ON THE DISTRIBUTION OF THE LATENT ROOTS OF A COMPLEX WISHART MATRIX (NON-CENTRAL CASE)*

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1. Introduction

N. G. Goodman [3], C. G. Khatri [7], [8] and A. T. James [6] have discussed some distribution problems of the complex multivariate normal distribution which appears in time series analysis and physics. The problems of complex variates can be treated in the same way as those of the real variates in the case of normal distributions. In this paper, we consider the distribution problems of the latent roots of a positive definite random Hermitian matrix which are extensions of the author [5] to the complex variates. We introduce the generalized Hermite and Laguerre polynomials with complex matrix argument to handle these problems.

2. Notations and useful results

We shall use the following notations which are given by James [6].

$$(1)\qquad \tilde{\Gamma}_m(a)=\pi^{m(m-1)/2}\prod_{\alpha=1}^{m}\Gamma(a-\alpha+1)\ .$$

$$(2)\qquad \tilde{\Gamma}_m(a,\kappa)=\pi^{m(m-1)/2}\prod_{\alpha=1}^{m}\Gamma(a+k_\alpha-\alpha+1)\ .$$

$$(3)\qquad [a]_\kappa=\prod_{\alpha=1}^{m}(a-\alpha+1)_{k_\alpha}=\tilde{\Gamma}_m(a,\kappa)/\tilde{\Gamma}_m(a)\ .$$

where $\kappa=(k_1,\cdots,k_m)$ is a partition of k into not more than m parts. The corresponding generalized hypergeometric functions are defined as

$$(4)\qquad {}_p\tilde{F}_q^{(m)}(a_1,\cdots,a_p,b_1,\cdots,b_q,A,B)$$
$$=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_p]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\,\frac{\tilde{C}_\kappa(A)\tilde{C}_\kappa(B)}{k!\tilde{C}_\kappa(I_m)}$$

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where $\tilde{C}_\kappa(A)$ is a zonal polynomial of a Hermitian matrix A corresponding to a partition κ of k. If either A or B is I_m, then we write as ${}_p\tilde{F}_q^{(m)}(\cdots, \cdots; A, I) = {}_p\tilde{F}_q(\cdots, \cdots; A)$.

The most fundamental properties of a zonal polynomial are of the same form as in the real case, so that

$$\int_{U(m)} \tilde{C}_\kappa(AUB\bar{U}')d(U) = \frac{\tilde{C}_\kappa(A)\tilde{C}_\kappa(B)}{\tilde{C}_\kappa(I_m)} \tag{5}$$

and

$$\int_{U(n)} \operatorname{etr}(XU + \bar{U}'\bar{X}')d(U) = {}_0\tilde{F}_1(n; X\bar{X}')\ , \tag{6}$$

where $d(U)$ in (5) and (6) is the invariant measure normalized to make the total measure unity on the unitary group $U(m)$ and $U(n)$ of order m and n, respectively, and A and B are Hermitian matrices, and X is an arbitrary $n \times n$ complex matrix, James [7].

HSU'S LEMMA. *Let $f(Z\bar{Z}')$ be a probability density function (p.d.f.) of Z $m \times n$ $(m \leqq n)$. Then the p.d.f. of the Hermitian matrix $R = Z\bar{Z}'$ is given by*

$$\frac{\pi^{mn}}{\tilde{\Gamma}_m(n)}(\det R)^{n-m} f(R)\ . \tag{7}$$

The p.d.f. of the latent roots $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$ of R is also given by

$$\frac{\pi^{m(n+m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}(\det \Lambda)^{n-m} f(\Lambda) \prod_{i<j} (\lambda_i - \lambda_j)^2\ , \tag{8}$$

if $f(U\Lambda\bar{U}') = f(\Lambda)$.

3. Generalized Hermite polynomials

Hayakawa [5] defined the generalized Hermite polynomial (g.H.p.) with a real matrix argument in order to discuss the distribution problems of the latent roots of the non-central Wishart matrix. In this section, we show that they can be extended to the complex variate case and they can be treated analogously as the real variate case.

Here we define the g.H.p. $\tilde{H}_\kappa(T)$ and the generalized Laguerre polynomial (g.L.p.) $\tilde{L}_\kappa(S)$, $\gamma > -1$, which correspond to the partition κ of k, in the following way:

$$\operatorname{etr}(-T\bar{T}')\tilde{H}_\kappa(T) = \frac{(-1)^k}{\pi^{mn}} \int_W \operatorname{etr}(-i(T\bar{W}' + W\bar{T}')) \cdot \operatorname{etr}(-W\bar{W}')\tilde{C}_\kappa(W\bar{W}')dW\ , \tag{9}$$

where T and W are $m\times n$ $(m\leqq n)$ arbitrary complex matrices, and for $\gamma>-1$,

$$\text{etr}\,(-S)\tilde{L}_\kappa^\gamma(S)=\int_{\bar{R}'=R>0}\tilde{A}_\gamma(RS)(\det R)^\gamma\,\text{etr}\,(-R)\tilde{C}_\kappa(R)dR\ , \tag{10}$$

where S and R are $m\times m$ Hermitian matrices and

$$\tilde{\Gamma}_m(\gamma+m)\tilde{A}_\gamma(RS)={}_0\tilde{F}_1(\gamma+m,\ -RS)\ , \qquad [2]\ .$$

The next theorem gives a relation between g.H.p. and g.L.p.

THEOREM 1

$$\tilde{H}_\kappa(T)=(-1)^k\tilde{L}_\kappa^{n-m}(T\bar{T}')\ . \tag{11}$$

PROOF. From the definition of g.H.p. and Hsu's Lemma in the complex case, we have

$$\begin{aligned}\text{etr}\,(-T\bar{T}')\tilde{H}_\kappa(T)&=\frac{(-1)^k}{\pi^{mn}}\int_W\int_{U(n)}\text{etr}\,\{-i(WU\bar{T}'+T\bar{U}'\bar{W}')\}\\&\qquad\cdot\,\text{etr}\,(-W\bar{W}')\tilde{C}_\kappa(W\bar{W}')dWd(U)\\&=\frac{(-1)^k}{\pi^{mn}}\int_W {}_0\tilde{F}_1(n\,;\,-T\bar{T}'W\bar{W}')\\&\qquad\cdot\,\text{etr}\,(-W\bar{W}')\tilde{C}_\kappa(W\bar{W}')dW\\&=(-1)^k\int_{\bar{R}'=R>0}\tilde{A}_{n-m}(T\bar{T}'R)(\det R)^{n-m}\\&\qquad\cdot\,\text{etr}\,(-R)\tilde{C}_\kappa(R)dR\\&=(-1)^k\,\text{etr}\,(-T\bar{T}')\tilde{L}_\kappa^{n-m}(T\bar{T}')\ ,\end{aligned}$$

which completes the proof.

We can show easily the following corollaries from the definition (8) and Hsu's Lemma.

COROLLARY 1.

$$\tilde{H}_\kappa(T)=\tilde{H}_\kappa(U_1T)=\tilde{H}_\kappa(TU_2)\ , \tag{12}$$

where $U_1\in U(m)$ *and* $U_2\in U(n)$, *respectively.*

COROLLARY 2.

$$|\tilde{H}_\kappa(T)|\leqq\text{etr}\,(T\bar{T}')[n]_\kappa\tilde{C}_\kappa(I)\ , \tag{13}$$

$$\tilde{H}_\kappa(0)=(-1)^k[n]_\kappa\,\text{etr}\,(T\bar{T}')\tilde{C}_\kappa(I)\ , \tag{14}$$

THEOREM 2. (*Generating function of g.H.p.'s*) *Let* S *and* T *be*

$m \times n$ ($m \leqq n$) arbitrary complex matrices. Then

$$\int_{U(m)} \int_{U(n)} \operatorname{etr}\{-S\bar{S}' + U_1 S U_2 \bar{T}' + T\bar{U}_2' \bar{S}' \bar{U}_1'\} d(U_2) d(U_1) \tag{15}$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{H}_\kappa(T) \tilde{C}_\kappa(S\bar{S}')}{k! [n]_\kappa \tilde{C}_\kappa(I)} ,$$

where $U_1 \in U(m)$ and $U_2 \in U(n)$, respectively. The right-hand side of (15) *converges absolutely.*

PROOF. We show (15) by the direct method, using (5), (6) and (9).

$$\text{R.H.S.} = \operatorname{etr}(T\bar{T}') \frac{1}{\pi^{mn}} \int_W \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k \tilde{C}_\kappa(S\bar{S}')}{k! [n]_\kappa \tilde{C}_\kappa(I)}$$

$$\cdot \operatorname{etr}\{-i(T\bar{W}' + W\bar{T}')\} \operatorname{etr}(-W\bar{W}') \tilde{C}_\kappa(W\bar{W}') dW$$

$$= \frac{\operatorname{etr}(T\bar{T}')}{\pi^{mn}} \int_W \int_{U(m)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k}{k! [n]_\kappa} \tilde{C}_\kappa(S\bar{S}' U_1 W \bar{W}' \bar{U}_1')$$

$$\cdot \operatorname{etr}\{-i(T\bar{W}' + W\bar{T}')\} \operatorname{etr}(-W\bar{W}') dW$$

$$= \frac{\operatorname{etr}(T\bar{T}')}{\pi^{mn}} \int_W \int_{U(m)} \int_{U(n)} \operatorname{etr}\{i(\bar{W}' U_1 S U_2 + \bar{U}_2' \bar{S}' \bar{U}_1' W)\}$$

$$\cdot \operatorname{etr}\{-i(T\bar{W}' + W\bar{T}')\} \operatorname{etr}(-W\bar{W}') dW$$

$$= \int_{U(m)} \int_{U(n)} \operatorname{etr}(-S\bar{S}' + U_1 S U_2 \bar{T}' + T\bar{U}_2' \bar{S}' \bar{U}_1') d(U_2) d(U_1) .$$

The absolute convergence of (15) is shown by using (13). This completes the proof.

THEOREM 3. (*Generating function of g.L.p.'s*) *Let S and Z be $m \times m$ positive definite Hermitian matrices. Then*

$$\det(I - Z)^{-\gamma - m} \int_{U(m)} \operatorname{etr}(-SUZ(I - Z)^{-1} \bar{U}') d(U) \tag{16}$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{L}_\kappa^\gamma(S) \tilde{C}_\kappa(Z)}{k! \tilde{C}_\kappa(I_m)} , \qquad \| Z \| < 1 ,$$

where $\| Z \|$ means the maximum of the absolute values of the characteristic roots of Z.

The proof runs similarly to that of Theorem 2, when we use (5), (7) and (10).

Remark 1. The invariance of $\tilde{H}_\kappa(T)$ with respect to $U(m)$ from left and $U(n)$ from right is obvious from Theorem 2, because $d(U_1)$ and $d(U_2)$ are invariant unitary measures on $U(m)$ and $U(n)$, respectively.

Remark 2. The relation between the generating functions of the g.H.p.'s and the g.L.p.'s when $\gamma=n-m$ is as follows. Multiply both sides of (15) by $\pi^{-mn}(\det Z)^{-n}\,\mathrm{etr}\,(-S\bar{S}'Z^{-1})$ and integrate with respect to S. Then we have (16) by replacing Z with $-Z$.

THEOREM 4. (*Mehler's formula*) *Let S and T be an* $m\times n$ *arbitrary complex matrix. Then*

$$\frac{1}{(1-u^2)^{mn}}\int_{U(m)}\int_{U(n)}\mathrm{etr}\left\{-\frac{u^2}{1-u^2}(S\bar{S}'+T\bar{T}')+\frac{u}{1-u^2}\right.$$
$$\left.\cdot\,(U_1SU_2\bar{T}'+T\bar{U}_2'\bar{S}'\bar{U}_1')\right\}d(U_2)d(U_1)$$
$$=\sum_{k=0}^{\infty}\frac{u^{2k}}{k!}\sum_{\kappa}\frac{\tilde{H}_\kappa(T)\tilde{H}_\kappa(S)}{[n]_\kappa\tilde{C}_\kappa(I_m)}\,,\qquad |u|<1\,. \tag{17}$$

PROOF. The proof is exactly similar to that of Theorem 2.

COROLLARY 3.

$$\sum_{\kappa}\tilde{L}_\kappa^\gamma(S)=L_k^{m(\gamma+m)-1}(\mathrm{tr}\,S)\,, \tag{18}$$

where the L.H.S. is a summation over all partition κ*'s of k and the R.H.S. is a univariate Laguerre polynomial.*

PROOF. The proof of (18) is very simple. If we set $Z=xI_m$, $|x|<1$ in (16), then

$$(1-x)^{-m(\gamma+m)}\exp\left(-\frac{x}{1-x}\,\mathrm{tr}\,S\right)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa}\tilde{L}_\kappa^\gamma(S)\,. \tag{19}$$

Since the L.H.S. of (19) gives a generating function of a univariate Laguerre polynomial of an argument $\mathrm{tr}\,S$, i.e.,

$$\text{L.H.S.}=\sum_{k=0}^{\infty}\frac{x^k}{k!}L_k^{m(\gamma+m)-1}(\mathrm{tr}\,S)\,,\qquad |x|<1\,,$$

by comparing the coefficient of x^k on the two sides of (19), we have (18), immediately.

4. Jacobians of the latent roots and integrals

In this section, we give some useful transformations for finding the p.d.f. of the maximum latent root of a positive definite Hermitian matrix, and of the latent roots of a non-central complex Wishart matrix with known covariance. We also give some related Beta-type integrals.

LEMMA 1. *Let S be an* $m\times m$ *positive definite random Hermitian matrix. We decompose S as follows,*

$$S=U\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}\bar{U}' , \tag{20}$$

where U *is an* $m\times m$ *unitary matrix which has only* $2(m-1)$ *independent variables* $u^R_{21},\cdots,u^R_{m1}$, $u^I_{21},\cdots,u^I_{m1}$, *the first column elements of* U *except* u_{11}, *and* V *is an* $(m-1)\times(m-1)$ *positive definite random Hermitian matrix which ranges* $\lambda_1 I_{m-1}>\bar{V}'=V>0$. *Then*

$$dS=\det(\lambda_1 I-V)^2 d\lambda_1 dV \prod_{\alpha=2}^{m} du^R_{\alpha 1} du^I_{\alpha 1} . \tag{21}$$

PROOF. The proof of this lemma is given in Appendix I.

Note. If we decompose S further such that

$$S=U_1\begin{bmatrix}1 & \\ & U_2\end{bmatrix}\cdots\begin{bmatrix}I_{k-1} & \\ & U_k\end{bmatrix}\begin{bmatrix}\Lambda_k & \\ & V_k\end{bmatrix}\begin{bmatrix}I_{k-1} & \\ & \bar{U}'_k\end{bmatrix}\cdots\begin{bmatrix}1 & \\ & \bar{U}'_2\end{bmatrix}\bar{U}'_1 ,$$

where U_ν is a unitary matrix of order $(m-\nu+1)$ whose first column vector contains independent variables, and V_k is an $(m-k)\times(m-k)$ Hermitian matrix which ranges $\lambda_k I>\bar{V}'_k=V_k>0$, then

$$dS=\prod_{i<j}^{k}(\lambda_i-\lambda_j)^2\prod_{i=1}^{k}\det(\lambda_i I_{m-k}-V_k)^2 d\Lambda_k dV_k \prod_{\beta=1}^{k-1}\prod_{\alpha=\beta+1}^{m} du^R_{\alpha\beta} du^I_{\alpha\beta} . \tag{22}$$

This can be shown, by induction, from Lemma 1.

LEMMA 2. *Let* X *be an* $m\times n$ $(m\leqq n)$ *arbitrary complex matrix which has* $2mn$ *independent elements,* U *an* $m\times m$ *unitary matrix whose diagonal elements are all real values and it has* m^2-m *independent elements, and* L *an* $n\times m$ *semi-unitary matrix (i.e.,* $\bar{L}'L=I_m$*) whose diagonal elements are all complex variables and it has* $2mn-m^2$ *independent elements, and* $\Lambda=\operatorname{diag}(\lambda_1,\cdots,\lambda_m)$, *where* λ^2_α*'s are real valued latent roots of* $X\bar{X}'$. *Then the Jacobian of the transformation*

$$X=U\Lambda\bar{L}' , \tag{23}$$

is given by

$$dX=\frac{\pi^{m(n+m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}(\det\Lambda)^{2(n-m)+1}\prod_{i<j}(\lambda_i^2-\lambda_j^2)^2 d\Lambda d(U) d(L) ,$$

where

$$d(U)=(\mathrm{A}2.6)=\frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(m)}h_2(U)dU ,$$

and

$$d(L)=(\mathrm{A}2.7)=\frac{\pi^{mn}}{\tilde{\Gamma}_m(m)}h_1(L)dL\ , \qquad \textit{Khatri}\ [7],\ [8]\ .$$

Proof. The proof is shown in Appendix II.

Note. If we decompose X as follows:

$$X=U_1\begin{bmatrix}1 & \\ & U_2\end{bmatrix}\cdots\begin{bmatrix}I_{k-1} & \\ & U_k\end{bmatrix}\begin{bmatrix}\Lambda_k & \\ & V_k\end{bmatrix}\bar{L}'\ ,$$

then we have

$$dX=c\left(\prod_{i=1}^{k}\lambda_i\right)^{2(n-m)+1}\prod_{i<j}^{k}(\lambda_i^2-\lambda_j^2)^2\prod_{i=1}^{k}\det(\lambda_i^2I_k-V_k\bar{V}_k')^2d\Lambda_kd(L)$$
$$\cdot\prod_{\beta=1}^{k-1}\prod_{\alpha=\beta+1}^{m}du_{\alpha\beta}^Rdu_{\alpha\beta}^IdV_k\ ,$$

where

$$c=\frac{\pi^{mn}}{\tilde{\Gamma}_m(n)}\ .$$

Theorem 5.

$$\int_{I_{m-1}>\bar{W}'=W>0}(\det W)^{\alpha-m}\det(I-W)^2\tilde{C}_\kappa\begin{pmatrix}1 & \\ & W\end{pmatrix}dW \tag{25}$$
$$=\frac{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(m)\Gamma(m)}{\pi^{m-1}\tilde{\Gamma}_m(\alpha+m)}(m\alpha+k)\frac{[\alpha]_\kappa}{[m+\alpha]_\kappa}\tilde{C}_\kappa(I_m)\ ,$$

$$\int_{1>\omega_2>\cdots>\omega_m>0}\left(\prod_{i=2}^{m}\omega_i\right)^{\alpha-m} \tag{26}$$
$$\cdot\prod_{i=2}^{m}(1-\omega_i)^2\prod_{2\leqq i<j\leqq m}(\omega_i-\omega_j)^2\tilde{C}_\kappa\begin{bmatrix}1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_m\end{bmatrix}\prod_{i=2}^{m}d\omega_i$$
$$=\frac{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(m)^2}{\pi^{m(m-1)}\tilde{\Gamma}_m(\alpha+m)}(m\alpha+k)\frac{[\alpha]_\kappa}{[\alpha+m]_\kappa}\tilde{C}_\kappa(I_m)\ .$$

Proof. The following formula is well-known (Khatri [9, (55)]),

$$\int_{I>\bar{S}'=S>0}(\det S)^{\alpha-m}\tilde{C}_\kappa(S)dS=\frac{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(m)}{\tilde{\Gamma}_m(\alpha+m)}\ \frac{[\alpha]_\kappa}{[\alpha+m]_\kappa}\tilde{C}_\kappa(I_m)\ . \tag{27}$$

To prove (25), we decompose S as follows:

$$S=U\begin{bmatrix}\lambda_1 & \\ & \lambda_1W\end{bmatrix}\bar{U}'\ .$$

Then, from Lemma 1, the integral element can be written as

$$dS=\lambda_1^{m^2-1}\det(I-W)^2 d\lambda_1 dW\prod_{i=2}^{m} du_{i1}^R du_{i1}^I .$$

Hence, inserting these results in (27), and noting that

$$\int_{\sum_{i=2}^{m} u_{i1}^{R^2}+\sum_{i=2}^{m} u_{i1}^{I^2}\leqq 1}\prod_{i=2}^{m} du_{i1}^R du_{i1}^I=\frac{\pi^{m-1}}{\Gamma(m)}$$

and

$$\int_0^1 \lambda_1^{m\alpha+k-1}d\lambda_1=\frac{1}{m\alpha+k} ,$$

we have (25). To prove (26), we decompose W further, so that

$$W=U_2\Lambda_\omega U_2 , \qquad U_2\in U(m-1) , \qquad \Lambda_\omega=\mathrm{diag}(\omega_2,\cdots,\omega_m) .$$

Then, by James [7] and Khatri [8], we have the following two equalities,

$$dW=\prod_{2\leqq i<j\leqq m}(\omega_i-\omega_j)^2 d\Lambda_\omega d^*(U_2) ,$$

and

$$\int_{U(m-1)} d^*(U_2)=\frac{\pi^{(m-1)(m-2)}}{\tilde{\Gamma}_{m-1}(m-1)} .$$

where $d^*(U_2)$ is an invariant measure on $U(m-1)$. Hence by inserting these equations into (25), we have (26).

5. The distribution of the latent roots of a complex non-central Wishart matrix with known covariance

The probability density function of the latent roots of a complex non-central Wishart matrix was given by James [6]. However, in some cases it is not convenient to treat the distribution problems of the related statistics. We here give another formula expressed in terms of g.H.p.'s.

THEOREM 6. *Let X $m\times n$ $(m\leqq n)$ be distributed with p.d.f.,*

$$\frac{1}{\pi^{mn}(\det\Sigma)^n}\,\mathrm{etr}\,[-\Sigma^{-1}(X-M)\overline{(X-M)}'] . \tag{28}$$

Then the p.d.f. of the latent roots of $\det(X\bar{X}'-\lambda\Sigma)=0$ *is given by*

$$\frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}\,\mathrm{etr}\,(-\Sigma^{-1}MM')(\det\Lambda)^{n-m}\prod_{i<j}(\lambda_i-\lambda_j)^2 \cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{H}_\kappa(\Sigma^{-1/2}M)\tilde{C}_\kappa(\Lambda)}{k![n]_\kappa\tilde{C}_\kappa(I_m)} , \tag{29}$$

where $\Lambda=\operatorname{diag}(\lambda_1,\cdots,\lambda_m)$.

PROOF. We decompose $\Sigma^{-1/2}X$ in (28) as follows:

$$Y=\Sigma^{-1/2}X=U_1\Lambda^{1/2}\bar{L}' \,, \tag{30}$$

where U_1, L and Λ are the same matrices as those of Lemma 2. Then

$$dY=\frac{\pi^{m(m+n-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}(\det\Lambda)^{n-m}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\Lambda d(U_1)d(L) \,. \tag{31}$$

Hence by inserting (30) and (31) into (28), we have the joint p.d.f. of Λ, U_1 and L;

$$\begin{aligned}&\frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}\operatorname{etr}(-\Sigma^{-1}MM')(\det\Lambda)^{n-m}\prod_{i<j}(\lambda_i-\lambda_j)^2\\ &\qquad\cdot\operatorname{etr}(-\Lambda^{1/2}\bar{L}'L\Lambda^{1/2}+U_1\Lambda^{1/2}\bar{L}'\bar{M}'\Sigma^{-1/2}+\Sigma^{-1/2}ML\Lambda^{1/2}\bar{U}_1') \,.\end{aligned} \tag{32}$$

If we set $\bar{L}'\to\bar{L}'U_2$, $U_2\in U(n)$, then $\bar{L}'U_2(\overline{\bar{L}'U_2})'=I_m$ and the semi unitary invariant measure $d(L)$ is unchanged under the unitary transformation. Therefore, we integrate (32) with respect to U_1 and L.

$$\begin{aligned}&\int_{\bar{L}'L=I_m}\int_{U(m)}\operatorname{etr}(-\Lambda^{1/2}\bar{L}'L\Lambda^{1/2}+U_1\Lambda^{1/2}\bar{L}'\bar{M}'\Sigma^{-1/2}+\Sigma^{-1/2}ML\Lambda^{1/2}\bar{U}_1')d(U_1)d(L)\\ &\quad=\int_{\bar{L}'L=I_m}\int_{U(m)}\int_{U(n)}\operatorname{etr}(-\Lambda^{1/2}\bar{L}'L\Lambda+U_1\Lambda^{1/2}\bar{L}'U_2\bar{M}'\Sigma^{-1/2}\\ &\qquad+\Sigma^{-1/2}M\bar{U}_2'L\Lambda^{1/2}\bar{U}_1')d(U_2)d(U_1)d(L) \,.\end{aligned}$$

The integral of the R.H.S. with respect to $U(m)$ and $U(n)$ is the same form as the generating function of g.H.p.'s if we set $S=\Lambda^{1/2}\bar{L}'$ and $T=\Sigma^{-1/2}M$. Thus we have

$$\begin{aligned}\text{R.H.S.}&=\int_{\bar{L}'L=I_m}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{H}_\kappa(\Sigma^{-1/2}M)\tilde{C}_\kappa(\Lambda)}{k![n]_\kappa\tilde{C}_\kappa(I_m)}d(L)\\ &=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{H}_\kappa(\Sigma^{-1/2}M)\tilde{C}_\kappa(\Lambda)}{k![n]_\kappa\tilde{C}_\kappa(I_m)} \,,\end{aligned}$$

which completes the proof.

COROLLARY 5. *If we set* $M=0$ *in* (29), *then we have the p.d.f. of* Λ *in the central case from* (14) [1].

THEOREM 7. *Let* Λ *be distributed with the p.d.f.* (29). *Then the p.d.f. of the maximum latent root* λ_1 *of* Λ *is given by*

$$\frac{\tilde{\Gamma}_m(m)}{\tilde{\Gamma}_m(n+m)}\operatorname{etr}(-\Sigma^{-1}M\bar{M}')\lambda_1^{mn-1}\sum_{k=0}^{\infty}\frac{(mn+k)}{k!}\lambda_1^k\sum_{\kappa}\frac{\tilde{H}_\kappa(\Sigma^{-1/2}M)}{[n+m]_\kappa} \,. \tag{33}$$

PROOF. If we set $\lambda_i=\lambda_1\omega_i$ $(i=2,\cdots,m)$ in (29), and integrate it with respect to $\omega_2,\cdots,\omega_m$, then from (26) we have (33) immediately.

COROLLARY 7. *The c.d.f. of* λ_1 *is given by*

$$\Pr\{\lambda_1<x\}=\frac{\tilde{\Gamma}_m(m)}{\tilde{\Gamma}_m(n+m)}\operatorname{etr}(-\Sigma^{-1}M\bar{M}')x^{mn}\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{\kappa}\frac{\tilde{H}_\kappa(\Sigma^{-1/2}M)}{[n+m]_\kappa}\,. \tag{34}$$

THEOREM 8. *Let* Λ *be distributed with p.d.f.* (29). *Then the p.d.f. of* $T=\operatorname{tr}\Lambda$ *is given by*

$$\frac{1}{\Gamma(mn)}\operatorname{etr}(-\Sigma^{-1}M\bar{M}')T^{mn-1}\sum_{k=0}^{\infty}\frac{T^k}{k!(mn)_k}\sum_{\kappa}\tilde{H}_\kappa(\Sigma^{-1/2}M)\,. \tag{35}$$

PROOF. We can obtain (35) as in the proof of Theorem 5 in [5].

Remark. (35) is of the same form as the p.d.f. of the $T=2\chi^2_{2mn}(\delta^2)$, where $\chi^2_{2mn}(\delta^2)$ is a non-central chi-square variable of $2mn$ degrees of freedom with non-central parameter $\delta^2=\operatorname{tr}\Sigma^{-1}M\bar{M}'$. Since from Theorem 1 and Corollary 4, the series term can be represented by

$$\sum_{k=0}^{\infty}\frac{(-T)^k L_k^{mn-1}(\operatorname{tr}\Sigma^{-1}M\bar{M}')}{k!(mn)_k}\,,$$

this can also be represented using a Bessel function as

$$\Gamma(mn)(-T\delta^2)^{-(mn-1)/2}\exp(-T)J_{mn-1}(2(-T\delta^2)^{1/2})\,,\qquad \delta^2=\operatorname{tr}\Sigma^{-1}M\bar{M}'\,.$$

However, as the Bessel function $J_{mn-1}(2(-T\delta^2)^{1/2})$ is written as

$$\frac{1}{\Gamma(mn)}\sum_{k=0}^{\infty}\frac{(T\delta^2)^k}{k!(mn)_k}(-T\delta^2)^{(mn-1)/2}\,,$$

the p.d.f. of T is written as

$$\frac{1}{\Gamma(mn)}\exp(-\delta^2-T)T^{mn-1}\sum_{k=0}^{\infty}\frac{(T\delta^2)^k}{k!(mn)_k}\,.$$

This is the p.d.f. of $2\chi^2_{2mn}(\delta^2)$.

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Appendix 1. (PROOF OF LEMMA 1)

Differentiating both sides of (20), we have

$$dS=dU\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}\bar{U}'+U\begin{bmatrix}d\lambda_1 & \\ & dV\end{bmatrix}\bar{U}'+U\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}d\bar{U}' . \tag{A1.1}$$

Multiply by $\bar{U}'$ from the left and U from the right to obtain

$$\bar{U}'dSU=\bar{U}'dU\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}+\begin{bmatrix}d\lambda_1 & \\ & dV\end{bmatrix}-\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}\bar{U}'dU ,$$

since $\bar{U}'dU=-d\bar{U}'U$. By setting $dW=(dW_{\alpha\beta})=\bar{U}'dSU$ and $dT=(dt_{\alpha\beta})=\bar{U}'dU$, we can see that dW is a Hermitian matrix whose diagonal elements are all real variates and dT is a skew Hermitian matrix whose diagonal elements are all pure imaginary. Hence we have

$$dW=dT\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}+\begin{bmatrix}d\lambda_1 & \\ & dV\end{bmatrix}-\begin{bmatrix}\lambda_1 & \\ & V\end{bmatrix}dT , \tag{A1.2}$$

$$dT=\bar{U}'dU . \tag{A1.3}$$

To obtain the Jacobian of this transformation, we shall use the exterior product of the differentials. First of all, we know that $\prod_{\alpha=1}^{m} d\omega_{\alpha\alpha}\prod_{\alpha>\beta}^{m} d\omega^R_{\alpha\beta}\cdot d\omega^I_{\alpha\beta}=\prod_{\alpha=1}^{m} dS_{\alpha\alpha}\prod_{\alpha>\beta}^{m} dS^R_{\alpha\beta}dS^I_{\alpha\beta}$, [7]. Before we calculate the exterior product of dT, we must consider the structure of U. The matrix U has only $2(m-1)$ independent elements. Hence we represent U as follows

$$U=\begin{array}{c}\left[\begin{array}{cc} u_{11} & \boldsymbol{a}^R_1+i\boldsymbol{a}^I_1 \\ \boldsymbol{u}^R_1+i\boldsymbol{u}^I_1 & U^R_{22}+iU^I_{22}\end{array}\right]\begin{array}{l}1\\ m-1\end{array} \\ \begin{array}{cc}\quad 1 \qquad\qquad & m-1\end{array}\end{array} , \tag{A1.4}$$

where

$$\begin{cases} u^2_{11}+\boldsymbol{u}^{R'}_1\boldsymbol{u}^R_1+\boldsymbol{u}^{I'}_1\boldsymbol{u}^I_1=u^2_{11}+\boldsymbol{a}^R_1\boldsymbol{a}^{R'}_1+\boldsymbol{a}^I_1\boldsymbol{a}^{I'}_1=1 , \\ u_{11}\boldsymbol{u}^{R'}_1+\boldsymbol{a}^R_1U^{R'}_{22}+\boldsymbol{a}^I_1U^{I'}_{22}=\boldsymbol{0}' , \qquad -u_{11}\boldsymbol{u}^{I'}_1-\boldsymbol{a}^R_1U^{I'}_{22}+\boldsymbol{a}^I_1U^{R'}_{22}=\boldsymbol{0}' , \\ \boldsymbol{u}^R_1\boldsymbol{u}^{R'}_1+\boldsymbol{u}^I_1\boldsymbol{u}^{I'}_1+U^R_{22}U^{R'}_{22}+U^I_{22}U^{I'}_{22}=I_{m-1} , \\ -\boldsymbol{u}^R_1\boldsymbol{u}^{I'}_1+\boldsymbol{u}^I_1\boldsymbol{u}^{R'}_1-U^R_{22}U^{I'}_{22}+U^I_{22}U^{R'}_{22}=\boldsymbol{0} . \end{cases} \tag{A1.5}$$

$\boldsymbol{u}^R_1$ and $\boldsymbol{u}^I_1$ are vectors of independent variables, and $\boldsymbol{a}^R_1$, $\boldsymbol{a}^I_1$, U^R_{22} and U^I_{22} are functional vectors and matrices of $\boldsymbol{u}^R_1$ and $\boldsymbol{u}^I_1$, and u_{11} is a real variable, respectively.

Let

$$dT=\begin{array}{cc}\left[\begin{array}{c:c} dt_{11} & \\ d\boldsymbol{t}_1^R+id\boldsymbol{t}_1^I & dT_2 \end{array}\right] & \begin{array}{l} 1 \\ m-1 \end{array} \\ \begin{array}{cc} 1 & \quad m-1 \end{array} & \end{array}=\bar{U}'\begin{array}{cc}\left[\begin{array}{c:c} du_{11} & \\ d\boldsymbol{u}_1^R+id\boldsymbol{u}_1^I & dU_2 \end{array}\right] & \begin{array}{l} 1 \\ m-1 \end{array} \\ \begin{array}{cc} 1 & \quad m-1 \end{array} & \end{array}.$$

Then

$$\begin{bmatrix} dt_{11} \\ d\boldsymbol{t}_1^R+id\boldsymbol{t}_1^I \end{bmatrix}=\bar{U}'\begin{bmatrix} du_{11} \\ d\boldsymbol{u}_1^R+id\boldsymbol{u}_1^I \end{bmatrix}.$$

Since dT is a skew Hermitian matrix, $dt_{11}=idt_{11}^I$. We will show later that dt_{11}^I can be represented by the linear combination of $dt_{21}^R,\cdots,dt_{m1}^R$, $dt_{21}^I,\cdots,dt_{m1}^I$. Therefore, we need only $dt_{21}^R,\cdots,dt_{m1}^I$. As the first column has a constraint such that $u_{11}^2+\boldsymbol{u}_1^{R\prime}\boldsymbol{u}_1^R+\boldsymbol{u}_1^{I\prime}\boldsymbol{u}_1^I=1$, the differential of the first column of dU is represented as follows;

$$\begin{bmatrix} -\dfrac{1}{u_{11}}(\boldsymbol{u}_1^{R\prime}d\boldsymbol{u}_1^R+\boldsymbol{u}_1^{I\prime}d\boldsymbol{u}_1^I) \\ d\boldsymbol{u}_1^R+id\boldsymbol{u}_1^I \end{bmatrix}.$$

Hence we have

$$\text{(A1.7)}\qquad \begin{bmatrix} d\boldsymbol{t}_1^R \\ d\boldsymbol{t}_1^I \end{bmatrix}=\begin{bmatrix} -\dfrac{1}{u_{11}}\boldsymbol{a}_1^{R\prime}\boldsymbol{u}_1^{R\prime}+U_{22}^{R\prime} & -\dfrac{1}{u_{11}}\boldsymbol{a}_1^{R\prime}\boldsymbol{u}_1^{I\prime}+U_{22}^{I\prime} \\ \dfrac{1}{u_{11}}\boldsymbol{a}_1^{I\prime}\boldsymbol{u}_1^{R\prime}-U_{22}^{I\prime} & \dfrac{1}{u_{11}}\boldsymbol{a}_1^{I\prime}\boldsymbol{u}_1^{I\prime}+U_{22}^{R\prime} \end{bmatrix}\begin{bmatrix} d\boldsymbol{u}_1^R \\ d\boldsymbol{u}_1^I \end{bmatrix}.$$

Thus by forming the exterior product of $dt_{21}^R,\cdots,dt_{m1}^I$, we have

$$\text{(A1.8)}\qquad \prod_{\alpha=2}^m dt_{\alpha1}^R\prod_{\alpha=2}^m dt_{\alpha1}^I=\det\begin{bmatrix} -\dfrac{1}{u_{11}}\boldsymbol{a}_1^{R\prime}\boldsymbol{u}_1^{R\prime}+U_{22}^{R\prime} & -\dfrac{1}{u_{11}}\boldsymbol{a}_1^{R\prime}\boldsymbol{u}_1^{I\prime}+U_{22}^{I\prime} \\ \dfrac{1}{u_{11}}\boldsymbol{a}_1^{I\prime}\boldsymbol{u}_1^{R\prime}-U_{22}^{I\prime} & \dfrac{1}{u_{11}}\boldsymbol{a}_1^{I\prime}\boldsymbol{u}_1^{I\prime}+U_{22}^{R\prime} \end{bmatrix}$$

$$\cdot\prod_{\alpha=2}^m du_{\alpha1}^R\prod_{\alpha=2}^m du_{\alpha1}^I .$$

We can easily rewrite the determinant (say J_1) as

$$\text{(A1.9)}\qquad J_1=\frac{1}{u_{11}}\det\begin{bmatrix} u_{11} & \boldsymbol{a}_1^R & -\boldsymbol{a}_1^I \\ \boldsymbol{u}_1^R & U_{22}^R & -U_{22}^I \\ \boldsymbol{u}_1^I & U_{22}^I & U_{22}^R \end{bmatrix}.$$

From (A1.5),

$$\text{(A1.10)}\qquad J_1^2=\frac{1}{u_{11}^2}\det\begin{bmatrix} 1 & \boldsymbol{0}' & \boldsymbol{0}' \\ 0 & I-\boldsymbol{u}_1^I\boldsymbol{u}_1^{I\prime} & \boldsymbol{u}_1^I\boldsymbol{u}_1^{R\prime} \\ 0 & \boldsymbol{u}_1^R\boldsymbol{u}_1^{I\prime} & I-\boldsymbol{u}_1^R\boldsymbol{u}_1^{R\prime} \end{bmatrix}$$

$$
\begin{aligned}
&=\frac{1}{u_{11}^2}\det\begin{bmatrix} 1 & \boldsymbol{u}_1^{I'} & -\boldsymbol{u}_1^{R'} \\ \boldsymbol{u}_1^I & I & 0' \\ -\boldsymbol{u}_1^R & 0 & I \end{bmatrix} \\
&=\frac{1}{u_{11}^2}(1-\boldsymbol{u}_1^{R'}\boldsymbol{u}_1^R-\boldsymbol{u}_1^{I'}\boldsymbol{u}_1^I) \\
&=1\ .
\end{aligned}
$$

From this calculation, $\prod_{\alpha=2}^{m} dt_{\alpha1}^R \prod_{\alpha=2}^{m} dt_{\alpha1}^I=\prod_{\alpha=2}^{m} du_{\alpha1}^R \prod_{\alpha=2}^{m} du_{\alpha1}^I$. Hence there exists a unimodular matrix D such that

$$[d\boldsymbol{t}_1^{R'} d\boldsymbol{t}_1^{I'}]=[d\boldsymbol{u}_1^{R'} d\boldsymbol{u}_1^{I'}]D'\ . \tag{A1.11}$$

dt_{11}^I of dT is represented by

$$
\begin{aligned}
dt_{11}^I&=-\boldsymbol{u}_1^{I'}d\boldsymbol{u}_1^R+\boldsymbol{u}_1^{R'}d\boldsymbol{u}_1^I \\
&=(-\boldsymbol{u}_1^{I'},\ \boldsymbol{u}_1^{R'})\begin{bmatrix} d\boldsymbol{u}_1^R \\ d\boldsymbol{u}_1^I \end{bmatrix} \\
&=(-\boldsymbol{u}_1^{I'},\ \boldsymbol{u}_1^{R'})D^{-1}\begin{bmatrix} d\boldsymbol{t}_1^R \\ d\boldsymbol{t}_1^I \end{bmatrix}\ .
\end{aligned}
$$

Therefore dt_{11}^I is a function of $d\boldsymbol{t}_1^R$ and $d\boldsymbol{t}_1^I$, which proves the previous assertion.

Secondly, we see the relation between dW and $d\lambda_1$, dV and dT. Here we must also note that in dT, dT_2 can be represented by $d\boldsymbol{t}_1^R$ and $d\boldsymbol{t}_1^I$. In fact, U_2 is a functional matrix of $\boldsymbol{u}_1^R$ and $\boldsymbol{u}_1^I$, i.e.,

$$u_{kl}=f_{kl}^R(\boldsymbol{u}_1^R,\ \boldsymbol{u}_1^I)+if_{kl}^I(\boldsymbol{u}_1^R,\ \boldsymbol{u}_1^I)\ , \qquad (1\leqq k\leqq m,\ 2\leqq l\leqq m)\ .$$

Therefore

$$du_{kl}=\sum_{r=2}^{m}\frac{\partial f_{kl}^R}{\partial u_{r1}^R}du_{r1}^R+\sum_{r=2}^{m}\frac{\partial f_{kl}^R}{\partial u_{r1}^I}du_{r1}^I+i\left\{\sum_{r=2}^{m}\frac{\partial f_{kl}^I}{\partial u_{r1}^R}du_{r1}^R+\sum_{r=2}^{m}\frac{\partial f_{kl}^I}{\partial u_{r1}^I}du_{r1}^R\right\}\ .$$

By setting

$$F_{kl}^R=\left[\frac{\partial f_{kl}}{\partial u_{21}^R},\cdots,\ \frac{\partial f_{kl}}{\partial u_{m1}^R}\right] \qquad \text{and} \qquad F_{kl}^I=\left[\frac{\partial f_{kl}}{\partial u_{21}^I},\cdots,\ \frac{\partial f_{kl}}{\partial u_{m1}^I}\right]\ ,$$

where $f_{kl}=f_{kl}^R+if_{kl}^I$, we have

$$dU_2=F^R\dot{\otimes}d\boldsymbol{u}_1^R+F^I\dot{\otimes}d\boldsymbol{u}_1^I\ ,$$

where $F^R=(F_{kl}^R)_{m\times(m-1)^2}$ and $F^I=(F_{kl}^I)_{m\times(m-1)^2}$. The notation $\dot{\otimes}$ means: $F^R\dot{\otimes}d\boldsymbol{u}_1^R=(F_{kl}^R d\boldsymbol{u}_1^R)$ and $F^I\dot{\otimes}d\boldsymbol{u}_1^I=(F_{kl}^I d\boldsymbol{u}_1^I)$, respectively.

Therefore we have from (A1.11),

$$\begin{aligned} dT_2=\bar{U}'dU_2=\bar{U}'[F^R\dot{\otimes}\{(D^{-1})_{11}d\boldsymbol{t}_1^R+(D^{-1})_{12}d\boldsymbol{t}_1^I\} \\ +F^I\dot{\otimes}\{(D^{-1})_{21}d\boldsymbol{t}_1^R+(D^{-1})_{22}d\boldsymbol{t}_1^I\}]\ , \end{aligned}$$

where $(D^{-1})_{ij}$ $(i, j=1, 2)$ is a submatrix of D^{-1} corresponding to $d\boldsymbol{t}_1^R$ and $d\boldsymbol{t}_1^I$, which establishes our assertion. Hence we need only $d\boldsymbol{t}_1^R$, $d\boldsymbol{t}_1^I$, dV and $d\lambda_1$.

From (A1.2), and $V=V^R+iV^I$,

$$\text{(A1.12)} \quad \begin{cases} d\omega_{11}=d\lambda_1\ , \\ d\boldsymbol{\omega}_1^R=(\lambda_1 I-V^R)d\boldsymbol{t}_1^R+V^I d\boldsymbol{t}_1^I\ , \\ d\boldsymbol{\omega}_1^I=-V^I d\boldsymbol{t}_1^R+(\lambda_1 I-V^R)d\boldsymbol{t}_1^I\ , \\ dW_{22}=dT_{22}V+dV-VdT_{22}\ , \end{cases}$$

where

$$dW=\begin{bmatrix} d\omega_{11} & d\bar{\boldsymbol{\omega}}_1' \\ d\boldsymbol{\omega}_1 & dW_{22} \end{bmatrix} \quad \text{and} \quad dT=\begin{bmatrix} dt_{11} & d\boldsymbol{t}_{12} \\ d\boldsymbol{t}_1^R+id\boldsymbol{t}_1^I & dT_{22} \end{bmatrix}.$$

Hence, forming the exterior product, we have

$$\begin{aligned} & d\omega_{11}\prod_{\alpha=2}^{m} d\omega_{\alpha 1}^R \prod_{\alpha=2}^{m} d\omega_{\alpha 1}^I \prod_{\alpha\geqq\beta\geqq 2} d\omega_{\alpha\beta}^R \prod_{\alpha\geqq\beta\geqq 2} d\omega_{\alpha\beta}^I \\ & =J_2 d\lambda_1 \prod_{\alpha=2}^{m} dt_{\alpha 1}^R \prod_{\alpha=2}^{m} dt_{\alpha 1}^I \prod_{\alpha\geqq\beta\geqq 2} dv_{\alpha\beta}^R \prod_{\alpha\geqq\beta\geqq 2} dv_{\alpha\beta}^I\ , \end{aligned}$$

$$\begin{aligned} J_2 &=\det\begin{bmatrix} 1 & 0' & 0' & 0' \\ 0 & \lambda_1 I-V^R & V^I & 0 \\ 0 & -V^I & \lambda_1 I-V^R & 0 \\ 0 & * & * & I \end{bmatrix} \\ &=\det(\lambda_1 I-V)^2\ . \end{aligned}$$

Summarizing these results, we have the Jacobian of the transformation as $J=\det(\lambda_1 I-V)^2$, which completes the proof.

Appendix 2. (PROOF OF LEMMA 2)

Differentiating both sides of (23), we have

$$\text{(A2.1)} \qquad dX=dU\Lambda\bar{L}'+Ud\Lambda\bar{L}'+U\Lambda d\bar{L}'\ .$$

Let $B_{n\times(n-m)}$ be a semi-unitary matrix such that $[L:B]$ is a unitary matrix. Multiplying both sides of (A2.1) by $\bar{U}'$ from the left and $[L:B]$ from the right, we have

$$\bar{U}'dX[L:B]=[\bar{U}'dU\Lambda:0]+[d\Lambda:0]+[\Lambda d\bar{L}'L:\Lambda d\bar{L}'B] \ . \tag{A2.2}$$

By setting L.H.D. of (A2.2) $dF=[\overset{m}{dF_1}:\overset{n-m}{dF_2}]m$, $dT=\bar{U}'dU$, $dP=-\bar{L}'dL$ and $dQ=\bar{L}'dB$, we have

$$dF_1=dT\Lambda+d\Lambda-\Lambda dP \tag{A2.3}$$

$$dF_2=-\Lambda dQ \ . \tag{A2.4}$$

From the construction of U, U can be expressed as $U=U_{m-1}U_{m-2}\cdots U_2U_1$, where

$$U_k=\begin{bmatrix} I_{k-1} & 0' & \\ & u_k & \boldsymbol{a}_k^R+i\boldsymbol{a}_k^I \\ 0 & & \\ & \boldsymbol{u}_k^R+i\boldsymbol{u}_k^I & W_k \end{bmatrix}, \qquad k=1,2,\cdots,m-1 \ ,$$

where U_k's are unitary matrices. U_k has $2(m-k)$ independent elements $\boldsymbol{u}_k^R$ and $\boldsymbol{u}_k^I$, and u_k, $\boldsymbol{a}_k^R+i\boldsymbol{a}_k^I$ and W_k are functions of $\boldsymbol{u}_k^R$ and $\boldsymbol{u}_k^I$, respectively. This implies that $dU=dU_{m-1}U_{m-2}\cdots U_1+\cdots+U_{m-1}\cdots U_2dU_1$. Hence the elements du_{kl}'s of dU are expressed as the functions of m^2-m independent elements $du_1^R,\cdots,du_{m-1}^R$, $du_1^I,\cdots,du_{m-1}^I$. Therefore, $dT=U'dU$ also has m^2-m independent elements since $(\det U)^2=1$. However, dT is a skew Hermitian matrix which has m^2 elements. On the other hand dT has only m^2-m independent elements. Therefore, we can assume that diagonal elements $dt_{\alpha\alpha}^I$'s of dT are represented by another ones of dT, i.e., $dt_{\alpha\alpha}^I=h_{\alpha\alpha}(dt_{21}^R,\cdots,dt_{m,m-1}^R,dt_{21}^I,\cdots,dt_{m,m-1}^I)$. Hence we have from (A2.3),

$$\begin{aligned}
&df_{\alpha\alpha}^{(1)R}=d\lambda_\alpha \\
&df_{\alpha\alpha}^{(1)I}=\lambda_\alpha\{h_{\alpha\alpha}(dt_{21}^R,\cdots,dt_{m,m-1}^R,dt_{21}^I,\cdots,dt_{m,m-1}^I)-dp_{\alpha\alpha}^I\} \ , \\
&df_{\alpha\beta}^{(1)R}=\lambda_\beta dt_{\alpha\beta}^R-\lambda_\alpha dp_{\alpha\beta}^R \ , \qquad (\alpha>\beta) \ , \\
&df_{\alpha\beta}^{(1)I}=\lambda_\beta dt_{\alpha\beta}^I-\lambda_\alpha dp_{\alpha\beta}^I \ , \qquad (\alpha>\beta) \ , \\
&df_{\beta\alpha}^{(1)R}=\lambda_\alpha dt_{\beta\alpha}^R-\lambda_\beta dp_{\beta\alpha}^R=-\lambda_\alpha dt_{\alpha\beta}^R+\lambda_\beta dp_{\alpha\beta}^I \ , \qquad (\alpha>\beta) \ , \\
&df_{\beta\alpha}^{(1)I}=\lambda_\alpha dt_{\beta\alpha}^I-\lambda_\beta dp_{\beta\alpha}^I=\lambda_\alpha dt_{\alpha\beta}^I-\lambda_\beta dp_{\alpha\beta}^I \ , \qquad (\alpha>\beta) \ ,
\end{aligned}$$

since dT and dP are skew Hermitian matrices. Hence, forming the exterior product, we have

$$\begin{aligned}
&\prod_{\alpha>\beta} df_{\alpha\beta}^{(1)R}df_{\beta\alpha}^{(1)R}\prod_{\alpha>\beta} df_{\alpha\beta}^{(1)I}df_{\beta\alpha}^{(1)I} \\
&\qquad =\prod_{\alpha>\beta}(\lambda_\alpha^2-\lambda_\beta^2)^2\prod_{\alpha>\beta} dp_{\alpha\beta}^R dt_{\alpha\beta}^R\prod_{\alpha>\beta} dp_{\alpha\beta}^I dt_{\alpha\beta}^I \ .
\end{aligned} \tag{A2.5}$$

We have from (A2.4)

$$df_{\alpha\beta}^{(2)R}=-\lambda_\alpha dq_{\alpha\beta}^R\ , \qquad df_{\alpha\beta}^{(2)I}=-\lambda_\alpha dq_{\alpha\beta}^I\ .$$

Hence

$$\prod_{\beta=1}^{n-m}\prod_{\alpha=1}^{m} df_{\alpha\beta}^{(2)R}df_{\alpha\beta}^{(2)I}=\left[\prod_{\alpha=1}^{m}\lambda_\alpha\right]^{2(n-m)}\prod_{\beta=1}^{n-m}\prod_{\alpha=1}^{m} dq_{\alpha\beta}^R dq_{\alpha\beta}^I\ .$$

Therefore we have

$$\begin{aligned} dF\overset{\text{def}}{\equiv}\prod_{\beta=1}^{n}\prod_{\alpha=1}^{m} df_{\alpha\beta}^R df_{\alpha\beta}^I=&\left[\prod_{\alpha=1}^{m}\lambda_\alpha\right]^{2(n-m)+1}\prod_{\alpha>\beta}(\lambda_\alpha^2-\lambda_\beta^2)^2\\ &\cdot\prod_{\alpha=1}^{m} d\lambda_\alpha\prod_{\alpha>\beta} dt_{\alpha\beta}^R dt_{\alpha\beta}^I\prod_{\alpha>\beta} dp_{\alpha\beta}^R dp_{\alpha\beta}^I\\ &\cdot\prod_{\alpha=1}^{m} dp_{\alpha\alpha}^I\prod_{\beta=1}^{n-m}\prod_{\alpha=1}^{m} dq_{\alpha\beta}^R dq_{\alpha\beta}^I\ .\end{aligned}$$

Since

$$\prod_{\beta=1}^{n}\prod_{\alpha=1}^{m} dx_{\alpha\beta}^R dx_{\alpha\beta}^I=\prod_{\beta=1}^{n}\prod_{\alpha=1}^{m} df_{\alpha\beta}^R df_{\alpha\beta}^I\ ,$$

(A2.6) $d^*(U)\overset{\text{def}}{\equiv}\prod_{\alpha>\beta} dt_{\alpha\beta}^R dt_{\alpha\beta}^I=h_2(U)dU$, in Khatri [8, (2.9)] ,

and $d^*(U)$ is also an invariant measure on a unitary group $U(m)$, (James [7]), and

(A2.7) $d^*(L)\overset{\text{def}}{\equiv}\prod_{\alpha=1}^{m} dp_{\alpha\alpha}^I\prod_{\alpha>\beta} dp_{\alpha\beta}^R dp_{\alpha\beta}^I\prod_{\beta=1}^{n-m}\prod_{\alpha=1}^{m} dq_{\alpha\beta}^R dq_{\alpha\beta}^I=h_1(L)dL$,

in Khatri [8, (2.6)], we finally have

(A2.8)
$$\begin{aligned} dX&\overset{\text{def}}{\equiv}\prod_{\beta=1}^{n}\prod_{\alpha=1}^{m} dx_{\alpha\beta}^R dx_{\alpha\beta}^I\ ,\\ &=(\det\Lambda)^{2(n-m)+1}\prod_{\alpha>\beta}(\lambda_\alpha^2-\lambda_\beta^2)^2 d\Lambda d^*(U)d^*(L)\ .\end{aligned}$$

Hence, we also have by Khatri [8, (2.7)],

$$\int_{U(m)} d^*(U)=\frac{\pi^{m(m-1)}}{\tilde{\Gamma}_m(m)}\qquad\text{and}\qquad\int_{\bar{L}'L=I_m} d^*(L)=\frac{\pi^{mn}}{\tilde{\Gamma}_m(n)}\ .\qquad \text{Q.E.D.}$$

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