# ON THE DERIVATION OF THE ASYMPTOTIC DISTRIBUTION OF THE GENERALIZED HOTELLING'S $T_0^2$*

TAKESI HAYAKAWA

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## 1. Introduction and summary

Let $S_1$, $S_2$ be independent $m\times m$ matrices on $n_1$ and $n_2$ degrees of freedom, respectively. $S_2$ having a Wishart distribution and $S_1$ having a non-central Wishart distribution with the same covariance matrix. Hotelling's generalized $T_0^2$ statistics is defined as

$$(1) \qquad T=n_2^{-1}T_0^2=\mathrm{tr}\, S_1S_2^{-1}\,.$$

When $n_2$ becomes large, the distribution of $T_0^2$ approaches that of $\chi^2$ based on $mn_1$ degrees of freedom. Ito [7] and Siotani [10] have independently derived asymptotic expansions for the cumulative distribution function (c.d.f.) of $T_0^2$ in the central case by using the idea of a perturbation as in physics. The non-central distribution was treated by Siotani [11] in the same way and later by Ito [8], who used the integral representation of the characteristic function of $T_0^2$ due to Hsu [6]. But neither could obtain the term of order $n_2^{-2}$. Recently Siotani [12], Fujikoshi [3] and Yoong-Sin Lee [14] have derived the asymptotic form of the non-central distribution of $T_0^2$ up to order $n_2^{-2}$ from the expansion of its characteristic function or Laplace transform $\varphi(t)$. However, the function which is expanded does not converge for all $t$, but it only converges for some restricted region. Therefore, the expanded function is not a characteristic or Laplace transform of the original statistic in the strict sense.

The exact distribution of $T$ over the region $0\leqq T<1$ has been obtained in the general case by Constantine [1], using the method of zonal polynomials and generalized Laguerre polynomials of matrix argument developed by James [9]. Constantine's solution has the form

$$(2) \qquad f(T)=\frac{\Gamma_m((n_1+n_2)/2)}{\Gamma_m(n_2/2)\Gamma(mn_1/2)}T^{mn_1/2-1}\,\mathrm{etr}\,(-S)$$

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$$\times\sum_{k=0}^{\infty}\frac{(-T)^k}{k!(mn_1/2)_k}\sum_{\kappa}\left(\frac{n_1+n_2}{2}\right)_{\kappa}L_{\kappa}^{n_1/2-p}(S)\ , \qquad |T|<1\ ,$$

where

$$\Gamma_m(a)=\pi^{m(m-1)/4}\prod_{\alpha=1}^{m}\Gamma\left(a-\frac{\alpha-1}{2}\right)\ , \qquad p=\frac{1}{2}(m+1)\ ,$$

$$(a)_{\kappa}=\prod_{\alpha=1}^{m}\left(a-\frac{\alpha-1}{2}\right)_{k_{\alpha}}\ , \qquad (a)_n=a(a+1)\cdots(a+n-1)\ ,$$

and $L^{n_1/2-p}(S)$ is a generalized Laguerre polynomial of matrix argument $S$, corresponding to a partition $\kappa$ of $k$ into not more than $m$ parts. Davis [2] suggested that the asymptotic expansion of the c.d.f. of $T_0^2$ can be obtained from (2) by changing the argument to $T=n_2^{-1}T_0^2$ and integrate term by term. In this paper, we give the exact bound of the convergence, $0\leqq T_0^2<n_2$, of the asymptotic form in terms of the probability density function (p.d.f.) of $T_0^2$ up to order $n_2^{-2}$ by preparing some formulas for the generalized Laguerre polynomials of matrix argument and for univariate Laguerre polynomials.

## 2. Some useful formulas for generalized Laguerre polynomials and the univariate Laguerre polynomials

Constantine defined the generalized Laguerre polynomial $L_{\kappa}^{\gamma}(S)$, ($\gamma>-1$), by the Hankel transform of matrix argument in the following way.

Let $S$ and $R$ be positive definite symmetric matrices of order $m$. Then $L_{\kappa}^{\gamma}(S)$ is defined by

$$\text{(3)} \qquad \operatorname{etr}(-S)L_{\kappa}^{\gamma}(S)=\int_{R>0}A_{\gamma}(RS)(\det R)^{\gamma}\operatorname{etr}(-R)\,C_{\kappa}(R)dR\ ,$$

where $A_{\gamma}(RS)$ is a Bessel function of matrix argument and $C_{\kappa}(R)$ is a zonal polynomial corresponding to a partition $\kappa$ of $k$ into not more than $m$ parts.

The following lemmas are very important for our argument.

LEMMA 1. (*Hayakawa* [3])

$$\text{(4)} \qquad \sum_{\kappa}L_{\kappa}^{n/2-p}(S)=L_{k}^{mn/2-1}(\operatorname{tr}S)\ , \qquad p=\frac{1}{2}(m+1)\ ,$$

*where the left-hand side (L.H.S.) is a summation over all partitions $\kappa$ of $k$ into not more than $m$ parts and the right-hand side (R.H.S.) is a univariate Laguerre polynomial.*

LEMMA 2. (*Sugiura and Fujikoshi* [12]) *Let $C_{\kappa}(S)$ be the zonal* *polynomial of degree $k$ corresponding to a partition $\kappa=\{k_1,\cdots,k_m\}$ of $k$ ($k_1\geqq k_2\geqq\cdots\geqq k_m\geqq 0$) for an $m\times m$ positive definite matrix $S$. Put*

$$a_1(\kappa)=\sum_{\alpha=1}^{m} k_\alpha(k_\alpha-\alpha)\ , \tag{5}$$

$$a_2(\kappa)=\sum_{\alpha=1}^{m} k_\alpha(4k_\alpha^2-6\alpha k_\alpha+3\alpha^2)\ . \tag{6}$$

*Then the following equalities hold*

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa} a_1(\kappa)C_\kappa(S)=x^2\operatorname{tr}S^2\operatorname{etr}(xS)\ , \tag{7}$$

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa} a_2(\kappa)C_\kappa(S)=[4x^3(\operatorname{tr}S^3)+3x^2\operatorname{tr}S^2+3x^2(\operatorname{tr}S)^2+x\operatorname{tr}S]\operatorname{etr}(xS)\ , \tag{8}$$

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa} a_1^2(\kappa)C_\kappa(S)=[x^4(\operatorname{tr}S^2)^2+4x^3\operatorname{tr}S^3+x^2\{\operatorname{tr}S^2+(\operatorname{tr}S)^2\}]\operatorname{etr}(xS)\ . \tag{9}$$

We can give a similar set of results for Laguerre polynomial $L_\kappa^{n/2-p}(S)$.

LEMMA 3. *Let $L_\kappa^{n/2-p}(S)$ be a generalized Laguerre polynomial corresponding to a partition $\kappa=\{k_1,\cdots,k_m\}$ of $k$ ($k_1\geqq\cdots\geqq k_m\geqq 0$) for an $m\times m$ positive definite matrix $S$, and $a_1(\kappa)$ and $a_2(\kappa)$ are given by* (5) *and* (6). *Then*

$$\begin{aligned}\sum_{\kappa} a_1(\kappa)L_\kappa^{n/2-p}(S)&=k(k-1)\Big[\frac{mn(n+m+1)}{4}L_{k-2}^{mn/2+1}(\operatorname{tr}S)-(n+m+1)\\&\quad\times\operatorname{tr}S\,L_{k-2}^{mn/2+2}(\operatorname{tr}S)+\operatorname{tr}S^2\,L_{k-2}^{mn/2+3}(\operatorname{tr}S)\Big]\ ,\end{aligned} \tag{10}$$

$$\begin{aligned}\sum_{\kappa} a_2(\kappa)L_\kappa^{n/2-p}(S)&=k\Big[\frac{mn}{2}L_{k-1}^{mn/2}(\operatorname{tr}S)-\operatorname{tr}S\,L_{k-1}^{mn/2+1}(\operatorname{tr}S)\Big]\\&\quad+3k(k-1)\Big[\frac{mn\{(m+1)n+m+3\}}{4}L_{k-2}^{mn/2+1}(\operatorname{tr}S)\\&\quad-\{(m+1)n+m+3\}\operatorname{tr}S\,L_{k-2}^{mn/2+2}(\operatorname{tr}S)\\&\quad+\{(\operatorname{tr}S)^2+\operatorname{tr}S^2\}L_{k-2}^{mn/2+3}(\operatorname{tr}S)\Big]\\&\quad+4k(k-1)(k-2)\Big[\frac{mn\{n^2+3(m+1)n+m^2+3m+4\}}{8}\end{aligned} \tag{11}$$

$$\times L_{k-3}^{mn/2+2}(\mathrm{tr}\, S)-\frac{3}{4}\{n^2+3(m+1)n+m^2+3m+4\}$$

$$\times \mathrm{tr}\, S\, L_{k-3}^{mn/2+3}(\mathrm{tr}\, S)+\frac{3}{2}\{(\mathrm{tr}\, S)^2+(n+m+2)\,\mathrm{tr}\, S^2\}$$

$$\times L_{k-3}^{mn/2+4}(\mathrm{tr}\, S)-\mathrm{tr}\, S^3\, L_{k-3}^{mn/2+5}(\mathrm{tr}\, S)\Big] .$$

(12) $\sum_\kappa a_1^2(\kappa)L_\kappa^{n/2-p}(S)$

$$=k(k-1)\Big[\frac{mn\{(m+1)n+m+3\}}{4}L_{k-2}^{mn/2+1}(\mathrm{tr}\, S)$$

$$-\{(m+1)n+m+3\}(\mathrm{tr}\, S)L_{k-2}^{mn/2+2}(\mathrm{tr}\, S)$$

$$+\{(\mathrm{tr}\, S)^2+\mathrm{tr}\, S^2\}L_{k-2}^{mn/2+3}(\mathrm{tr}\, S)\Big]$$

$$+4k(k-1)(k-2)\Big[\frac{mn}{8}\{n^2+3(m+1)n+m^2+3m+4\}$$

$$\times L_{k-3}^{mn/2+2}(\mathrm{tr}\, S)-\frac{3}{4}\{n^2+3(m+1)n+m^2+3m+4\}$$

$$\times \mathrm{tr}\, S\, L_{k-3}^{mn/2+3}(\mathrm{tr}\, S)+\frac{3}{2}\{(\mathrm{tr}\, S)^2+(n+m+2)\,\mathrm{tr}\, S^2\}$$

$$\times L_{k-3}^{mn/2+4}(\mathrm{tr}\, S)-\mathrm{tr}\, S^3\, L_{k-3}^{mn/2+5}(\mathrm{tr}\, S)\Big]$$

$$+k(k-1)(k-2)(k-3)\Big[\frac{mn}{16}\{mn^3+2(m^2+m+4)n^2$$

$$+(m^3+2m^2+21m+20)n+4(2m^2+5m+5)\}L_{k-4}^{mn/2+3}(\mathrm{tr}\, S)$$

$$-\frac{1}{2}\{mn^3+2(m^2+m+4)n^2+(m^3+2m^2+21m+20)n$$

$$+4(2m^2+5m+5)\}\,\mathrm{tr}\, S\, L_{k-4}^{mn/2+4}(\mathrm{tr}\, S)$$

$$+\frac{1}{2}\{2[(m+n+1)^2+6](\mathrm{tr}\, S)^2+[(mn+20)(n+m+1)$$

$$+12]\,\mathrm{tr}\, S^2\}L_{k-4}^{mn/2+5}(\mathrm{tr}\, S)-\{2(n+m+1)\,\mathrm{tr}\, S\,\mathrm{tr}\, S^2$$

$$+8\,\mathrm{tr}\, S^3\}L_{k-4}^{mn/2+6}(\mathrm{tr}\, S)+(\mathrm{tr}\, S^2)^2L_{k-4}^{mn/2+7}(\mathrm{tr}\, S)\Big] .$$

PROOF. From the definition of $L_\kappa^{n/2-p}(S)$ and Lemma 2, (7), the generating function of $\sum_\kappa a_1(\kappa)L_\kappa^{n/2-p}(S)$ is given by

$$\mathrm{etr}\,(-S)\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_\kappa a_1(\kappa)L_\kappa^{n/2-p}(S)\ , \qquad |x|<1\ .$$

$$=\int_{R>0}A_{n/2-p}(RS)(\det R)^{n/2-p}\,\mathrm{etr}\,(-R)\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_\kappa a_1(\kappa)C_\kappa(R)dR$$

$$=x^2\int_{R>0} A_{n/2-p}(RS)(\det R)^{n/2-p}\operatorname{etr}(-(1-x)R)\operatorname{tr} R^2\, dR$$
$$=\frac{x^2}{(1-x)^{mn/2+2}}\int_{R>0} A_{n/2-p}\left(\frac{S}{1-x}R\right)(\det R)^{n/2-p}\operatorname{etr}(-R)$$
$$\times\left\{C_{(2)}(R)-\frac{1}{2}C_{(1^2)}(R)\right\}dR\ .$$

Hence by the definition of $L_\kappa^{n/2-p}(S)$, again, the R.H.S. is

$$\frac{x^2}{(1-x)^{mn/2+2}}\operatorname{etr}\left(-\frac{S}{1-x}\right)\left\{L_{(2)}^{n/2-p}\left(\frac{S}{1-x}\right)-\frac{1}{2}L_{(1^2)}^{n/2-p}\left(\frac{S}{1-x}\right)\right\}\ .$$

Therefore, we have

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa} a_1(\kappa)L_\kappa^{n/2-p}(S) \tag{13}$$
$$=x^2(1-x)^{-mn/2-2}\exp\left(-\frac{x}{1-x}\operatorname{tr} S\right)$$
$$\times\left[L_{(2)}^{n/2-p}\left(\frac{S}{1-x}\right)-\frac{1}{2}L_{(1^2)}^{n/2-p}\left(\frac{S}{1-x}\right)\right]\ .$$

Since from the tables of Constantine [1],

$$L_{(2)}^{n/2-p}(S)-\frac{1}{2}L_{(1^2)}^{n/2-p}(S)=\frac{mn(n+m+1)}{4}-(n+m+1)\operatorname{tr} S+\operatorname{tr} S^2\ ,$$

the R.H.S. of (13) becomes

$$x^2\exp\left(-\frac{x}{1-x}\operatorname{tr} S\right)\left[\frac{mn(n+m+1)}{4}(1-x)^{-mn/2-2}\right.$$
$$\left.-(n+m+1)\operatorname{tr} S(1-x)^{-mn/2-3}+\operatorname{tr} S^2(1-x)^{-mn/2-4}\right]\ .$$

Hence by the use of the generating function of a univariate Laguerre polynomial, i.e.

$$(1-x)^{-\alpha-1}\exp\left(-\frac{x}{1-x}z\right)=\sum_{k=0}^{\infty}\frac{x^k}{k!}L_k^\alpha(z)\ ,\qquad (|x|<1)\ ,$$

we can expand the R.H.S. of (13) as a power series in $x$, and comparing the coefficient of $x^k$ on both sides of (13), we have (10).

The proofs of (11) and (12) are done completely the same way as the one of (10). In these cases, as we need the explicit forms of Laguerre polynomials corresponding to $\operatorname{tr} R$, $\operatorname{tr} R^2+(\operatorname{tr} R)^2$, $\operatorname{tr} R^3$ and $(\operatorname{tr} R^2)^2$, we write them here and will omit the details of the proof.

$$L_{(1)}^{n/2-p}(S)=\frac{mn}{2}-\operatorname{tr} S\ . \tag{14}$$

$$(15)\quad 2L_{(2)}^{n/2-p}(S)+\frac{1}{2}L_{(1^2)}^{n/2-p}(S)$$
$$=\frac{mn\{(m+1)n+m+3\}}{4}-\{(m+1)n+m+3\}\operatorname{tr}S$$
$$+\operatorname{tr}S^2+(\operatorname{tr}S)^2\,.$$

$$(16)\quad L_{(3)}^{n/2-p}(S)-\frac{1}{4}L_{(21)}^{n/2-p}(S)+\frac{1}{4}L_{(1^3)}^{n/2-p}(S)$$
$$=\frac{mn}{8}\{n^2+3(m+1)n+m^2+3m+4\}-\frac{3}{4}\{n^2+3(m+1)n$$
$$+m^2+3m+4\}\operatorname{tr}S+\frac{3}{2}\{(\operatorname{tr}S)^2+(n+m+2)\operatorname{tr}S^2\}-\operatorname{tr}S^3\,.$$

$$(17)\quad L_{(4)}^{n/2-p}(S)-\frac{1}{6}L_{(31)}^{n/2-p}(S)+\frac{7}{12}L_{(2^2)}^{n/2-p}(S)$$
$$-\frac{1}{6}L_{(21^2)}^{n/2-p}(S)+\frac{1}{4}L_{(1^4)}^{n/2-p}(S)$$
$$=\frac{mn}{16}[mn^3+2(m^2+m+4)n^2+(m^3+2m^2+21m+20)n$$
$$+4(2m^2+5m+5)]$$
$$-\frac{1}{2}[mn^3+2(m^2+m+1)n^2+(m^3+2m^2+21m+20)n$$
$$+4(2m^2+5m+5)]\operatorname{tr}S$$
$$+\frac{1}{2}\{2[(n+m+1)^2+6](\operatorname{tr}S)^2+[(mn+20)(n+m+1)+12]$$
$$\times\operatorname{tr}S^2\}-2\{(n+m+1)\operatorname{tr}S\operatorname{tr}S^2+8\operatorname{tr}S^3\}+(\operatorname{tr}S^2)^2\,.$$

LEMMA 4. *Put*

$$h_{2\alpha+2j}(x,z)=h_{2\alpha+2j}=\exp\left(-\frac{x}{2}\right)\frac{x^{\alpha+j-1}}{2^{\alpha+j}\Gamma(\alpha+j)}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+j)_k}\,.$$

*and put*

$$g_{2\alpha+2j}(x,z)=g_{2\alpha+2j}=\exp(-z)h_{2\alpha+2j}(x,z)\,,$$

*i.e.,* $g_{2\alpha+2j}(x,z)$ *is the probability density function of non-central* $\chi^2$ *distribution with* $2\alpha+2j$ *degrees of freedom and non-centrality parameter* $z$*, and, for convenience*

$$(\alpha)(x)=\frac{x^{\alpha-1}}{2^\alpha\Gamma(\alpha)}\,.$$

*Then the following equalities hold*

$$(20)\qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha-1}(z)}{k!(\alpha)_k}=h_{2\alpha}$$

$$(21)\qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha}(z)}{k!(\alpha)_k}=-h_{2\alpha+2}+h_{2\alpha}$$

$$(22)\qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{k!(\alpha)_k}=h_{2\alpha+4}-2h_{2\alpha+2}+h_{2\alpha}$$

$$(23)\qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+2}(z)}{k!(\alpha)_k}=-h_{2\alpha+6}+3h_{2\alpha+4}-3h_{2\alpha+2}+h_{2\alpha}$$

$$(24)\qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+3}(z)}{k!(\alpha)_k}=h_{2\alpha+8}-4h_{2\alpha+6}+6h_{2\alpha+4}-4h_{2\alpha+2}+h_{2\alpha}$$

$$(25)\qquad (\alpha)(x)\sum_{k=1}^{\infty}\frac{(-x/2)^k L_k^{\alpha-1}(z)}{(k-1)!(\alpha)_k}=-zh_{2\alpha+4}+(z-\alpha)h_{2\alpha+2}$$

$$(26)\qquad (\alpha)(x)\sum_{k=2}^{\infty}\frac{(-x/2)^k L_k^{\alpha-1}(z)}{(k-2)!(\alpha)_k}=z^2h_{2\alpha+8}+2\{(\alpha+1)z-z^2\}h_{2\alpha+6}$$
$$+\{z^2-2(\alpha+1)z+\alpha(\alpha+1)\}h_{2\alpha+4}$$

$$(27)\qquad (\alpha)(x)\sum_{k=1}^{\infty}\frac{(-x/2)^k L_k^{\alpha}(z)}{(k-1)!(\alpha)_k}=zh_{2\alpha+6}-(2z-\alpha-1)h_{2\alpha+4}+(z-\alpha-1)h_{2\alpha+2}$$

$$(28)\qquad (\alpha)(x)\sum_{k=1}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{(k-1)!(\alpha)_k}=-zh_{2\alpha+8}+(3z-\alpha-2)h_{2\alpha+6}$$
$$-(3z-2\alpha-4)h_{2\alpha+4}+(z-\alpha-2)h_{2\alpha+2}\,.$$

PROOF. (20) is another type of generating function for $L_k^{\alpha-1}(z)$. We can show (21), (22), (23) and (24) in the same way. We prove here as an example, (22).

$$\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{k!(\alpha)_k}=\frac{1}{(\alpha)_2}\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{k!(\alpha+2)_k}\{k(k-1)+2(\alpha+1)k+\alpha(\alpha+1)\}$$
$$=\frac{1}{(\alpha)_2}\sum_{k=2}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{(k-2)!(\alpha+2)_k}$$
$$+\frac{2(\alpha+1)}{(\alpha)_2}\sum_{k=1}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{(k-1)!(\alpha+2)_k}+\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{k!(\alpha+2)_k}\,.$$

Hence the third term is obtained from (20) by replacing $\alpha-1$ to $\alpha+1$.

$$(29)\qquad \sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{k!(\alpha+2)_k}=\exp\left(-\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+2)_k}\,.$$

By differentiating both sides of (29) with respect to $x$ and multiplying both sides by $x$, then

$$\sum_{k=1}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{(k-1)!(\alpha+2)_k}=\exp\left(-\frac{x}{2}\right)\left[-\frac{x}{2}\sum_{k=1}^{\infty}\frac{(xz/2)^k}{k!(\alpha+2)_k}+\sum_{k=1}^{\infty}\frac{(xz/2)^k}{(k-1)!(\alpha+2)_k}\right].$$

Thus, the second term is obtained.

By differentiating both sides of (29) twice with respect to $x$ and multiplying both sides by $x^2$, then

$$\sum_{k=2}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{(k-2)!(\alpha+2)_k}=\exp\left(-\frac{x}{2}\right)\left[\frac{x^2}{2^2}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+2)_k}\right.$$
$$\left.-2\cdot\frac{x}{2}\sum_{k=1}^{\infty}\frac{(xz/2)^k}{(k-1)!(\alpha+2)_k}+\sum_{k=2}^{\infty}\frac{(xz/2)^k}{(k-2)!(\alpha+2)_k}\right].$$

Adding these results, we have

$$\exp\left(-\frac{x}{2}\right)\left[\frac{x^2}{2^2(\alpha)_2}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+2)_k}-2\frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{(k-1)!(\alpha+2)_k}\left\{\frac{1}{\alpha+1}+\frac{1}{k}\right\}\right.$$
$$\left.+\sum_{k=2}^{\infty}\frac{(xz/2)^k}{(k-2)!(\alpha+2)_k}\left\{\frac{1}{(\alpha)_2}+2\frac{1}{(k-1)\alpha}+\frac{1}{k(k-1)}\right\}\right]$$
$$=\exp\left(-\frac{x}{2}\right)\left[\frac{x^2}{2^2(\alpha)_2}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+2)_k}-2\frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+1)_k}+\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha)_k}\right].$$

Hence we have (22) by multiplying both sides by $(\alpha)(x)$.

Formulas (25), (26), (27) and (28) can be obtained in the same way. As an example, we prove (27). From (21),

$$\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha}(z)}{k!(\alpha)_k}=\left[-\frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+1)_k}+\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha)_k}\right]\exp\left(-\frac{x}{2}\right).$$

By differentiating both sides of the above equation with respect to $x$ and multiplying both side by $x$, we have

$$\sum_{k=1}^{\infty}\frac{(-x/2)^k L_k^{\alpha}(z)}{(k-1)!(\alpha)_k}=\exp\left(-\frac{x}{2}\right)\left[\frac{x^2}{2^2\alpha}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+1)_k}-\frac{x}{2}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha)_k}\right.$$
$$-\frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+1)_k}-\frac{x}{2\alpha}\sum_{k=1}^{\infty}\frac{(xz/2)^k}{(k-1)!(\alpha+1)_k}$$
$$\left.+\sum_{k=1}^{\infty}\frac{(xz/2)^k}{(k-1)!(\alpha)_k}\right].$$

Since

$$\frac{x^2}{2^2\alpha}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+1)_k}=(\alpha+1)\frac{x^2}{2^2(\alpha)_2}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+2)_k}+z\frac{x^3}{2^3(\alpha)_3}\sum_{k=1}^{\infty}\frac{(xz/2)^{k-1}}{(k-1)!(\alpha+3)_{k-1}},$$

$$\frac{x}{2}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha)_k}=\alpha\frac{x}{2\alpha}\sum_{k=0}^{\infty}\frac{(xz/2)^k}{k!(\alpha+1)_k}+z\frac{x^2}{2^2(\alpha)_2}\sum_{k=1}^{\infty}\frac{(xz/2)^{k-1}}{(k-1)!(\alpha+2)_{k-1}},$$

(27) follows by simple calculation.

*Note.* In Section 3, we sometimes need the following type of summations. For example,

$$\frac{x^{\alpha-1}}{2^\alpha\Gamma(\alpha)}\sum_{k=4}^{\infty}\frac{(-x/2)^k L_{k-4}^{\alpha+5}(z)}{(k-4)!(\alpha)_k}\,,\qquad \frac{x^{\alpha-1}}{2^\alpha\Gamma(\alpha)}\sum_{k=3}^{\infty}\frac{(-x/2)^k L_{k-2}^{\alpha+3}(z)}{(k-3)!(\alpha)_k}\,,\quad \text{etc.}$$

The first sum is obtained in the following way.

$$\begin{aligned}\frac{x^{\alpha+3}}{2^{\alpha+4}\Gamma(\alpha+4)}\sum_{k=4}^{\infty}\frac{(-x/2)^{k-4}L_{k-4}^{\alpha+5}(z)}{(k-4)!(\alpha+4)_{k-4}}&=(\alpha+4)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+5}(z)}{k!(\alpha+4)_k}\\&=h_{2\alpha+12}-2h_{2\alpha+10}+h_{2\alpha+8}\,.\end{aligned}$$

The second sum is obtained in the following way.

$$\begin{aligned}\frac{x^{\alpha+1}}{2^{\alpha+2}\Gamma(\alpha+2)}\sum_{k=3}^{\infty}\frac{(-x/2)^k L_{k-2}^{\alpha+3}(z)}{(k-3)!(\alpha+2)_{k-2}}&=(\alpha+2)(x)\sum_{k=1}^{\infty}\frac{(-x/2)^k L_k^{\alpha+3}(z)}{(k-1)!(\alpha+2)_k}\\&=-zh_{2\alpha+12}+(3z-\alpha-4)h_{2\alpha+10}\\&\quad-(3z-2\alpha-8)h_{2\alpha+8}+(z-\alpha-4)h_{2\alpha+6}\,.\end{aligned}$$

## 3. Derivation of the asymptotic probability density function of $T_0^2$

In this section, we denote $n_1$ by $n$.
Let us write

$$x=n_2T=T_0^2\,. \tag{30}$$

Then the p.d.f. $f(x)$ of $x=T_0^2$ is represented by (31),

$$\begin{aligned}&\frac{n_2^{-mn/2}\Gamma_m((n+n_2)/2)}{\Gamma_m(n_2/2)\Gamma(mn/2)}\operatorname{etr}(-S)x^{mn/2-1}\\&\quad\times\sum_{k=0}^{\infty}\frac{1}{k!(mn/2)_k}\left(-\frac{x}{n_2}\right)^k\sum_{\kappa}\left(\frac{n+n_2}{2}\right)_{\kappa}L_{\kappa}^{n/2-p}(S)\,.\end{aligned}\tag{31}$$

This series is convergent for $|x|<n_2$.

Using a Stirling-type asymptotic expansion, we have

$$\begin{aligned}\frac{n_2^{-mn/2}\Gamma_m((n+n_2)/2)}{\Gamma_m(n_2/2)\Gamma(mn/2)}=\frac{1}{2^{mn/2}\Gamma(mn/2)}\Big[&1+\frac{mn}{4n_2}(n-m-1)+\frac{mn}{96n_2^2}\\&\times\{3m^3n-2m^2(3n^2-3n+4)\\&+3m(n^3-2n^2+5n-4)-8n^2+12n+4\}\\&+O(1/n_2^3)\Big]\,,\end{aligned}\tag{32}$$

$$\left(\frac{n+n_2}{2}\right)_{\kappa}=\left(\frac{n_2}{2}\right)^k\Big[1+\frac{1}{n_2}\{a_1(\kappa)+nk\} \tag{33}$$

$$+\frac{1}{6n_2^2}\{3a_1^2(\kappa)-a_2(\kappa)+6n(k-1)a_1(\kappa)+3n^2k(k-1)+k\}$$
$$+O(1/n_2^3)\Big]\,.$$

Therefore, by inserting (32) and (33) into (31), we obtain the asymptotic expansion of the p.d.f. of $x$ up to order 2.

$$\text{(34)}\qquad \operatorname{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\sum_{k=0}^{\infty}\frac{1}{k!(mn/2)_k}\left(-\frac{x}{2}\right)^k\Big[\sum_{\kappa}L_{\kappa}^{n/2-p}(S)$$
$$+\frac{1}{4n_2}\Big\{\sum_{\kappa}\{mn(n-m-1)+4(a_1(\kappa)+nk)\}L_{\kappa}^{n/2-p}(S)\Big\}$$
$$+\frac{1}{96n_2^2}\Big\{\sum_{\kappa}\{mn\{3m^3n-2m^2(3n^2-3n+4)$$
$$+3m(n^3-2n^2+5n-4)-8n^2+12n+4\}+24mn(n-m-1)$$
$$\times\{a_1(\kappa)+nk\}+16\{3a_1^2(\kappa)-a_2(\kappa)+6n(k-1)a_1(\kappa)$$
$$+3n^2k(k-1)+k\}L_{\kappa}^{n/2-p}(S)\Big\}+O(1/n_2^3)\Big]\,.$$

The next problem is to calculate each term by using the previous lemmas in Section 2.

(i) *The first term* is obvious from (20) and Lemma 1.

$$\text{(35)}\qquad \operatorname{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\sum_{k=0}^{\infty}\frac{(-x/2)^k}{k!(mn/2)_k}\sum_{\kappa}L_{\kappa}^{n/2-p}(S)$$
$$=\operatorname{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\exp\left(-\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{(x\operatorname{tr}S/2)^k}{k!(mn/2)_k}$$
$$=g_{mn}(x;\operatorname{tr}S)\,.$$

(ii) *The term of order* $1/n_2$:

From (10), (20), (21) and (22),

$$\text{(36)}\qquad \operatorname{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\sum_{k=0}^{\infty}\frac{(-x/2)^k}{k!(mn/2)_k}\sum_{\kappa}a_1(\kappa)L_{\kappa}^{n/2-p}(S)$$
$$=\frac{1}{4}\{mn(n+m+1)-4(n+m+1)\operatorname{tr}S+4\operatorname{tr}S^2\}g_{mn+4}$$
$$+\{(n+m+1)\operatorname{tr}S-2\operatorname{tr}S^2\}g_{mn+6}+\operatorname{tr}S^2g_{mn+8}\,,$$

and from Lemma 4, (25),

$$\text{(37)}\qquad \operatorname{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\sum_{k=0}^{\infty}\frac{(-x/2)^k}{k!(mn/2)_k}\sum_{\kappa}kL_{\kappa}^{n/2-p}(S)$$
$$=-\operatorname{tr}Sg_{mn+4}+\left(\operatorname{tr}S-\frac{mn}{2}\right)g_{mn+2}\,.$$

Hence, by combining these results, we have the term of order 1.

$$
\begin{aligned}
(38)\qquad A_1=&mn(n-m-1)g_{mn}-2n(mn-2\operatorname{tr} S)g_{mn+2}\\
&+\{mn(n+m+1)-4(2n+m+1)\operatorname{tr} S+4\operatorname{tr} S^2\}g_{mn+4}\\
&+4\{(n+m+1)\operatorname{tr} S-2\operatorname{tr} S^2\}g_{mn+6}+4\operatorname{tr} S^2 g_{mn+8}\,.
\end{aligned}
$$

(iii) *The term of order* 2:

From Lemma 3, Lemma 4 and Note, we have the following formulas.

$$
\begin{aligned}
(39)\qquad &\operatorname{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\sum_{k=0}^{\infty}\frac{(-x/2)^k}{k!(mn/2)_k}\sum_{\kappa} a_1^2(\kappa)L_\kappa^{n/2-p}(S)\\
&=\frac{1}{4}[mn\{(m+1)n+m+3\}-4\{(m+1)n+m+3\}\operatorname{tr} S\\
&\quad+4\operatorname{tr} S^2+4(\operatorname{tr} S)^2]g_{mn+4}-\frac{1}{2}[mn\{n^2+3(m+1)n+m^2\\
&\quad+3m+4\}-2\{3n^2+10(m+1)n+3m^2+10m+15\}\operatorname{tr} S\\
&\quad+4\{4(\operatorname{tr} S)^2+(3m+3n+7)\operatorname{tr} S^2\}-8\operatorname{tr} S^3]g_{mn+6}\\
&+\frac{1}{16}[mn\{mn^3+2(m^2+m+4)n^2+(m^3+2m^2+20m+21)n\\
&\quad+4(2m^2+5m+5)\}-8\{mn^3+2(m^2+m+7)n^2\\
&\quad+(m^3+2m^2+39m+38)n+2(7m^2+19m+22)\}\operatorname{tr} S\\
&\quad+16\{n^2+2(m+1)n+m^2+2m+20\}(\operatorname{tr} S)^2\\
&\quad+8\{mn^2+(m^2+m+44)n+44m+82\}\operatorname{tr} S^2\\
&\quad-32\{(n+m+1)\operatorname{tr} S\operatorname{tr} S^2+10\operatorname{tr} S^3\}+16(\operatorname{tr} S^2)^2]g_{mn+8}\\
&+\frac{1}{2}[\{mn^3+2(m^2+m+4)n^2+(m^3+2m^2+21m+20)n\\
&\quad+4(2m^2+5m+5)\}\operatorname{tr} S-4\{n^2+2(m+1)n+m^2+2m+10\}\\
&\quad\times(\operatorname{tr} S)^2-2\{mn^2+(m^2+m+26)n+26m+44\}\operatorname{tr} S^2\\
&\quad+12\{(m+n+1)\operatorname{tr} S\operatorname{tr} S^2+6\operatorname{tr} S^3\}-8(\operatorname{tr} S^2)^2]g_{mn+10}\\
&+\frac{1}{2}[2\{n^2+2(m+1)n+m^2+2m+7\}(\operatorname{tr} S)^2\\
&\quad+\{mn^2+(m^2+m+20)n+20m+32\}\operatorname{tr} S^2\\
&\quad-4\{3(n+m+1)\operatorname{tr} S\operatorname{tr} S^2+14\operatorname{tr} S^3\}+12(\operatorname{tr} S^2)^2]g_{mn+12}\\
&+2[(n+m+1)\operatorname{tr} S\operatorname{tr} S^2+4\operatorname{tr} S^3-2(\operatorname{tr} S^2)^2]g_{mn+14}\\
&+(\operatorname{tr} S^2)^2 g_{mn+16}\,.
\end{aligned}
$$

$$
\begin{aligned}
(40)\qquad &\operatorname{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\sum_{k=0}^{\infty}\frac{(-x/2)^k}{k!(mn/2)_k}\sum_{\kappa} a_2(\kappa)L_\kappa^{n/2-p}(S)\\
&=-\frac{1}{2}\{mn-2\operatorname{tr} S\}g_{mn+2}+\frac{1}{4}[3mn\{(m+1)n+m+3\}
\end{aligned}
$$

$$-4\{3(m+1)n+3m+10\}\ \mathrm{tr}\ S+12\{(\mathrm{tr}\ S)^2+\mathrm{tr}\ S^2\}]g_{mn+4}$$
$$-\frac{1}{2}[mn\{n^2+3(m+1)n+m^2+3m+4\}-6\{n^2+4(m+1)n$$
$$+m^2+4m+7\}\ \mathrm{tr}\ S+12\{2(\mathrm{tr}\ S)^2+(n+m+3)\ \mathrm{tr}\ S^2\}$$
$$-8\ \mathrm{tr}\ S^3]g_{mn+6}-3[\{n^2+3(m+1)n+m^2+3m+4\}\ \mathrm{tr}\ S$$
$$-\{5(\mathrm{tr}\ S)^2+(4m+4n+9)\ \mathrm{tr}\ S^2\}+4\ \mathrm{tr}\ S^3]g_{mn+8}$$
$$-6[(\mathrm{tr}\ S)^2+(n+m+2)\ \mathrm{tr}\ S^2-2\ \mathrm{tr}\ S^3]g_{mn+10}-4\ \mathrm{tr}\ S^3 g_{mn+12}\ .$$

(41)
$$\mathrm{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\sum_{k=0}^{\infty}\frac{(-x/2)^k}{k!(mn/2)_k}k(k-1)\sum_{\kappa}L_\kappa^{n/2-p}(S)$$
$$=(\mathrm{tr}\ S)^2 g_{mn+8}+\{(mn+2)\ \mathrm{tr}\ S-2(\mathrm{tr}\ S)^2\}g_{mn+6}$$
$$+\frac{1}{4}\{mn(mn+2)-4(mn+2)\ \mathrm{tr}\ S+4(\mathrm{tr}\ S)^2\}g_{mn+4}\ .$$

(42)
$$\mathrm{etr}(-S)\frac{x^{mn/2-1}}{2^{mn/2}\Gamma(mn/2)}\sum_{k=0}^{\infty}\frac{(-x/2)^k}{k!(mn/2)_k}(k-1)\sum_{\kappa}a_1(\kappa)L_\kappa^{n/2-p}(S)$$
$$=\frac{1}{4}[mn(n+m+1)-4(n+m+1)\ \mathrm{tr}\ S+4\ \mathrm{tr}\ S^2]g_{mn+4}$$
$$-\frac{1}{8}[mn(n+m+1)(mn+4)-2(n+m+1)(3mn+16)\ \mathrm{tr}\ S$$
$$+4\{2(n+m+1)(\mathrm{tr}\ S)^2+(mn+12)\ \mathrm{tr}\ S^2\}-8\ \mathrm{tr}\ S\ \mathrm{tr}\ S^2]g_{mn+6}$$
$$-\frac{1}{4}[3(n+m+1)(mn+4)\ \mathrm{tr}\ S-8(n+m+1)(\mathrm{tr}\ S)^2$$
$$-4(mn+9)\ \mathrm{tr}\ S^2+12\ \mathrm{tr}\ S\ \mathrm{tr}\ S^2]g_{mn+8}$$
$$-\frac{1}{2}[2(n+m+1)(\mathrm{tr}\ S)^2+(mn+8)\ \mathrm{tr}\ S^2-6\ \mathrm{tr}\ S\ \mathrm{tr}\ S^2]g_{mn+10}$$
$$-\mathrm{tr}\ S\ \mathrm{tr}\ S^2 g_{mn+12}\ .$$

Combining (35), (36), (37), (39), (40), (41) and (42) with (34), and arranging them appropriately, we obtain the term of order 2 in the following form.

(43)
$$A_2=mn\{3mn^3-2(3m^2+3m+4)n^2+3(m^3+2m^2+5m+4)n-8m^2$$
$$-12m+4\}g_{mn}-12mn^2(n-m-1)(mn-2\ \mathrm{tr}\ S)g_{mn+2}$$
$$+6n[mn\{3mn^2+8n-(m+1)(m^2+m-4)\}-4\{4mn^2$$
$$-(m^2+m-8)n-(m^3+2m^2-3m-4)\}\ \mathrm{tr}\ S+8n(\mathrm{tr}\ S)^2$$
$$+4\{mn-(m^2+m-4)\}\ \mathrm{tr}\ S^2]g_{mn+4}-4[mn\{3mn^3$$
$$+(3m^2+3m+16)n^2+24(m+1)n+4(m^2+3m+4)\}$$
$$-6\{6mn^3+3(m^2+m+8)n^2-(m^3+2m^2-27m-28)n$$
$$+4(m^2+3m+4)\}\ \mathrm{tr}\ S+24\{2n^2+(m+1)n+2\}(\mathrm{tr}\ S)^2$$
$$+12\{2mn^2-(m^2+m-16)n+4(m+2)\}\ \mathrm{tr}\ S^2-24n\ \mathrm{tr}\ S\ \mathrm{tr}\ S^2$$

$$
\begin{aligned}
&-32 \operatorname{tr} S^3]g_{mn+6}+3[mn\{mn^3+2(m^2+m+4)n^2 \\
&+(m^3+2m^2+21m+20)n+4(2m^2+5m+5)\}-8\{4mn^3 \\
&+(5m^2+5m+24)n^2+(m^3+2m^2+45m+44)n \\
&+4(3m^2+8m+9)\} \operatorname{tr} S+16\{6n^2+6(m+1)n+m^2 \\
&+2m+15\}(\operatorname{tr} S)^2+16\{3mn^2+36n+18m+32\} \operatorname{tr} S^2 \\
&-32\{4n+m+1\} \operatorname{tr} S \operatorname{tr} S^2-256 \operatorname{tr} S^3+16(\operatorname{tr} S^2)^2]g_{mn+8} \\
&+24[\{mn^3+2(m^2+m+4)n^2+(m^3+2m^2+21m+20)n \\
&+4(2m^2+5m+5)\} \operatorname{tr} S-4\{2n^2+3(m+1)n+m^2 \\
&+2m+9\}(\operatorname{tr} S)^2-2\{2mn^2+(m^2+m+32)n+8(3m+5)\} \operatorname{tr} S^2 \\
&+12(2n+m+1) \operatorname{tr} S \operatorname{tr} S^2+64 \operatorname{tr} S^3-8(\operatorname{tr} S^2)^2]g_{mn+10} \\
&+8[6\{n^2+2(m+1)n+m^2+2m+7\}(\operatorname{tr} S)^2+3\{mn^2 \\
&+(m^2+m+20)n+20m+32\} \operatorname{tr} S^2-12(4n+3m+3) \operatorname{tr} S \operatorname{tr} S^2 \\
&-160 \operatorname{tr} S^3+36(\operatorname{tr} S^2)^2]g_{mn+12}+96[(n+m+1) \operatorname{tr} S \operatorname{tr} S^2 \\
&+4 \operatorname{tr} S^3-2(\operatorname{tr} S^2)^2]g_{mn+14}+48(\operatorname{tr} S^2)^2 g_{mn+16}\ .
\end{aligned}
$$

Hence we have obtained an asymptotic expansion of the probability density function of $T_0^2$ as far as the term in $n_2^{-2}$.

We summarize our results in

THEOREM. *Let $S_1$ and $S_2$ be independent Wishart matrix with $n$ and $n_2$ degrees of freedom, and let $S_1$ have a non-centrality parameter matrix $S$. Then the asymptotic expansion of the probability density function of a generalized Hotelling's $T_0^2=n_2 \operatorname{tr} S_1S_2^{-1}$ which is only valid for $0\leqq T_0^2<n_2$ is given by*

$$f(x)=g_{mn}(x, \operatorname{tr} S)+\frac{A_1}{4n_2}+\frac{A_2}{96n_2^2}+O(1/n_2^3)\ ,$$

*where $g_{mn}(x, \operatorname{tr} S)$ is a probability density function of a non-central $\chi^2$ variable with $mn$ degrees of freedom and non-centrality parameter* $\operatorname{tr} S$. *$A_1$ and $A_2$ are given by* (38) *and* (43), *respectively.*

*Note.* It should be noted that the coefficients of each chi-square probability density function are in a slightly different form from the ones of Siotani [12]. However, by rearranging the variables of each coefficients, we have the same results as Siotani's.

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DEPARTMENT OF STATISTICS, UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL
INSTITUTE OF STATISTICAL MATHEMATICS, TOKYO

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