n(k)}^0)$ as $n\to\infty$, with definitions $s_{n(k)}^0=(s_{n1}^0,\cdots,s_{nk}^0)'$ and $S_{n(k)}^0=(1/n)||s_{nij}||$, $s_{nij}=l_{ni}^0(1-l_{nj}^0)/f_n(s_{ni}^0)f_n(s_{nj}^0)$, $i\leqq j$; $i, j=1,\cdots,k$, in the sense that

$$\lim_{n\to\infty}\left|\int_{E_{(k)}} h_n(z_{(k)})dz_{(k)}-\int_{E_{(k)}} h_n^0(z_{(k)})dz_{(k)}\right|=0$$

for any measurable subset $E_{(k)}$ of the k-dimensional Euclidean space, where h_n and h_n^0 denote the pdf's of the distributions under consideration. It should be remarked that this notion of asymptotic equivalence is of type $(\boldsymbol{B})_d$ (see Lemma 1.3.2 of [1]).

In the present article, the authors set forth the problem in more general situation, and derive conditions under which X_{nn_i}'s are jointly asymptotically $(\boldsymbol{B})_d$ normally distributed as $n\to\infty$. For this, in the following section, we give some results on type $(\boldsymbol{B})_d$ asymptotic equivalence which are necessary for discussions in later sections.

In Section 3, we treat a special case of equal basic distributions, where the sample are taken from a uniform distribution over $(0, 1)$, and give an interesting result (Theorem 3.1) which is fundamental to the studies in subsequent sections. General case of equal basic distributions is considered in Section 4, and finally in Section 5 the case of unequal basic distributions is handled, where the Weiss result [8] is improved.

2. Some results on type $(\boldsymbol{B})_d$ asymptotic equivalence

Let $\{X_s\}$ $(s=1, 2, \cdots)$ and $\{Y_s\}$ $(s=1, 2, \cdots)$ be two sequences of random variables, where for each s X_s and Y_s are distributed over a measurable space $(R_s, \boldsymbol{B}_s)$, $\boldsymbol{B}_s$ being a σ-field of subsets of any given abstract space R_s. Type $(\boldsymbol{B})_d$ or uniform asymptotic equivalence of these two sequences, denoted simply by $X_s \sim Y_s$ $(\boldsymbol{B})_d$, as $s \to \infty$, has been defined [1, 2] by the condition

$$\sup_{E \in \boldsymbol{B}_s} \left| P^{X_s}(E) - P^{Y_s}(E) \right| \to 0 , \qquad (s \to \infty) , \tag{2.1}$$

where P^{X_s} and P^{Y_s} designate the probability measures corresponding to the random variables X_s and Y_s respectively.

If for each s both X_s and Y_s are absolutely continuous with respect to μ_s, a σ-finite measure over $(R_s, \boldsymbol{B}_s)$, then the condition (2.1) is equivalent to

$$V(X_s, Y_s) = \int_{R_s} |f_s - g_s| d\mu_s \to 0 , \qquad (s \to \infty) , \tag{2.2}$$

where f_s and g_s denote the gpdf (μ_s) of X_s and Y_s, respectively. In such a case, it has been shown [1] that the condition

$$\rho(X_s, Y_s) = \int_{R_s} \sqrt{f_s g_s} d\mu_s \to 1 , \qquad (s \to \infty) , \tag{2.3}$$

is necessary and sufficient, and any one of the conditions

$$I(X_s : Y_s) = \int_{R_s} f_s \log (f_s/g_s) d\mu_s \to 0 , \qquad (s \to \infty) , \tag{2.4}$$

and

$$I(Y_s : X_s) = \int_{R_s} g_s \log (g_s/f_s) d\mu_s \to 0 , \qquad (s \to \infty) , \tag{2.4$'$}$$

is sufficient, for the condition (2.2).

In the following two lemmas, we shall extend these two criteria for type $(\boldsymbol{B})_d$ asymptotic equivalence to a more general situation: Suppose that X_s and Y_s are dominated by μ_s over some measurable subset A_s of R_s, and let f_s^* and g_s^* be the density functions of X_s and Y_s with respect to μ_s such that $f_s^* > 0$ and $g_s^* > 0$ over A_s, and

$$P^{X_s}(E_s) = \int_{E_s} f_s^* d\mu_s \qquad \text{and} \qquad P^{Y_s}(E_s) = \int_{E_s} g_s^* d\mu_s$$

for any measurable subset E_s of A_s. Outside the set A_s, the variables X_s and Y_s are allowable to be or not to be dominated by μ_s.

Under this situation, we first prove the following

LEMMA 2.1. *The condition*

$$\rho^*(X_s, Y_s)=\int_{A_s} \sqrt{f_s^* g_s^*}\, d\mu_s \to 1 , \qquad (s\to\infty) , \tag{2.5}$$

implies that $X_s \sim Y_s$ $(\boldsymbol{B})_d$ *as* $s\to\infty$.

PROOF. Let us put

$$\xi_s=P^{X_s}(A_s)=\int_{A_s} f_s^* d\mu_s \quad \text{and} \quad \eta_s=P^{Y_s}(A_s)=\int_{A_s} g_s^* d\mu_s \tag{2.6}$$

for each s. Then, by using the Schwarz inequality, we get

$$\int_{A_s} |f_s^*-g_s^*|\, d\mu_s \leqq 2\sqrt{((\xi_s+\eta_s)/2)^2-\rho^*(X_s, Y_s)^2} . \tag{2.7}$$

Since $0<\xi_s$, $\eta_s<1$ for each s, the condition (2.5) implies that

$$\xi_s\to 1,\ \eta_s\to 1 \quad \text{and} \quad V^*(X_s, Y_s)=\int_{A_s} |f_s^*-g_s^*|\, d\mu_s\to 0 , \quad (s\to\infty) . \tag{2.8}$$

Since

$$2 \sup_{E\in \boldsymbol{B}_s} \left|P^{X_s}(E)-P^{Y_s}(E)\right| \leqq V^*(X_s, Y_s)+(2-\xi_s-\eta_s) ,$$

it follows that (2.8) implies (2.1), which completes the proof of the lemma.

The criterion (2.4) or (2.4)′ works only when the carrier of f_s is contained in that of g_s or vice versa, up to the measure μ_s. The following lemma requires no such assumptions.

LEMMA 2.2. *Under the same situation as in the preceding lemma, the simultaneous conditions*

$$P^{X_s}(A_s)\to 1 , \qquad (s\to\infty) , \tag{2.9}$$

and

$$I^*(X_s : Y_s)=\int_{A_s} f_s^* \log (f_s^*/g_s^*) d\mu_s\to 0 , \qquad (s\to\infty) , \tag{2.10}$$

imply that $X_s\sim Y_s$ $(\boldsymbol{B})_d$ *as* $s\to\infty$.

PROOF. Since the function f_s^*/ξ_s, ξ_s being defined by (2.6), gives a gpdf (μ_s) over the set A_s, Jensen's inequality can be applied to get

$$I^*(X_s : Y_s)\geqq -2\xi_s \log \rho^*(X_s, Y_s)+2\xi_s \log \xi_s ,$$

or equivalently,

$$I^*(X_s : Y_s)-2\xi_s \log \xi_s \geqq -2\xi_s \log \rho^*(X_s, Y_s)\geqq 0 , \tag{2.11}$$

for each s.

Hence, the conditions (2.9) and (2.10) simultaneously imply that $\rho^*(X_s, Y_s)\to 1$ as $s\to\infty$, from which it follows by the preceding lemma that $X_s \sim Y_s$ $(\boldsymbol{B})_d$ as $s\to\infty$.

This completes the proof of the lemma.

In the next place, we shall state a result on type $(\boldsymbol{B})_d$ asymptotic equivalence of induced probability measures: Let, for each s, t_s be a measurable transformation from a certain measurable subset A_s of R_s into another measurable space $(\bar{R}_s, \bar{\boldsymbol{B}}_s)$, and let $\bar{X}_s$ and $\bar{Y}_s$ be any given random variables defined over $(\bar{R}_s, \bar{\boldsymbol{B}}_s)$ such that

$$P^{\bar{X}_s}(\bar{E}_s)=P^{X_s}(t_s^{-1}(\bar{E}_s)) \qquad \text{and} \qquad P^{\bar{Y}_s}(\bar{E}_s)=P^{Y_s}(t_s^{-1}(\bar{E}_s))$$

for every measurable subset $\bar{E}_s$ of $\bar{A}_s=t_s(A_s)$. Then, we can see the following

LEMMA 2.3. *Suppose that the condition*

$$P^{X_s}(A_s)\to 1\ , \qquad (s\to\infty)\ , \tag{2.12}$$

is satisfied. Then, $X_s\sim Y_s$ $(\boldsymbol{B})_d$ *implies that*

$$P^{\bar{X}_s}(\bar{A}_s)\to 1,\ P^{\bar{Y}_s}(\bar{A}_s)\to 1 \quad \textit{and} \quad \bar{X}_s\sim \bar{Y}_s\ (\boldsymbol{B})_d\ , \qquad (s\to\infty)\ . \tag{2.13}$$

The proof of this lemma is easy and will be omitted.

In the final place, we shall consider the case of real probability distributions, and derive conditions under which two given sequences of multi-dimensional normal random variables are asymptotically equivalent $(\boldsymbol{B})_d$, where in general the dimension increases under the limiting process.

Let, for each positive integer s, $X_{(n_s)}$ and $Y_{(n_s)}$ be non-degenerate, n_s-dimensional random variables distributed as $N(a_{(n_s)}, A_{(n_s)})$ and $N(b_{(n_s)}, B_{(n_s)})$, respectively. The dispersion matrices $A_{(n_s)}$ and $B_{(n_s)}$ are therefore positive definite.

We now prove the following

LEMMA 2.4. *In order that*

$$X_{(n_s)}\sim Y_{(n_s)}\ (\boldsymbol{B})_d\ , \qquad (s\to\infty)\ , \tag{2.14}$$

it is necessary and sufficient that the simultaneous conditions

$$\operatorname{tr}(A_{(n_s)}^{-1}B_{(n_s)}-I_{(n_s)})+\operatorname{tr}(A_{(n_s)}B_{(n_s)}^{-1}-I_{(n_s)})\to 0\ , \qquad (s\to\infty)\ , \tag{2.15}$$

and

$$(a_{(n_s)}-b_{(n_s)})'(A_{(n_s)}+B_{(n_s)})^{-1}(a_{(n_s)}-b_{(n_s)})\to 0\ , \qquad (s\to\infty)\ , \tag{2.16}$$

are satisfied, where $I_{(n_s)}$ stands for the unit matrix of order n_s.

PROOF. For the sake of notational simplicity, we shall delete the suffix (n_s) from vectors and matrices in the proof below.

As is seen in [4], the affinity defined by (2.3) is calculated as

$$\rho(X, Y)=\frac{|A^{-1}B^{-1}|^{1/4}}{|(A^{-1}+B^{-1})/2|^{1/2}} \exp\Big[\frac{-1}{4}\{a'A^{-1}a+b'B^{-1}b - (A^{-1}a+B^{-1}b)'(A^{-1}+B^{-1})^{-1}(A^{-1}a+B^{-1}b)\}\Big] .$$

Since $(A^{-1}+B^{-1})^{-1}=A-A(A+B)^{-1}A=B-B(A+B)^{-1}B$, the above quantity becomes

$$(2.17) \quad \rho(X, Y)=\Big[\frac{|A^{-1}B^{-1}|}{|(A^{-1}+B^{-1})/2|^2}\Big]^{1/4} \exp\Big[\frac{-1}{4}(a-b)'(A+B)^{-1}(a-b)\Big] .$$

By the inequality $|\lambda H+(1-\lambda)K|\geqq|H|^{\lambda}|K|^{1-\lambda}$ for any positive definite matrices H and K and any λ $(0\leqq\lambda\leqq1)$, the first factor of the right-hand side of (2.17) does not exceed unity, and the same is seen with the second factor, too. Hence, the condition

$$(2.18) \qquad \rho(X, Y)\rightarrow 1 , \qquad (s\rightarrow\infty) ,$$

is equivalent to the simultaneous conditions

$$(2.19) \qquad |(A^{-1}+B^{-1})/2|^2/|A^{-1}B^{-1}|\rightarrow 1 , \qquad (s\rightarrow\infty) ,$$

and

$$(2.20) \qquad (a-b)'(A+B)^{-1}(a-b)\rightarrow 0 , \qquad (s\rightarrow\infty) ,$$

the latter of which is the same as (2.16).

We now show that the condition (2.19) is equivalent to (2.15).

Since A and B are positive definite, there exist non-singular matrices C and D such that $A^{-1}=C'C$ and $B^{-1}=D'D$. By using these, it is easily verified that

$$|(A^{-1}+B^{-1})/2|^2/|A^{-1}B^{-1}|=|(H+H^{-1}+2I)/4| ,$$

where we have put $H=CBC'$. Let $\alpha_1,\cdots,\alpha_{n_s}$ be the characteristic roots of H. Then,· there exists an orthogonal matrix P such that

$$P(H+H^{-1}+2I)P'=\mathrm{Diag}\,(\alpha_1+\alpha_1^{-1}+2,\cdots,\alpha_{n_s}+\alpha_{n_s}^{-1}+2) ,$$

from which it follows that the left-hand member of (2.19) is equal to $\prod_{i=1}^{n_s}\{1+(\alpha_i+\alpha_i^{-1}-2)/4\}$. But, since $\alpha_i>0$ and $\alpha_i+\alpha_i^{-1}-2>0$, $i=1,\cdots,n_s$, the condition (2.19) is satisfied if and only if

$$\sum_{i=1}^{n_s}(\alpha_i+\alpha_i^{-1}-2)\to 0\ , \qquad (s\to\infty)\ , \tag{2.21}$$

which is the same condition as (2.15).

This completes the proof of the lemma.

We conclude this section by stating that each of the conditions

$$(a_{(n_s)}-b_{(n_s)})'A_{(n_s)}^{-1}(a_{(n_s)}-b_{(n_s)})\to 0\ , \qquad (s\to\infty)\ , \tag{2.22}$$

and

$$(a_{(n_s)}-b_{(n_s)})'B_{(n_s)}^{-1}(a_{(n_s)}-b_{(n_s)})\to 0\ , \qquad (s\to\infty)\ , \tag{2.23}$$

implies (2.16), because the matrices $A_{(n_s)}^{-1}-(A_{(n_s)}+B_{(n_s)})^{-1}$ and $B_{(n_s)}^{-1}-(A_{(n_s)}+B_{(n_s)})^{-1}$ are non-negative definite.

3. The case of uniform distribution

Let $U_{n1}<U_{n2}<\cdots<U_{nn}$ be order statistics of a random sample of size n from a uniform distribution over (0, 1), and let us choose $k=k(n)$ variables, $U_{nn_1}<U_{nn_2}<\cdots<U_{nn_k}$, whose joint variable is denoted by $U_{n(k)}=(U_{nn_1},\cdots,U_{nn_k})'$. Then, mean vector and dispersion matrix $U_{n(k)}$ are given respectively by

$$l_{n(k)}=(l_{n1},l_{n2},\cdots,l_{nk})' \tag{3.1}$$

and

$$L_{n(k)}=\frac{1}{n+2}\begin{bmatrix} l_{n1}(1-l_{n1}) & l_{n1}(1-l_{n2}) & \cdots & l_{n1}(1-l_{nk}) \\ l_{n1}(1-l_{n2}) & l_{n2}(1-l_{n2}) & \cdots & l_{n2}(1-l_{nk}) \\ \cdots & \cdots & \cdots & \cdots \\ l_{n1}(1-l_{nk}) & \cdots & \cdots & l_{nk}(1-l_{nk}) \end{bmatrix}. \tag{3.2}$$

where $l_{ni}=n_i/(n+1)$, $i=1,\cdots,k$. It is known that the inverse matrix of $L_{n(k)}$ is given by

$$L_{n(k)}^{-1}=(n+2)\begin{bmatrix} \frac{l_{n2}-l_{n0}}{(l_{n2}-l_{n1})(l_{n1}-l_{n0})} & \frac{-1}{l_{n2}-l_{n1}} & 0\cdots\cdots & 0 \\ \frac{-1}{l_{n2}-l_{n1}} & \frac{l_{n3}-l_{n1}}{(l_{n3}-l_{n2})(l_{n2}-l_{n1})} & \frac{-1}{l_{n3}-l_{n2}} & 0\cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0\cdots\cdots & & \frac{-1}{l_{nk}-l_{nk-1}} & \frac{l_{nk+1}-l_{nk-1}}{(l_{nk+1}-l_{nk})(l_{nk}-l_{k-1})} \end{bmatrix} \tag{3.3}$$

(see, for example, [6]), where $l_{n0}=0$ and $l_{nk+1}=1$.

Now, let $Z_{n(k)}=(Z_{n1},\cdots,Z_{nk})'$ be a k-dimensional normal random variable whose mean vector and dispersion matrix are $l_{n(k)}$ and $L_{n(k)}$ defined by (3.1) and (3.2), respectively.

Under this situation, the following theorem gives us a sufficient condition for $U_{n(k)}$ and $Z_{n(k)}$ to be asymptotically equivalent $(\boldsymbol{B})_d$.

THEOREM 3.1. *If the condition*

$$k \Big/ \min_{1 \le i \le k+1} (n_i - n_{i-1}) \to 0 , \qquad (n \to \infty) , \tag{3.4}$$

is satisfied, then

$$U_{n(k)} \sim Z_{n(k)} \ (\boldsymbol{B})_d , \qquad (n \to \infty) . \tag{3.5}$$

PROOF. It is sufficient for (3.5) to hold that the K–L information $I(U_{n(k)} : Z_{n(k)})$ tends to zero as $n \to \infty$.

The pdf's of $U_{n(k)}$ and $Z_{n(k)}$ are given by

$$h_n(z_{(k)}) = \left\{ n! \Big/ \prod_{i=1}^{k+1} (d_i!) \right\} \prod_{i=1}^{k+1} (z_i - z_{i-1})^{d_i} , \tag{3.6}$$

$$(0 = z_0 < z_1 < \cdots < z_k < z_{k+1} = 1) ,$$

and

$$g_n(z_{(k)}) = (2\pi)^{-k/2} |L_{n(k)}|^{-1/2} \exp\left[\frac{-1}{2} (z_{(k)} - l_{n(k)})' L_{n(k)}^{-1} (z_{(k)} - l_{n(k)}) \right] \tag{3.7}$$

$$(-\infty < z_i < \infty, \ i = 1, \cdots, k) ,$$

respectively, where $d_i = n_i - n_{i-1} - 1$, $i = 1, \cdots, k+1$, with the convention $n_0 = 0$ and $n_{k+1} = n+1$, and $z_{(k)} = (z_1, \cdots, z_k)'$. Hence the K–L information is given by

$$\begin{aligned} I(U_{n(k)} : Z_{n(k)}) &= \mathcal{E}[\log \{h_n(U_{n(k)}) / g_n(U_{n(k)})\}] \\ &= \log \left[(2\pi)^{k/2} n! \, |L_{n(k)}|^{1/2} \Big/ \prod_{i=1}^{k+1} (d_i!) \right] \\ &\quad + \sum_{i=1}^{k+1} d_i \, \mathcal{E}[\log (U_{nn_i} - U_{nn_{i-1}})] \\ &\quad + \frac{1}{2} \mathcal{E}[(U_{n(k)} - l_{n(k)})' L_{n(k)}^{-1} (U_{n(k)} - l_{n(k)})] . \end{aligned} \tag{3.8}$$

From (3.2) we have

$$|L_{n(k)}| = (n+2)^{-k} \prod_{i=1}^{k+1} (l_{ni} - l_{ni-1}) ,$$

and also

$$\mathcal{E}[(U_{n(k)} - l_{n(k)})' L_{n(k)}^{-1} (U_{n(k)} - l_{n(k)})] = k .$$

It is also seen that

$$\mathcal{E}[\log (U_{nn_i} - U_{nn_{i-1}})] = - \sum_{j=1}^{n-d_i} \frac{1}{d_i + j}$$

$$=\log d_i-\log n+\frac{1}{2d_i}-\frac{1}{2n}-\frac{T(d_i)}{d_i}+\frac{T(n)}{n}\ ,$$

where $T(m)$ is defined, for any integer $m\geqq 2$, by

$$T(m)=\sum_{i=1}^{\infty}\frac{a_{i+1}}{(m+1)\cdots(m+i)}\ ,$$

with

$$a_r=\frac{1}{r}\int_0^1 z(1-z)(2-z)\cdots(r-1-z)dz\ , \qquad (r\geqq 2)$$

(see Lemma 1.1 of [3]).

Using Stirling's formula and the above results, we can get

$$\begin{aligned}(3.9)\qquad I(U_{n(k)}:Z_{n(k)})=&\frac{k}{2}\log\left(1-\frac{2}{n+2}\right)+\frac{k+1}{2}\log\left(1-\frac{1}{n+1}\right)\\ &+\frac{1}{2}\sum_{i=1}^{k+1}\log\left(1+\frac{1}{d_i}\right)+\frac{k}{2n}+\left(1-\frac{k}{n}\right)T(n)\\ &-\sum_{i=1}^{k+1}T(d_i)+\frac{c(n)}{n}+\sum_{i=1}^{k+1}\frac{c_i(n)}{d_i}\ ,\end{aligned}$$

where $c(n)=O(1)$ and $\max\{c_i(n);\ i=1,\cdots,k+1\}=O(1)$ as $n\to\infty$. But, since the K-L information is always non-negative, non-positive terms can be deleted from the right-hand side of (3.9), which gives us

$$(3.10)\qquad I(U_{n(k)}:Z_{n(k)})\leqq\frac{k+1}{2d}+\frac{k}{2n}+T(n)+\left(\frac{1}{n}+\frac{k+1}{d}\right)c\ ,$$

where we have put $d=\min\{d_i;\ i=1,\cdots,k+1\}$, and c is a positive constant.

As is easily verified, it holds that $T(m)\to 0$ as $m\to\infty$. Thus, by (3.10), it is seen that the condition (3.4) implies the vanishing of $I(U_{n(k)}:Z_{n(k)})$ as $n\to\infty$, which guarantees the validity of (3.5).

This completes the proof of the theorem.

This theorem plays a fundamental role in subsequent discussions.

Now, let $Z^0_{n(k)}=(Z^0_{n1},\cdots,Z^0_{nk})'$ be a k-dimensional normal random variable distributed as $N(l^0_{n(k)},L^0_{n(k)})$, where

$$(3.11)\qquad l^0_{n(k)}=(l^0_{n1},\cdots,l^0_{nk})'\qquad\text{with}\qquad l^0_{ni}=n_i/n,\ i=1,\cdots,k\ ,$$

and

$$(3.12)\qquad L^0_{n(k)}=\frac{1}{n}\begin{bmatrix} l^0_{n1}(1-l^0_{n1}) & l^0_{n1}(1-l^0_{n2}) & \cdots & l^0_{n1}(1-l^0_{nk})\\ l^0_{n1}(1-l^0_{n2}) & l^0_{n2}(1-l^0_{n2}) & \cdots & l^0_{n2}(1-l^0_{nk})\\ \cdots & \cdots & \cdots & \cdots\\ l^0_{n1}(1-l^0_{nk}) & \cdots & \cdots & l^0_{nk}(1-l^0_{nk})\end{bmatrix}.$$

Then, by using Lemma 2.4 it is easily seen that $Z_{n(k)} \sim Z^0_{n(k)}$ $(\boldsymbol{B})_d$ as $n \rightarrow \infty$, provided the condition (3.4) of the theorem. Hence the following theorem is immediate from the above theorem.

THEOREM 3.2. *If the condition* (3.4) *is satisfied, then it holds that*

$$U_{n(k)} \sim Z^0_{n(k)} \ (\boldsymbol{B})_d \,, \qquad (n \rightarrow \infty) \,. \tag{3.13}$$

By these two theorems, we can see the following

COROLLARY 3.1. (a) *If k is fixed independently of n, then the condition*

$$\min_{1 \leqq i \leqq k+1} (n_i - n_{i-1}) \rightarrow \infty \,, \qquad (n \rightarrow \infty) \,, \tag{3.14}$$

implies (3.5) *and* (3.13).

(b) *The mth order statistic, U_{nm}, is asymptotically $(\boldsymbol{B})_d$ normally distributed as $N(m/(n+1), m(n-m+1)/((n+2)(n+1)^2))$, or as $N(m/n, m(n-m)/n^3)$ according as $n \rightarrow \infty$, provided that $m \rightarrow \infty$ and $n-m \rightarrow \infty$.*

It has been shown [3] that $(U_{n1}, \cdots, U_{nm_1})'$, $(U_{nh}, \cdots, U_{nh+v-1})'$ and $(U_{nn-m_2+1}, \cdots, U_{nn})'$ constitute an asymptotically independent $(\boldsymbol{B})_d$ set of size 3 as $n \rightarrow \infty$, if $m_1/n \rightarrow 0$, $m_2/n \rightarrow 0$, $h/n \rightarrow \lambda$ and $v/n \rightarrow \mu$ for any given λ and μ such that $0 < \lambda \leqq \lambda + \mu < 1$. Hence, in the case where the spacing of n_i's is such that

$$\begin{cases} l_{n1}, \cdots, l_{ns} \rightarrow 0 \,, \\ \gamma \leqq l_{ns+1}, \cdots, l_{nk-t} \leqq 1 - \gamma \,, \\ l_{nk-t+1}, \cdots, l_{nk} \rightarrow 1 \,, \end{cases} \tag{3.15}$$

as $n \rightarrow \infty$, for some positive number γ independent of n, s and t being allowable to depend on n, we may take the dispersion matrix of $Z_{n(k)}$ in Theorem 3.1 to be of slightly different form: Let $\bar{Z}_{n(k)}$ be a k-dimensional normal random variable distributed as $N(l_{n(k)}, \bar{L}_{n(k)})$, where

$$\bar{L}_{n(k)} = \begin{bmatrix} \bar{L}_{n(s)1} & & 0 \\ & \bar{L}_{n(u)2} & \\ 0 & & \bar{L}_{n(t)3} \end{bmatrix} , \qquad (u = k - s - t) \,, \tag{3.16}$$

Here, $\bar{L}_{n(s)1}$, $\bar{L}_{n(u)2}$ and $\bar{L}_{n(t)3}$ are the dispersion matrices of the first s, the second u and the last t components of $Z_{n(k)}$, respectively. Then, under the assumption (3.15), it holds that $U_{n(k)} \sim \bar{Z}_{n(k)}$ $(\boldsymbol{B})_d$, as $n \rightarrow \infty$, provided the condition (3.4). Analogous result is obtained with Theorem 3.2, where the dispersion matrix $L^0_{n(k)}$ may be replaced by $\bar{L}^0_{n(k)}$ obtained analogously to (3.16).

These will be summarized in the following

COROLLARY 3.2. *Under the assumption* (3.15), *the condition* (3.4) *implies that* $U_{n(k)}$ *is asymptotically* $(\boldsymbol{B})_d$ *normally distributed as* $N(l_{n(k)}, \bar{L}_{n(k)})$, *or as* $N(l^0_{n(k)}, \bar{L}^0_{n(k)})$, *as* $n\to\infty$.

In the second half of this section, we shall consider the following situation: Suppose that for each n we are given a positive integer $k=k(n)$ and a set of k spacings $0<\lambda_{n1}<\lambda_{n2}<\cdots<\lambda_{nk}<1$. Sample quantiles corresponding to these λ_{ni}'s are $U_{nn_1}<U_{nn_2}<\cdots<U_{nn_k}$, where as usual $n_i=[n\lambda_{ni}]+1$, $i=1,\cdots,k$. As before, let us denote by $U_{n(k)}$ the joint variable of U_{nn_i}'s.

Let $\tilde{Z}_{n(k)}=(\tilde{Z}_{n1},\cdots,\tilde{Z}_{nk})'$ be a k-dimensional normal random variable whose mean vector and dispersion matrix are given by

$$\lambda_{n(k)}=(\lambda_{n1},\cdots,\lambda_{nk})' \tag{3.17}$$

and

$$\Lambda_{n(k)}=\frac{1}{n}\begin{bmatrix} \lambda_{n1}(1-\lambda_{n1}) & \lambda_{n1}(1-\lambda_{n2}) & \cdots & \lambda_{n1}(1-\lambda_{nk}) \\ \lambda_{n1}(1-\lambda_{n2}) & \lambda_{n2}(1-\lambda_{n2}) & \cdots & \lambda_{n2}(1-\lambda_{nk}) \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{n1}(1-\lambda_{nk}) & \cdots & \cdots & \lambda_{nk}(1-\lambda_{nk}) \end{bmatrix}, \tag{3.18}$$

respectively. Put $\lambda_{n0}=0$ and $\lambda_{nk+1}=1$.

Let $Z_{n(k)}$ be defined the same as in the beginning part of this section with n_i's given above. Then, it is not so difficult to see that under the condition (3.4) two variables, $Z_{n(k)}$ distributed as $N(l_{n(k)}, L_{n(k)})$ and $\tilde{Z}_{n(k)}$ as $N(\lambda_{n(k)}, \Lambda_{n(k)})$, are asymptotically equivalent $(\boldsymbol{B})_d$ as $n\to\infty$; Indeed, the condition (2.15) and (2.16) of Lemm 2.4 are easily seen to be satisfied. Thus, we have the following theorem, in which the condition (3.4) of Theorem 3.1 is rewritten as (3.19).

THEOREM 3.3. *If* k *and* λ_{ni}*'s satisfy the condition*

$$k\Big/\Big\{n\min_{1\leq i\leq k+1}(\lambda_{ni}-\lambda_{ni-1})\Big\}\to 0\,,\qquad (n\to\infty)\,, \tag{3.19}$$

then it holds that

$$U_{n(k)}\sim\tilde{Z}_{n(k)}\ (\boldsymbol{B})_d\,,\qquad (n\to\infty)\,. \tag{3.20}$$

When k is fixed independently of n, the following corollary is an immediate consequence of the above theorem.

COROLLARY 3.3. *If* k *is fixed independently of* n, *then the condition*

$$n\min_{1\leq i\leq k+1}(\lambda_{ni}-\lambda_{ni-1})\to\infty\,,\qquad (n\to\infty)\,, \tag{3.21}$$

implies (3.20). *If, moreover,* $\lambda_{ni}=\lambda_i$, $i=1,\cdots,k$, *are fixed independently of* n, *then* $U_{n(k)}$ *is asymptotically* $(\boldsymbol{B})_d$ *normally distributed according to* $N(\lambda_{(k)}, (1/n)\Lambda_{(k)})$ *as* $n\to\infty$, *where*

$$\lambda_{(k)}=(\lambda_1, \lambda_2, \cdots, \lambda_k)' \tag{3.22}$$

and

$$\Lambda_{(k)}=\begin{bmatrix} \lambda_1(1-\lambda_1) & \lambda_1(1-\lambda_2) \cdots & \lambda_1(1-\lambda_k) \\ \lambda_1(1-\lambda_2) & \lambda_2(1-\lambda_2) \cdots & \lambda_2(1-\lambda_k) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \lambda_1(1-\lambda_k) & \cdots\cdots & \lambda_k(1-\lambda_k) \end{bmatrix}. \tag{3.23}$$

In the second half of this corollary, the condition (3.19) or (3.21) is not necessary, because the condition is automatically satisfied.

4. The case of equal basic distributions

Let $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics of a sample of size n drawn from a real, continuous distribution whose pdf and cdf are given by $f(x)$ and $F(x)$, respectively. If we define U_{ni} by $U_{ni}=F(X_{ni})$, $i=1, \cdots, n$, then $U_{n1}<U_{n2}<\cdots<U_{nn}$ are regarded as order statistics from a uniform distribution over (0, 1), for which the asymptotic $(\boldsymbol{B})_d$ normality of sample quantiles has been discussed in the preceding section. In order to treat the same problem for the case of general distributions in the present and the subsequent section, we make use of Lemma 2.3. For this, we need an assumption which guarantees the non-singularity of probability integral transformation.

To avoid the complexity of discussion, we make the following assumption which is simple but fairly common to practical applications.

ASSUMPTION 4.1. $D(f)=\{x : f(x)>0\}$ is an open interval on the real line.

Under this assumption, $F^{-1}(z)$ is a measurable and one-to-one transformation from the interval (0, 1) onto the interval $D(f)$.

Now, let us consider asymptotic $(\boldsymbol{B})_d$ normality of the joint variable $X_{n(k)}=(X_{nn_1}, X_{nn_2}, \cdots, X_{nn_k})'$ of sample quantiles X_{nn_i}'s with $n_1<n_2<\cdots<n_k$, where $k=k(n)$ and $n_i=n_i(n)$ may in general be dependent on n. Let us put $l_{ni}=n_i/(n+1)$, $s_{ni}=F^{-1}(l_{ni})$ and $f_{ni}=f(s_{ni})$, $i=1,\cdots,k$.

Let $Y_{n(k)}=(Y_{n1}, \cdots, Y_{nk})'$ be a k-dimensional normal random variable with mean vector and dispersion matrix defined by

$$s_{n(k)}=(s_{n1}, \cdots, s_{nk})' \tag{4.1}$$

and

$$(4.2)\quad S_{n(k)}=\frac{1}{n+2}\begin{bmatrix} l_{n1}(1-l_{n1})/f_{n1}^2 & l_{n1}(1-l_{n2})/f_{n1}f_{n2} & \cdots & l_{n1}(1-l_{nk})/f_{n1}f_{nk} \\ l_{n1}(1-l_{n2})/f_{n1}f_{n2} & l_{n2}(1-l_{n2})/f_{n2}^2 & \cdots & l_{n2}(1-l_{nk})/f_{n2}f_{nk} \\ \cdots & \cdots & \cdots & \cdots \\ l_{n1}(1-l_{nk})/f_{n1}f_{nk} & \cdots & \cdots & l_{nk}(1-l_{nk})/f_{nk}^2 \end{bmatrix}$$

respectively. Note that $\mathrm{Diag}\,(f_{n1},\cdots,f_{nk})S_{n(k)}\,\mathrm{Diag}\,(f_{n1},\cdots,f_{nk})=L_{n(k)}$, and hence $|L_{n(k)}|=|S_{n(k)}|\left\{\prod_{i=1}^{k} f_{ni}\right\}^2$.

Now, let us define a random variable $V_{n(k)}=(V_{n1},\cdots,V_{nk})'$ by $V_{ni}=F(Y_{ni})$, $i=1,\cdots,k$. Then, $V_{n(k)}$ is distributed over the closure of a k-dimensional open cube $Q_k=(0,1)\times\cdots\times(0,1)$, and is discontinuous on the boundary of this set unless $D(f)=(-\infty,\infty)$. Over the domain Q_k, $V_{n(k)}$ is absolutely continuous with respect to the Lebesgue measure over the k-dimensional Euclidean space, and has the density

$$(4.3)\quad p_n(z_{(k)})=(2\pi)^{-k/2}|S_{n(k)}|^{-1/2}\left\{\prod_{i=1}^{k} f(F^{-1}(z_i))\right\}^{-1} \times\exp\left[\frac{-1}{2}(F^{-1}(z_{(k)})-s_{n(k)})'S_{n(k)}^{-1}(F^{-1}(z_{(k)})-s_{n(k)})\right],\quad (z_{(k)}\in Q_k),$$

where we have put $F^{-1}(z_{(k)})=(F^{-1}(z_1),\cdots,F^{-1}(z_k))'$.

Since, under the condition (3.4),

$$\frac{1}{l_{n1}}\sqrt{\frac{l_{n1}(1-l_{n1})}{n+2}}<\sqrt{\frac{1}{n_1}}\to 0$$

and

$$\frac{1}{1-l_{nk}}\sqrt{\frac{l_{nk}(1-l_{nk})}{n+2}}<\sqrt{\frac{1}{n-n_k+1}}\to 0$$

as $n\to\infty$, there exist sequences of positive numbers $\{\rho_n\}$ $(n=1,2,\cdots)$ and $\{\rho_n'\}$ $(n=1,2,\cdots)$ such that $\rho_n\to\infty$, $\rho_n'\to\infty$ and

$$\delta_n=\rho_n\sqrt{\frac{l_{n1}(1-l_{n1})}{n+2}}<l_{n1}\qquad\text{and}\qquad \delta_n'=\rho_n'\sqrt{\frac{l_{nk}(1-l_{nk})}{n+2}}<1-l_{nk}$$

for all n, and further $\delta_n\to 0$ and $\delta_n'\to\infty$ as $n\to\infty$. Hence, the set

$$(4.4)\quad Q_{n,k}=\{z_{(k)}:0<l_{n1}-\delta_n<z_1<\cdots<z_k<l_{nk}+\delta_n'<1\}$$

is well defined for every n, provided (3.4), and by Theorem 3.1 and by using the Chebycheff inequality, we can see easily that

$$(4.5)\quad P^{Z_{n(k)}}(Q_{n,k})\to 1\,,\qquad (n\to\infty)\,.$$

Hence, if the condition (3.4) is satisfied, Lemma 2.2 assures us that

$$(4.6)\qquad I^*(Z_{n(k)}:V_{n(k)})=\int_{Q_{n,k}} g_n \log\{g_n/p_n\}\,dz_{(k)}\to 0\ ,\qquad (n\to\infty)\ ,$$

implies

$$(4.7)\qquad Z_{n(k)}\sim V_{n(k)}\ (\boldsymbol{B})_d\ ,\qquad (n\to\infty)\ ,$$

and consequently, by Lemma 2.3 it holds that $X_{n(k)}\sim Y_{n(k)}\ (\boldsymbol{B})_d$ as $n\to\infty$.

We now derive conditions which guarantee the condition (4.6). For this, we make the following

Assumption 4.2. $f(x)$ is differentiable once and $f'(x)$ is continuous over $D(f)$. Then, by (3.7) and (4.3) it is seen that

$$(4.8)\qquad \log\{g_n(z_{(k)})/p_n(z_{(k)})\}=\sum_{i=1}^{k}\varphi(z_{ni}^{**})(z_i-l_{ni})-(1/2)w'_{n(k)}L_{n(k)}^{-1}(z_{(k)}-l_{n(k)})$$
$$+(1/8)w'_{n(k)}L_{n(k)}^{-1}w_{n(k)}\ ,$$

where

$$(4.9)\qquad \begin{aligned}&\varphi(z)=f'(F^{-1}(z))/f^2(F^{-1}(z))\ , &&(0<z<1)\ ,\\ &\phi(z;l)=f(F^{-1}(l))/f(F^{-1}(z))\ , &&(0<z,l<1)\ ,\end{aligned}$$

$$(4.10)\qquad \begin{aligned}&w_{n(k)}=(w_{n1},\cdots,w_{nk})'\ ,\\ &w_{ni}=\varphi(z_{ni}^*)\phi(z_{ni}^*;l_{ni})(z_i-l_{ni})^2\ , &&i=1,\cdots,k\ ,\end{aligned}$$

and z_{ni}^{**} and z_{ni}^* are some functions of z_i which lie between z_i and l_{ni} (denoted by $z_{ni}^{**}, z_{ni}^*\in((z_i, l_{ni})))$ for each i.

Let us designate the integral operator $\int_{Q_{n,k}} g_n dz_{(k)}$ by $E^*[\cdot]$. Then, we have

$$(4.11)\qquad \left|E^*\left[\sum_{i=1}^{k}\varphi(z_{ni}^{**})(z_i-l_{ni})\right]\right|\leqq\sum_{i=1}^{k}\{E^*[\varphi(z_{ni}^{**})^2]E^*[(z_i-l_{ni})^2]\}^{1/2}\ ,$$

and, by using the Cauchy-Schwarz inequality and its integral version,

$$(4.12)\qquad |E^*[w'_{n(k)}L_{n(k)}^{-1}(z_{(k)}-l_{n(k)})]|\leqq\sqrt{k}\,\{E^*[w'_{n(k)}L_{n(k)}^{-1}w_{n(k)}]\}^{1/2}\ .$$

Also,

$$(4.13)\qquad E^*[w'_{n(k)}L_{n(k)}^{-1}w_{n(k)}]$$
$$\leqq(n+2)\left\{\sum_{i=1}^{k}\frac{l_{ni+1}-l_{ni-1}}{(l_{ni+1}-l_{ni})(l_{ni}-l_{ni-1})}E^*[\varphi^2(z_{ni}^*)\phi^2({}^*_{ni};l_{ni})(z_i-l_{ni})^4]\right.$$
$$+2\sum_{i=1}^{k-1}\frac{1}{l_{ni+1}-l_{ni}}E^*[|\varphi(z_{ni}^*)\varphi(z_{ni+1}^*)|\phi(z_{ni}^*;l_{ni})\phi(z_{ni+1}^*;l_{ni+1})$$
$$\left.\times(z_i-l_{ni})^2(z_{i+1}-l_{ni+1})^2]\right\}\ .$$

It is difficult to evaluate the right-hand members of (4.11), (4.12) and (4.13) in general situation. Hence, we shall consider here the simplest case where the functions φ and ϕ are uniformly bounded over $Q_{n,k}$, in which case we can show the following

THEOREM 4.1. *Under the assumptions* 4.1 *and* 4.2, *assume that the condition*

$$(4.14) \qquad \sup_{z_{(k)}\in Q_{n,k}} \max_{1\leqq i\leqq k} \sup_{z_i^*\in((z_i,\,l_{ni}))} \max\{|\varphi(z_i^*)|,\ \phi(z_i^*\,;\,l_{ni})\}\leqq M\ ,$$

is satisfied for some positive constant M uniformly for all n. Then, the condition

$$(4.15) \qquad k^2\Big/\min_{1\leqq i\leqq k+1}(n_i-n_{i-1})\to 0\ , \qquad (n\to\infty)\ ,$$

implies that

$$(4.16) \qquad X_{n(k)}\sim Y_{n(k)}\ (\boldsymbol{B})_d\ , \qquad (n\to\infty)\ .$$

PROOF. From (4.11) and (4.14) it follows that

$$\left|E^*\left[\sum_{i=1}^{k}\varphi(z_{ni}^{**})(z_i-l_{ni})\right]\right|\leqq M\sqrt{k^2/(n+2)}\ ,$$

and from (4.13) and (4.14)

$$E^*[w'_{n(k)}L_{n(k)}^{-1}w_{n(k)}]\leqq 12M^4k\Big/\min_{1\leqq i\leqq k+1}(n_i-n_{i-1})\ .$$

Hence, by (4.6) and (4.8), we obtain

$$(4.17) \qquad I^*(Z_{n(k)}:V_{n(k)})\leqq M\sqrt{\frac{k^2}{n+2}}+\sqrt{3}\,M^2\sqrt{\frac{k^2}{\min\limits_{1\leqq i\leqq k+1}(n_i-n_{i-1})}} + \frac{3}{2}M^4\frac{k}{\min\limits_{1\leqq i\leqq k+1}(n_i-n_{i-1})}\ .$$

Thus, the condition (4.15) implies the condition (4.6), and consequently (4.16), which proves the theorem.

It is not so easy in general to check whether the condition (4.14) is fulfilled or not. We therefore consider three simpler cases for which the condition (4.14) is satisfied.

Case I. k is fixed independently of n, and $l_{ni}\to\lambda_i$, $i=1,\cdots,k$, as $n\to\infty$ for some fixed $0<\lambda_1<\cdots<\lambda_k<1$.

In this case, it is evident that the condition (4.14) is fulfilled, and the condition (4.15) is automatically satisfied. Therefore, we can state the following

COROLLARY 4.1. *If k is fixed independently of n, and $l_{ni}\to\lambda_i$, $i=1,\cdots,k$, as $n\to\infty$, for some fixed $0<\lambda_1<\cdots<\lambda_k<1$, then, under the assumptions* 4.1 *and* 4.2, *$X_{n(k)}$ is asymptotically $(\boldsymbol{B})_d$ normally distributed according to $N(s_{n(k)}, S_{n(k)})$ as $n\to\infty$.*

Case II. $0<\gamma\leqq l_{n1}<\cdots<l_{nk}\leqq 1-\gamma<1$ for some fixed number γ independent of n.

In this case, since $|\varphi(z)|$ and $\phi(z;l)$ are uniformly bounded for all z and l such that $\gamma/2\leqq z$, $l\leqq 1-\gamma/2$, the condition (4.14) is satisfied, provided (3.4). Thus, we have the following

COROLLARY 4.2. *If the spacing of l_{ni}'s is such that*

$$0<\gamma\leqq l_{n1}<\cdots<l_{nk}\leqq 1-\gamma<1\ , \tag{4.18}$$

for some fixed number γ independent of n, then the condition (4.15) *implies* (4.16), provided the assumptions 4.1 and 4.2.

Case III. $f(x)\geqq M_1$ and $|f'(x)|\leqq M_2$ for some positive M_1 and M_2 independent of n.

In this case, $D(f)$ is necessarily bounded, and the situation is essentially the same as in Case II. Thus, the following corollary is an immediate consequence of the theorem.

COROLLARY 4.3. *Suppose that there exist positive numbers M_1 and M_2 which are independent of n such that $f(x)\geqq M_1$ and $|f'(x)|\leqq M_2$ uniformly for all x in $D(f)$, provided the assumptions* 4.1 *and* 4.2. *Then, the condition* (4.15) *implies* (4.16).

In the second half of this section, we shall consider the case where spacings are chosen first and then the corresponding sample quantiles. For this case, the argument is quite similar to that of the first half of this section, and so we state the results briefly.

Let $0<\lambda_{n1}<\cdots<\lambda_{nk}<1$ be a set of k spacings for each n, where k and λ_{ni}'s may vary with increasing n. The corresponding sample quantiles $X_{nn_1},\cdots,X_{nn_k}$ are chosen in the usual manner as $n_i=[n\lambda_{ni}]+1$, $i=1,\cdots,k$. As before, let $X_{n(k)}$ be their joint variable.

On the other hand, let $\tilde{Y}_{n(k)}=(Y_{n1},\cdots,Y_{nk})'$ be a normal random variable whose mean vector and dispersion matrix are given respectively by

$$\zeta_{n(k)}=(\zeta_{n1},\cdots,\zeta_{nk})' \tag{4.19}$$

and

$$(4.20)\quad \Sigma_{n(k)}=\frac{1}{n}\begin{bmatrix} \lambda_{n1}(1-\lambda_{n1})/\tilde{f}_{n1}^2 & \lambda_{n1}(1-\lambda_{n2})/\tilde{f}_{n1}\tilde{f}_{n2} & \cdots & \lambda_{n1}(1-\lambda_{nk})/\tilde{f}_{n1}\tilde{f}_{nk} \\ \lambda_{n1}(1-\lambda_{n2})/\tilde{f}_{n1}\tilde{f}_{n2} & \lambda_{n2}(1-\lambda_{n2})/\tilde{f}_{n2}^2 & \cdots & \lambda_{n2}(1-\lambda_{nk})/\tilde{f}_{n2}\tilde{f}_{nk} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{n1}(1-\lambda_{nk})/\tilde{f}_{n1}\tilde{f}_{nk} & \cdots & \cdots & \lambda_{nk}(1-\lambda_{nk})/\tilde{f}_{nk}^2 \end{bmatrix},$$

where $\zeta_{ni}=F^{-1}(\lambda_{ni})$ and $\tilde{f}_{ni}=f(\zeta_{ni})$, $i=1,\cdots,k$.

Then, analogously to Theorem 4.1, we can state the following

THEOREM 4.2. *Under the assumptions* 4.1 *and* 4.2, *suppose that the condition*

$$(4.21)\qquad \sup_{z_{(k)}\in\tilde{Q}_{n,k}}\ \max_{1\leqq i\leqq k}\ \sup_{z_i^*\in((z_i,\lambda_{ni}))}\max\{|\varphi(z_i^*)|,\ \phi(z_i^*;\lambda_{ni})\}\leqq M\,,$$

is satisfied for some positive number M independent of n, where

$$(4.22)\qquad \tilde{Q}_{n,k}=\{z_{(k)}:0<\lambda_{n1}-\tilde{\delta}_n<z_1<\cdots<z_k<\lambda_{nk}+\tilde{\delta}'_n<1\}\,,$$

with $\tilde{\delta}_n$ *and* $\tilde{\delta}'_n$ *defined analogously to* δ_n *and* δ'_n *in* (4.4) *by changing* l_{ni}*'s to* λ_{ni}*'s. Then, the condition*

$$(4.23)\qquad k^2\Big/\Big\{n\min_{1\leqq i\leqq k+1}(\lambda_{ni}-\lambda_{ni-1})\Big\}\to 0\,,\qquad (n\to\infty)\,,$$

implies that

$$(4.24)\qquad X_{n(k)}\sim\tilde{Y}_{n(k)}\ (\boldsymbol{B})_d\,,\qquad (n\to\infty)\,.$$

Corresponding to the three cases mentioned before, we can state the following

COROLLARY 4.4. (a) *If k and* λ_{ni}*'s are fixed independently of n, then, under the assumptions* 4.1 *and* 4.2, *the asymptotic equivalence* (4.24) *always holds.*

(b) *If there exists a positive number* γ *independent of n such that*

$$(4.25)\qquad 0<\gamma\leqq\lambda_{n1}<\cdots<\lambda_{nk}\leqq 1-\gamma<1$$

uniformly for all n, then under the assumptions 4.1 *and* 4.2 *the condition* (4.23) *implies* (4.24).

(c) *If* $f(x)\geqq M_1$ *and* $|f'(x)|\leqq M_2$ *uniformly for all x in* $D(f)$ *for some fixed positive numbers* M_1 *and* M_2, *then under the assumptions* 4.1 *and* 4.2, *the condition* (4.23) *implies* (4.24).

It should be noted, in the final place, that there are many variations of the theorems 4.1 and 4.2, which are obtained from those theorems by slightly changing the mean vector and the dispersion matrix of the asymptotic normal distribution of each theorem. Among those, we shall merely state the following theorem, whose proof can be made in a quite analogous manner to that of Theorem 4.1.

THEOREM 4.3. *Under the assumptions* 4.1 *and* 4.2, *suppose that the condition*

$$(4.26)\qquad \sup_{z_{(k)}\in Q^0_{n,k}} \max_{1\leq i\leq k} \sup_{z_i^*\in((z_i,\, l^0_{ni}))} \max\{|\varphi(z_i^*)|,\ \phi(z_i^*\,;\, l^0_{ni})\}\leqq M\ ,$$

is satisfied for some positive M uniformly for all n. Then, the condition (4.15) *implies that*

$$(4.27)\qquad X_{n(k)}\sim Y^0_{n(k)}\ (\boldsymbol{B})_d\ ,\qquad (n\to\infty)\ ,$$

where $Q^0_{n,k}$ *is the set obtained by the same definition as* (4.4) *but using* $l^0_{ni}=n_i/n$ *for* l_{ni}, $i=1,\cdots,k$, *and* $Y^0_{n(k)}$ *stands for a k-dimensional normal random variable with mean vector* $s^0_{n(k)}$ *and dispersion matrix* $S^0_{n(k)}$, $s^0_{n(k)}$ *and* $S^0_{n(k)}$ *being defined by*

$$(4.28)\qquad s^0_{n(k)}=(s^0_{n1},\cdots,s^0_{nk})'\ ,\qquad s^0_{ni}=F^{-1}(l^0_{ni})\ ,\quad i=1,\cdots,k\ ,$$

and

$$(4.29)\quad S^0_{n(k)}=\frac{1}{n}\begin{bmatrix} l^0_{n1}(1-l^0_{n1})/f^{02}_{n1} & l^0_{n1}(1-l^0_{n2})/f^0_{n1}f^0_{n2} & \cdots & l^0_{n1}(1-l^0_{nk})/f^0_{n1}f^0_{nk} \\ l^0_{n1}(1-l^0_{n2})/f^0_{n1}f^0_{n2} & l^0_{n2}(1-l^0_{n2})/f^{02}_{n2} & \cdots & l^0_{n2}(1-l^0_{nk})/f^0_{n2}f^0_{nk} \\ \cdots & \cdots & \cdots & \cdots \\ l^0_{n1}(1-l^0_{nk})/f^0_{n1}f^0_{nk} & \cdots & \cdots & l^0_{nk}(1-l^0_{nk})/f^{02}_{nk} \end{bmatrix}$$

with $f^0_{ni}=f(F^{-1}(l^0_{ni}))$, $i=1,\cdots,k$.

5. The case of unequal basic distributions

In this final section, we consider the case of unequal basic distributions.

Let $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics of a random sample of size n from a continuous distribution over the real line, whose pdf and cdf are given by $f_n(x)$ and $F_n(x)$ respectively. For any given $n_1<n_2<\cdots<n_k$, let $X^n_{n(k)}$ be the joint random variable of $X_{nn_1}<X_{nn_2}<\cdots<X_{nn_k}$, and let us put $l_{ni}=n_i/(n+1)$, $i=1,\cdots,k$, as before.

We shall make the following

ASSUMPTION 5.1. $D(f_n)=\{x:f_n(x)>0\}$ is an open interval on the real line, for each n.

ASSUMPTION 5.2. For each n, $f_n(x)$ is differentiable once and $f_n'(x)$ is continuous over $D(f_n)$.

Under these two assumptions, let $s_{ni}=F_n^{-1}(l_{ni})$ and $f_{ni}=f_n(s_{ni})$, $i=1,\cdots,k$. Further, let $s_{n(k)}$ and $S_{n(k)}$ be defined to be the same as (4.1) and (4.2) with the definitions of l_{ni}'s, s_{ni}'s and f_{ni}'s given above, and

$Y^n_{n(k)}$ be a k-dimensional normal random variable distributed as $N(s_{n(k)}, S_{n(k)})$.

Then, the argument of the preceding section, through which Theorem 4.1 was derived, is still valid in the present case, because the transformation t_s in Lemma 2.3 may be dependent on the underlying parameter s. Thus, we can state a result parallel to Theorem 4.1, for which we need more definitions: Let $Q_{n,k}$ be defined to be the same as (4.4) with l_{ni}'s given above, and put

$$\begin{aligned} &\varphi_n(z)=f_n'(F_n^{-1}(z))/f_n^2(F_n^{-1}(z)) \,, && (0<z<1) \,, \\ &\phi_n(z;l)=f_n(F_n^{-1}(l))/f_n(F_n^{-1}(z)) \,, && (0<z,\ l<1) \,, \end{aligned} \tag{5.1}$$

for each n.

THEOREM 5.1. *Under the assumptions* 5.1 *and* 5.2, *assume that the condition*

$$\sup_{z_{(k)}\in Q_{n,k}} \max_{1\leq i\leq k} \sup_{z_i^*\in((z_i,\, l_{ni}))} \max\{|\varphi_n(z_i^*)|,\, \phi_n(z_i^*;l_{ni})\}\leqq M \,, \tag{5.2}$$

is satisfied uniformly for all n, when M is some positive number independent of n. Then, the condition (4.15) *implies that*

$$X^n_{n(k)}\sim Y^n_{n(k)}\ (\boldsymbol{B})_d \,, \qquad (n\to\infty) \,. \tag{5.3}$$

Corresponding to Theorem 4.3, we also have the following result: Let $Y^{n0}_{n(k)}$ be a k-dimensional normal random variable with mean vector $s^0_{n(k)}$ and dispersion matrix $S^0_{n(k)}$, $s^0_{n(k)}$ and $S^0_{n(k)}$ being defined to be the same as in (4.28) and (4.29) with the definitions $l^0_{ni}=n_i/n$, $s^0_{ni}=F_n^{-1}(l^0_{ni})$ and $f^0_{ni}=f_n(s^0_{ni})$, $i=1,\cdots,k$. Then, we can state the following

THEOREM 5.2. *Under the assumptions* 5.1 *and* 5.2, *suppose that the condition*

$$\sup_{z_{(k)}\in \tilde{Q}_{n,k}} \max_{1\leq i\leq k} \sup_{z_i^*\in((z_i,\, l^0_{ni}))} \max\{|\varphi_n(z_i^*)|,\, \phi_n(z_i^*;l_{ni})\}\leqq M \tag{5.4}$$

is satisfied for some positive M uniformly for all n. Then, the condition (4.15) *implies that*

$$X^n_{n(k)}\sim Y^{n0}_{n(k)}\ (\boldsymbol{B})_d \,, \qquad (n\to\infty) \,. \tag{5.5}$$

We can also have a result corresponding to Theorem 4.2, which will be omitted. In general, it is quite difficult to check whether the condition (5.2) or (5.4) is valid or not, except for the case where the following assumption is fulfilled.

ASSUMPTION 5.3. For some positive numbers M_1 and M_2, (5.6) $$\inf_{x \in D(f_n)} f_n(x) \geqq M_1 \quad \text{and} \quad \sup_{x \in D(f_n)} |f_n'(x)| \leqq M_2$$

uniformly for all n.

Then, we have the following

COROLLARY 5.1. *Under the assumptions* 5.1, 5.2 *and* 5.3, *the condition* (4.15) *implies* (5.3) *and* (5.5).

It should be noted that this result implies the Weiss' [8]: Indeed, under the assumptions 5.1, 5.2 and 5.3, if we put $k=n^{1-\delta}-1$ and $n_i=in^\delta$, $i=1,\cdots,k$, (assuming these are integers), for any given δ such that $2/3<\delta<1$, then these k and n_i's satisfy the condition (4.15), and consequently, $X_{nn_1},\cdots,X_{nn_k}$ are jointly asymptotically $(\boldsymbol{B})_d$ normally distributed according to $N(s^0_{n(k)}, S^0_{n(k)})$ as $n\to\infty$.

It is also remarked that, if the basic distributions are uniform distributions and satisfy the Assumption 5.3, then the condition (3.4) implies (5.3) and (5.5).

REFERENCES

[1] Ikeda, S. (1963). Asymptotic equivalence of probability distributions with applications to some problems of asymptotic independence, *Ann. Inst. Statist. Math.*, **15**, 87-116.

[2] Ikeda, S. (1968). Asymptotic equivalence of real probability distributions, *Ann. Inst. Statist. Math.*, **20**, 339-362.

[3] Ikeda, S. and Matsunawa, T. (1970). On asymptotic independence of order statistics, *Ann. Inst. Statist. Math.*, **22**, 435-449.

[4] Matusita, K. (1966). Distance and related statistics in multivariate analysis, *Multivariate Analysis*, (Proc. on an Intern. Symp. held in Dayton, Ohio, June 14-19, 1965), 187-200.

[5] Mosteller, F. (1946). On some useful "inefficient" statistics, *Ann. Math. Statist.*, **17**, 377-407.

[6] Sarhan, A. E. and Greenberg, B. G. (ed.) (1962). *Contributions to Order Statistics*, John Wiley & Sons, New York.

[7] Walker, A. M. (1968). A note on the asymptotic distribution of sample quantiles, *Jour. Roy. Statist. Soc.* (B), **30**, 570-575.

[8] Weiss, L. (1969). The asymptotic joint distribution of an increasing number of sample quantiles, *Ann. Inst. Statist. Math.*, **21**, 257-263.