ASYMPTOTIC EXPANSIONS OF THE NON-NULL DISTRIBUTIONS OF TWO CRITERIA FOR THE LINEAR HYPOTHESIS CONCERNING COMPLEX MULTIVARIATE NORMAL POPULATIONS[1]

YASUNORI FUJIKOSHI[2]

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1. Introduction and summary

Some distribution problems of the complex multivariate normal distribution appear in time series analysis (see e.g. Goodman [4], Hannan [5], p. 295–307). It is known that some distributions in the case of complex multivariate normal distributions can be obtained in the same manner as those in the case of real multivariate normal distributions. Such examples have been seen in the papers, e.q. Goodman [4], James [8] and Khatri [9], [10]. Recently, Hayakawa [6] has extended the asymptotic formulas for some distributions obtained by Fujikoshi [2] and Sugiura and Fujikoshi [14] to the complex variates by deriving some formulas of weighted sums of zonal polynomials of hermitian matrices. In this paper we derive asymptotic expansions of the distributions of Pillai's and Hotelling's criteria for the linear hypothesis concerning complex normal populations which are extensions of author [3] to the complex variates. For the purpose, we give the needed results on complex variates in a way parallel to one method of obtaining the asymptotic distributions of real multivariate analysis due to author [3].

2. Preliminaries

We list some necessary results on zonal polynomials of hermitian matrices and others which are used in the present paper. The hypergeometric function of matrix argument in complex case is defined by James [8] as follows:

$$(2.1)\qquad {}_r\tilde{F}_s(a_1,\cdots,a_r;b_1,\cdots,b_s;S,Z)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_r]_\kappa}{[b_1]_\kappa\cdots[b_s]_\kappa}\frac{\tilde{C}_\kappa(S)\tilde{C}_\kappa(Z)}{k!\tilde{C}_\kappa(I)}\,,$$

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2) Now at the University of Kobe, Kobe, Japan.

where $\tilde{C}_\kappa(Z)$ is a zonal polynomial of a hermitian matrix Z corresponding to a partition κ of k, i.e., $\kappa=\{k_1, k_2, \cdots, k_p\}$, $k_1\geqq k_2\geqq\cdots\geqq k_p\geqq 0$, $k_1+k_2+\cdots+k_p=k$ and is defined by

$$\tilde{C}_\kappa(Z)=\chi_{[\kappa]}(1)\chi_{\{\kappa\}}(Z)\ , \tag{2.2}$$

in which $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group on k symbols and $\chi_{\{\kappa\}}(Z)$ is the character of the representation $\{\kappa\}$ of the linear group $GL(p)$. Further

$$[a]_\kappa=\prod_{\alpha=1}^{p}(a-\alpha+1)_{k_\alpha}\ , \qquad (a)_k=a(a+1)\cdots(a+k-1)\ . \tag{2.3}$$

If either S or Z is I, then we write as $\tilde{F}(\cdots;\cdots;I, Z)=\tilde{F}(\cdots;\cdots;Z)$. Special cases of ${}_r\tilde{F}_s$ are

$${}_0\tilde{F}_0(Z)=\sum_{k=0}^{\infty}\sum_{\kappa}\tilde{C}_\kappa(Z)/k!=\mathrm{etr}\, Z\ , \tag{2.4}$$

$${}_1\tilde{F}_0(a; Z)=\sum_{k=0}^{\infty}\sum_{\kappa}[a]_\kappa\tilde{C}_\kappa(Z)/k!=|I-Z|^{-a}\ . \tag{2.5}$$

The most properties of a zonal polynomial are of the same form as in the real case, i.e.,

$$\int_{U(p)}\tilde{C}_\kappa(\bar{U}'SUZ)d\tilde{\mu}(U)=\tilde{C}_\kappa(S)\tilde{C}_\kappa(Z)/\tilde{C}_\kappa(I)\ , \tag{2.6}$$

$$\int_{\bar{S}'=S>0}\{\mathrm{etr}\,(-ZS)\}|S|^{t-p}\tilde{C}_\kappa(ST)dS=\tilde{\Gamma}_p(t)[t]_\kappa|Z|^{-t}\tilde{C}_\kappa(TZ^{-1})\ , \tag{2.7}$$

$$\int_{I>\bar{S}'=S>0}|S|^{a-p}|I-S|^{b-p}\tilde{C}_\kappa(ST)dS=\frac{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b)[a]_\kappa}{\tilde{\Gamma}_p(a+b)[a+b]_\kappa}\tilde{C}_\kappa(T)\ , \tag{2.8}$$

where $\tilde{\mu}$ is the invariant measure on the unitary group $U(p)$ normalized to make the total measure unity, the second formula (2.7) holds for any positive definite hermitian matrix Z and any hermitian matrix T with $\mathcal{R}(t)>p-1$, and the complex multivariate gamma function $\tilde{\Gamma}_p(t)$ is defined by

$$\tilde{\Gamma}_p(t)=\pi^{p(p-1)/2}\prod_{\alpha=1}^{p}\Gamma(t-\alpha+1)\ . \tag{2.9}$$

The differential relations for $\tilde{C}_\kappa(Z)$, which are derived by a slight modification of Fujikoshi [3], are (see Hayakawa [6]),

$$\{\tilde{a}_1(\kappa)+k\}\tilde{C}_\kappa(Z)=\mathrm{tr}\,(\Lambda\partial)^2\tilde{C}_\kappa(\Sigma)|_{\Sigma=\Lambda}\ , \tag{2.10}$$

$$\begin{aligned}&\{3\tilde{a}_1(\kappa)^2-2\tilde{a}_2(\kappa)+6\tilde{a}_1(\kappa)(k-1)+3k^2-2k\}\tilde{C}_\kappa(Z)\\&\quad=[3\{\mathrm{tr}\,(\Lambda\partial)^2\}^2+8\,\mathrm{tr}\,(\Lambda\partial)^3]\tilde{C}_\kappa(\Sigma)|_{\Sigma=\Lambda}\ ,\end{aligned} \tag{2.11}$$

where ∂ denotes the matrix of differential operators having $(1+\delta_{rs})/2\cdot\partial/\partial\sigma^R_{rs}+i((1-\delta_{rs})/2)\cdot\partial/\partial\sigma^I_{rs}$ as its (r, s) element for a hermitian matrix $\Sigma=(\sigma^R_{rs})+i(\sigma^I_{rs})$, $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$ is a diagonal matrix with p characteristic roots of Z as its non-zero elements, and $\tilde{a}_j(\kappa)$ $(j=1, 2)$ are defined by

$$(2.12) \qquad \tilde{a}_1(\kappa)=\sum_{\alpha=1}^p k_\alpha(k_\alpha-2\alpha)\ , \qquad \tilde{a}_2(\kappa)=2\sum_{\alpha=1}^p k_\alpha(k_\alpha^2-3\alpha k_\alpha+3\alpha^2)\ .$$

Let $f(\Sigma)$ be a real valued function of hermitian matrix Σ and analytic about $\Sigma=\Lambda$. Then the following formula with the same notations as (2.10) and (2.11) has been recognized by Hayakawa [6]:

$$(2.13) \qquad \frac{n^{pn}}{\tilde{\Gamma}_p(n)|Z|^n}\int_{\bar{S}'=S>0}\{\mathrm{etr}\,(-nZ^{-1}S)\}|S|^{n-p}f(S)dS$$
$$=\left[1+\frac{1}{2n}\,\mathrm{tr}\,(\Lambda\partial)^2+\frac{1}{24n^2}\{3(\mathrm{tr}\,(\Lambda\partial)^2)^2 +8\,\mathrm{tr}\,(\Lambda\partial)^3\}+O(n^{-3})\right]f(\Sigma)\Big|_{\Sigma=\Lambda}\ ,$$

where Z is a hermitian positive definite matrix.

The Laguerre polynomial of matrix argument in complex case is defined by

$$(2.14) \quad \tilde{L}^\gamma_\kappa(\Omega)=\frac{\mathrm{etr}\,\Omega}{\tilde{\Gamma}_p(\gamma+p)}\int_{\bar{S}'=S>0}\{\mathrm{etr}\,(-S)\}|S|^\gamma\tilde{C}_\kappa(S)_0\tilde{F}_1(\gamma+p;\,-S\Omega)dS\ ,$$

where $\gamma>-1$.

3. Some useful formulas

We will use the following abbreviated notations:

$$(3.1) \qquad \tilde{L}_Z[\{\ \}]=\frac{|Z|^q}{\tilde{\Gamma}_p(q)}\int_{\bar{S}'=S>0}\{\mathrm{etr}\,(-ZS)\}|S|^{q-p}\{\ \}dS\ ,$$

for any matrix $Z=X+iY$ such that X is a hermitian positive definite matrix and Y is a hermitian matrix, and

$$(3.2) \qquad \tilde{I}[\{\ \}]=\frac{\tilde{\Gamma}_p(q)2^{p(p-1)}}{(2\pi i)^{p^2}}\int_{Z=X_0+iY}(\mathrm{etr}\,Z)|Z|^{-q}\{\ \}dZ\ ,$$

where q is any number satisfying $\mathcal{R}(q)>p-1$. In (3.2) the integration is taken over $Z=X_0+iY$ with X_0 fixed positive definite hermitian matrix and Y ranges over all hermitian matrices. From (2.7) we can get

$$(3.3) \qquad \tilde{L}_Z[\tilde{C}_\kappa(\Omega S)]=[q]_\kappa\tilde{C}_\kappa(\Omega Z^{-1})\ ,$$

for any hermitian matrix Ω. Applying the general Cauchy inverse formula (see Bochner [1]) to (3.3) and using the same method as in the derivation of (3.4) in Fujikoshi [3] with the help of (2.6), we can write

$$\tilde{I}[\tilde{C}_\kappa(\Omega Z^{-1})]=\tilde{C}_\kappa(\Omega)/[q]_\kappa\,. \tag{3.4}$$

By using (3.3), (3.4) and the explicit formulas of zonal polynomials up to order 4 in Appendix 1, we get the following Lemmas 1 and 2.

LEMMA 1. *Put* $\tilde{s}_j=\mathrm{tr}\,(\Omega S)^j$ *and* $\tilde{z}_j=\mathrm{tr}\,(\Omega Z^{-1})^j$ $(j=1,2,3,4)$, *then the following identities hold:*

$$\tilde{L}_Z[\tilde{s}_1]=q\tilde{z}_1\,, \tag{3.5}$$

$$\begin{bmatrix}\tilde{L}_Z[\tilde{s}_1^2]\\ \tilde{L}_Z[\tilde{s}_2]\end{bmatrix}=q\begin{bmatrix}q & 1\\ 1 & q\end{bmatrix}\begin{bmatrix}\tilde{z}_1^2\\ \tilde{z}_2\end{bmatrix}, \tag{3.6}$$

$$\begin{bmatrix}\tilde{L}_Z[\tilde{s}_1^3]\\ \tilde{L}_Z[\tilde{s}_1\tilde{s}_2]\\ \tilde{L}_Z[\tilde{s}_3]\end{bmatrix}=q\begin{bmatrix}q^2 & 3q & 2\\ q & q^2+2 & 2q\\ 1 & 3q & q^2+1\end{bmatrix}\begin{bmatrix}\tilde{z}_1^3\\ \tilde{z}_1\tilde{z}_2\\ \tilde{z}_3\end{bmatrix}, \tag{3.7}$$

$$\begin{bmatrix}\tilde{L}_Z[\tilde{s}_1^4]\\ \tilde{L}_Z[\tilde{s}_1^2\tilde{s}_2]\\ \tilde{L}_Z[\tilde{s}_2^2]\\ \tilde{L}_Z[\tilde{s}_1\tilde{s}_3]\\ \tilde{L}_Z[\tilde{s}_4]\end{bmatrix}=q\begin{bmatrix}q^3 & 6q^2 & 3q & 8q & 6\\ q^2 & q(q^2+5) & q^2+2 & 4(q^2+1) & 6q\\ q & 2(q^2+2) & q(q^2+2) & 8q & 2(2q^2+1)\\ q & 3(q^2+1) & 3q & q(q^2+7) & 3(q^2+1)\\ 1 & 6q & 2q^2+1 & 4(q^2+1) & q(q^2+5)\end{bmatrix}\begin{bmatrix}\tilde{z}_1^4\\ \tilde{z}_1^2\tilde{z}_2\\ \tilde{z}_2^2\\ \tilde{z}_1\tilde{z}_3\\ \tilde{z}_4\end{bmatrix}. \tag{3.8}$$

LEMMA 2. *The following identities hold:*

$$q\tilde{I}[\tilde{z}_1]=\omega_1\,, \tag{3.9}$$

$$d_1\begin{bmatrix}\tilde{I}[\tilde{z}_1^2]\\ \tilde{I}[\tilde{z}_2]\end{bmatrix}=\begin{bmatrix}q & -1\\ -1 & q\end{bmatrix}\begin{bmatrix}\omega_1^2\\ \omega_2\end{bmatrix}, \tag{3.10}$$

$$d_2\begin{bmatrix}\tilde{I}[\tilde{z}_1^3]\\ \tilde{I}[\tilde{z}_1\tilde{z}_2]\\ \tilde{I}[\tilde{z}_3]\end{bmatrix}=\begin{bmatrix}q^2-2 & -3q & 4\\ -q & q^2+2 & -2q\\ 2 & -3q & q^2\end{bmatrix}\begin{bmatrix}\omega_1^3\\ \omega_1\omega_2\\ \omega_3\end{bmatrix}, \tag{3.11}$$

$$d_3\begin{bmatrix}\tilde{I}[\tilde{z}_1^4]\\ \tilde{I}[\tilde{z}_1^2\tilde{z}_2]\\ \tilde{I}[\tilde{z}_2^2]\\ \tilde{I}[\tilde{z}_1\tilde{z}_3]\\ \tilde{I}[\tilde{z}_4]\end{bmatrix}=\begin{bmatrix}q^4-8q^2+6 & -6q(q^2-4) & 3(q^2+6)\\ -q(q^2-4) & q^2(q^2+1) & -q(q^2+6)\\ q^2+6 & -2q(q^2+6) & q^4-6q^2+18\\ 2q^2-3 & -3q(q^2+1) & 3(2q^2-3)\\ -5q & 10q^2 & -q(2q^2-3)\end{bmatrix} \tag{3.12}$$

$$\left.\begin{matrix} 8(2q^2-3) & -30q \\ -4q(q^2+1) & 10q^2 \\ 8(2q^2-3) & -2q(2q^2-3) \\ q^4+3q^2+12 & -3q(q^2+1) \\ -4q(q^2+1) & q^2(q^2+1) \end{matrix}\right] \begin{bmatrix} \omega_1^4 \\ \omega_1^2\omega_2 \\ \omega_2^2 \\ \omega_1\omega_3 \\ \omega_4 \end{bmatrix}$$

where $\omega_j = \mathrm{tr}\,\Omega^j$ $(j=1, 2, 3, 4)$ *and* d_j *are given by*

$$d_1 = q(q^2-1)\,, \qquad d_2 = q(q^2-1)(q^2-4)\,, \qquad d_3 = q^2(q^2-1)(q^2-4)(q^2-9)\,. \tag{3.13}$$

The following Lemma 3 obtained by Hayakawa [6] has been used for obtaining some asymptotic distributions in the complex variates.

LEMMA 3. *Put* $z_j = \mathrm{tr}\,Z^j$ $(j=1, 2, 3)$, *then the following identities hold*:

$$\sum_{k=l}^{\infty}\sum_{\kappa} \tilde{C}_\kappa(Z)/(k-l)! = z_1^l e^{z_1}\,, \tag{3.14}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} \tilde{C}_\kappa(Z)\tilde{a}_1(\kappa)/k! = (z_2-z_1)e^{z_1}\,, \tag{3.15}$$

$$\sum_{k=1}^{\infty}\sum_{\kappa} \tilde{C}_\kappa(Z)\tilde{a}_1(\kappa)/(k-1)! = (-z_1-z_1^2+2z_2+z_1z_2)e^{z_1}\,, \tag{3.16}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} \tilde{C}_\kappa(Z)\tilde{a}_1(\kappa)^2/k! = (z_1+3z_1^2-4z_2-2z_1z_2+4z_3+z_2^2)e^{z_1}\,, \tag{3.17}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} \tilde{C}_\kappa(Z)\tilde{a}_2(\kappa)/k! = (2z_1+3z_1^2-3z_2+2z_3)e^{z_1}\,. \tag{3.18}$$

By using the above Lemmas 1, 2 and 3, we obtain the following Lemmas 4 and 5 which play an important role in derivation of asymptotic expansions of the distributions of Pillai's and Hotelling's criteria for the linear hypothesis in the complex variates, respectively.

LEMMA 4. *Let* Z *be any hermitian matrix such that all the absolute values of the characteristic roots are less than one and put* $v_j = \mathrm{tr}\,\{Z(I-Z)^{-1}\}^j$ $(j=1, 2, 3, 4)$. *Then the following identities hold*:

$$\sum_{k=1}^{\infty}\sum_{\kappa} [q]_\kappa \tilde{C}_\kappa(Z)/(k-1)! = qv_1|I-Z|^{-q}\,, \tag{3.19}$$

$$\sum_{k=2}^{\infty}\sum_{\kappa} [q]_\kappa C_\kappa(Z)/(k-2)! = q(qv_1^2+v_2)|I-Z|^{-q}\,, \tag{3.20}$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} [q]_\kappa \tilde{C}_\kappa(Z)\tilde{a}_1(\kappa)/k! = q(-v_1+v_1^2+qv_2)|I-Z|^{-q}\,, \tag{3.21}$$

(3.22)
$$\sum_{k=1}^{\infty}\sum_{\kappa}[q]_{\kappa}\tilde{C}_{\kappa}(Z)\tilde{a}_1(\kappa)/(k-1)!$$
$$=q\{-v_1-(q-2)v_1^2+(2q-1)v_2+qv_1^3+(q^2+2)v_1v_2 +2qv_3\}|I-Z|^{-q},$$

(3.23)
$$\sum_{k=0}^{\infty}\sum_{\kappa}[q]_{\kappa}\tilde{C}_{\kappa}(Z)\tilde{a}_1(\kappa)^2/k!$$
$$=q\{v_1+(3q-4)v_1^2-(4q-3)v_2-2(q-2)v_1^3 -2(q^2-6q+2)v_1v_2+4(q^2-q+1)v_3+qv_1^4 +2(q^2+2)v_1^2v_2+q(q^2+2)v_2^2+8qv_1v_3 +2(2q^2+1)v_4\}|I-Z|^{-q},$$

(3.24)
$$\sum_{k=0}^{\infty}\sum_{\kappa}[q]_{\kappa}\tilde{C}_{\kappa}(Z)\tilde{a}_2(\kappa)/k!$$
$$=q\{2v_1+3(q-1)v_1^2-3(q-1)v_2+2v_1^3 +6qv_1v_2+2(q^2+1)v_3\}|I-Z|^{-q}.$$

PROOF. It is sufficient to prove that the formulas (3.19)~(3.24) hold for $\mathcal{R}(q)>p-1$. From (2.7) and (3.15) we can get the following reductions for (3.21):

(3.25)
$$\sum_{k=0}^{\infty}\sum_{\kappa}[q]_{\kappa}\tilde{C}_{\kappa}(Z)\tilde{a}_1(\kappa)/k!$$
$$=\sum_{k=0}^{\infty}\sum_{\kappa}\tilde{L}_I[\tilde{C}_{\kappa}(ZS)]\tilde{a}_1(\kappa)/k!$$
$$=\tilde{L}_{I-Z}[-\mathrm{tr}\, ZS-(\mathrm{tr}\, ZS)^2+2\,\mathrm{tr}\,(SZ)^2 +(\mathrm{tr}\, ZS)\,\mathrm{tr}\,(ZS)^2].$$

By Lemma 1 we can see that (3.21) is true. Similarly, we can derive the another formulas by using Lemmas 1 and 3.

LEMMA 5. *Let x be any number such that $|x|<1$ and put $\gamma=q-p>-1$, $y=x(1-x)^{-1}$ and $t_j=\mathrm{tr}\{(1-x)^{-1}\Omega\}^j$ $(j=1, 2, 3)$. Then the following identities hold:*

(3.26)
$$\sum_{k=0}^{\infty}\sum_{\kappa}x^k\tilde{L}_{\kappa}^{\gamma}(\Omega)/k!=(1-x)^{-pq}e^{-xt_1},$$

(3.27)
$$\sum_{k=1}^{\infty}\sum_{\kappa}x^k\tilde{L}_{\kappa}^{\gamma}(\Omega)/(k-1)!=(1-x)^{-pq}y(pq-t_1)e^{-xt_1},$$

(3.28)
$$\sum_{k=2}^{\infty}\sum_{\kappa}x^k\tilde{L}_{\kappa}^{\gamma}(\Omega)/(k-2)!$$
$$=(1-x)^{-pq}y^2\{pq(pq+1)-2(pq+1)t_1+t_1^2\}e^{-xt_1},$$

(3.29)
$$\sum_{k=0}^{\infty}\sum_{\kappa}x^k\tilde{L}_{\kappa}^{\gamma}(\Omega)\tilde{a}_1(\kappa)/k!$$

$$=(1-x)^{-pq}y[pq\{-1+(p+q)y\}+\{1-2(p+q)y\}t_1 + yt_2\}e^{-xt_1} ,$$

(3.30)
$$\sum_{k=1}^{\infty}\sum_{\kappa} x^k\tilde{L}_\kappa^I(\Omega)\tilde{a}_1(\kappa)/(k-1)!$$
$$=(1-x)^{-pq}y[-pq\{1+((q-2)p-2q+1)y -(qp^2+(q^2+2)p+2q)y^2\}+\{1+2((q-2)p-2q+1)y -3(qp^2+(q^2+2)p+2q)y^2\}t_1+\{-1+2(p+y)y\}yt_1^2 +\{2+(pq+4)y\}yt_2-y^2t_1t_2]e^{-xt_1} ,$$

(3.31)
$$\sum_{k=0}^{\infty}\sum_{\kappa} x^k\tilde{L}_\kappa^I(\Omega)\tilde{a}_1(\kappa)^2/k!$$
$$=(1-x)^{-pq}y[pq\{1+((3q-4)p-4q+3)y -2((q-2)p^2+(q^2-6q+2)p-2(q^2-q+1))y^2 +(qp^3+2(q^2+2)p^2+q(q^2+10)p+2(2q^2+1))y^3\} -\{1+2((3q-4)p-4q+3)y-6((q-2)p^2 +(q^2-6q+2)p-2(q^2-q+1))y^2+4(qp^3+2(q^2+2)p^2 +q(q^2+10)p+2(2q^2+1))y^3\}t_1+\{3-4(p+q-3)y +4(p^2+2pq+q^2+3)y^2\}yt_1^2+2\{-2-((q-6)p -6q+4)y+(qp^2+(q^2+10)p+10q)y^2\}yt_2 +2\{1-2(p+q)y\}y^2t_1t_2-4(1+2y)y^2t_3+y^3t_2^2]e^{-xt_1} ,$$

(3.32)
$$\sum_{k=0}^{\infty}\sum_{\kappa} x^k\tilde{L}_\kappa^I(\Omega)\tilde{a}_2(\kappa)/k!$$
$$=(1-x)^{-pq}y[pq\{2+3((q-1)p-q+1)y+2(p^2+3pq +q^2+1)y^2\}-2\{1+3((q-1)p-q+1)y +3(p^2+3pq+q^2+1)y^2\}t_1+3(1+2y)yt_1^2 +3\{-1+2(p+q)y\}yt_2-2y^2t_3]e^{-xt_1} .$$

PROOF. The proof is done completely same way as one of Lemma 8 Fujikoshi [3] with the help of Lemmas 1, 2 and 3.

LEMMA 6. *The following identities hold:*

(3.33)
$${}_1\tilde{F}_1(a;b;S)=\frac{\tilde{\Gamma}_p(b)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b-a)}\int_{I>\bar{B}'=B>0}(\mathrm{etr}\, SB)|B|^{a-p}|I-B|^{b-a-p}dB ,$$

(3.34)
$$\tilde{L}_I[{}_r\tilde{F}_s(a_1,\cdots,a_r;b_1,\cdots,b_s;ST,U)] ={}_{r+1}\tilde{F}_s(a_1,\cdots,a_r,q;b_1,\cdots,b_s;T,U) ,$$

(3.35)
$$\tilde{I}[{}_r\tilde{F}_s(a_1,\cdots,a_r;b_1,\cdots,b_s;Z^{-1}S,U)] ={}_r\tilde{F}_{s+1}(a_1,\cdots,a_r;b_1,\cdots,b_s,q;S,U) .$$

PROOF. The above formulas follow from (2.8), (3.3) and (3.4), respectively.

4. Asymptotic expansion of the non-null distribution of Pillai's criterion

The multivariate linear hypothesis model in the complex variates is reduced to the following canonical form: Let the each column vector of $p\times N$ matrix $X'=(X_1'(p+q), X_2'(p\times \overline{N-s}), X_3'(p\times \overline{s-q}))$ with $q\leqq s$ have the complex p-variate normal distribution with the common $p\times p$ hermitian positive definite covariance matrix Σ and expectations given by

$$E[X_1]=M(q\times p)\,,\qquad E[X_2]=0(\overline{N-s}\times p)\,,\qquad E[X_3]=\Gamma(\overline{s-q}\times p)\,.$$

Then the hypothesis H and the alternative K are specified by

$$H: M=0\,,\qquad K: M\neq 0\,. \tag{4.1}$$

The Pillai's criterion with an appropriate constant multiplier is expressed by

$$V=2m\ \mathrm{tr}\, S_h(S_h+S_e)^{-1}\,, \tag{4.2}$$

where $m=N-s+q$, $S_h=X_1'\bar{X}_1$ and $S_e=X_2'\bar{X}_2$. Under K, S_e has the complex Wishart distribution with $n=N-s$ degrees of freedom and S_h has the non-central complex Wishart distribution with q degrees of freedom and non-centrality parameter matrix $\Omega=\Sigma^{-1/2}M'\bar{M}\Sigma^{-1/2}$. The Laplace transform of a density function of V under alternative K with $p\leqq q$ is is expressed as follows:

$$M(t)=\{\mathrm{etr}\,(-\Omega)\}\tilde{I}\Big[\frac{1}{\tilde{\Gamma}_p(m)}\int_{\bar{S}'=S>0}\{\mathrm{etr}\,(-S)\}|S|^{m-p} \cdot {}_1\tilde{F}_1(q;m;-2mtI+\Omega^{1/2}S\Omega^{1/2}Z^{-1})dS\Big]\,, \tag{4.3}$$

where $\tilde{I}[\ ]$ is defined by (3.2). This expression is obtained by the same method as in Pillai [13] with the help of Lemma 6. Now we expand (4.3) as before. By Lemma 4, expanding ${}_1\tilde{F}_1$ in the above expression as $|(1+2t)I-\Omega^{1/2}m^{-1}S\Omega^{1/2}Z^{-1}|^{-q}\{1-q(2m)^{-1}U_1(m^{-1}S, Z)+q(24m^2)^{-1}U_2(m^{-1}S, Z)+O(m^{-3})f(m^{-1}S, Z)\}$, and using (2.13), we can write the part in the brackets [ ] in (4.3) as

$$\phi^{pq}|Z|^q|Z-\phi\Omega|^{-q}\Big[1-\frac{1}{2m}\{qU_1(I, Z) - |Z-\phi\Omega|^q\ \mathrm{tr}\,\partial^2|Z-\phi\Omega^{1/2}\Sigma\Omega^{1/2}|^{-q}|_{\Sigma=I}\}+\frac{1}{24m^2}\{qU_2(I, Z) \tag{4.4}$$

$$-6q|Z-\phi\Omega|^q \operatorname{tr} \partial^2 |Z-\phi\Omega^{1/2}\Sigma\Omega^{1/2}|^{-q} U_1(\Sigma, Z)|_{\Sigma=I}$$
$$+|Z-\phi\Omega|^q(8 \operatorname{tr} \partial^3+3(\operatorname{tr} \partial^2)^2)|Z-\phi\Omega^{1/2}\Sigma\Omega^{1/2}|^{-q}|_{\Sigma=I}\}+O(m^{-3})\Big] ,$$

which holds for sufficiently small $|t|$ such that all the absolute values of the characteristic roots of $-2tI+\Omega^{1/2}\Sigma\Omega^{1/2}Z^{-1}$ are less than one and large m, where $\phi=(1+2t)^{-1}$ and

$$U_1(\Sigma, Z)=w_1^2+qw_2 , \tag{4.5}$$

$$U_2(\Sigma, Z)=12qw_1^2+12w_2+16w_1^3+48qw_1w_2+16(q^2+1)w_3+3qw_1^4 \\ +6(q^2+2)w_1^2w_2+3q(q^2+2)w_2^2+24qw_1w_3+6(2q^2+1)w_4 \tag{4.6}$$

with $w_j=\operatorname{tr}[(-2tI+\Omega^{1/2}\Sigma\Omega^{1/2}Z^{-1})\{(1+2t)I-\Omega^{1/2}\Sigma\Omega^{1/2}Z^{-1}\}^{-1}]^j$ $(j=1, 2, 3, 4)$. From the formulas on ∂ in Appendix 2 and Lemma 2, we finally obtain the following asymptotic formula for the Laplace transform of a density function of V:

$$M(t)=(1+2t)^{-pq}\exp\left(\frac{-2t}{1+2t}\omega_1\right)\Big[1-\frac{1}{2m}\{pq(p+q)-2pq(p+q)\phi \\ +(pq(p+q)-2(p+q)\omega_1-\omega_2)\phi^2+2(p+q)\omega_1\phi^3+\omega_2\phi^4\} \\ +\frac{1}{24m^2}\left\{\sum_{\alpha=0}^{8} A_\alpha\phi^\alpha\right\}+O(m^{-3})\Big] , \tag{4.7}$$

which holds for all $t=x+iy$ satisfying $|x|<x_0$ for some small x_0 $\left(0<x_0<\frac{1}{2}\right)$, where $\omega_j=\operatorname{tr}\Omega^j$ and A_α $(\alpha=0, 1, \cdots, 8)$ are given by

$$A_0=pqh_0 , \qquad A_1=-pqh_1 , \tag{4.8}$$
$$A_2=pqh_2-h_1\omega_1-6pq(p+q)\omega_2 ,$$
$$A_3=-pqh_3+2h_2\omega_1+12(p+q)(pq+2)\omega_2+8\omega_3 ,$$
$$A_4=pqh_4-3h_3\omega_1+12(p^2+2pq+q^2-1)\omega_1^2-12(p+q)\omega_2 \\ +12(p+q)\omega_1\omega_2+3\omega_2^2 ,$$
$$A_5=4h_4\omega_1-24(p^2+2pq+q^2+1)\omega_1^2-12(qp^2+(q^2+6)p+6q)\omega_2 \\ -12(p+q)\omega_1\omega_2-24\omega_3 ,$$
$$A_6=12(p^2+2pq+q^2+3)\omega_1^2+6(qp^2+(q^2+10)p+10q)\omega_2 \\ -12(p+q)\omega_1\omega_2-8\omega_3-6\omega_2^2 ,$$
$$A_7=12(p+q)\omega_1\omega_2+24\omega_3 , \qquad A_8=3\omega_2^2 ,$$

with h_α $(\alpha=0, 1, \cdots, 4)$ given by

$$h_0=3qp^3+2(3q^2-2)p^2+3q(q^2-2)p-4q^2+2 , \tag{4.9}$$
$$h_1=12pq(p+q)^2 ,$$

$$h_2=6\{3qp^3+2(3q^2+2)p^2+q(3q^2+8)p+4q^2\}\ ,$$
$$h_3=4\{3qp^3+2(3q^2+4)p^2+3q(q^2+6)p+2(4q^2+1)\}\ ,$$
$$h_4=3\{qp^3+2(q^2+2)p^2+q(q^2+10)p+2(2q^2+1)\}\ .$$

We note that the asymptotic expansion of $M(t)$ in the case of $q<p$ is also given by (4.7). By inverting (4.7) we obtain the following,

THEOREM 1. *The non-null distribution of Pillai's criterion* (4.2) *for the multivariate linear hypothesis* (4.1) *can be approximated asymptotically up to order* m^{-2} *by*

$$(4.10)\quad \begin{aligned} P(V<x)=&P(\chi_f^2(\delta^2)<x)-\frac{1}{2m}[pq(p+q)P(\chi_f^2(\delta^2)<x)\\ &-2pq(p+q)P(\chi_{f+2}^2(\delta^2)<x)+\{pq(p+q)\\ &-2(p+q)\omega_1-\omega_2\}P(\chi_{f+4}^2(\delta^2)<x)\\ &+2(p+q)\omega_1P(\chi_{f+6}^2(\delta^2)<x)+\omega_2P(\chi_{f+8}^2(\delta^2)<x)]\\ &+\frac{1}{24m^2}\left\{\sum_{\alpha=0}^{8}A_\alpha P(\chi_{f+2\alpha}^2(\delta^2)<x)\right\}+O(m^{-3})\ , \end{aligned}$$

where $m=N-s+q$, $f=2pq$, $\delta^2=\omega_1=\mathrm{tr}\,\Omega$ *and the coefficients* h_α $(\alpha=0,1,\cdots,4)$ *and* A_α $(\alpha=0,1,\cdots,8)$ *are given by* (4.8) *and* (4.9), *respectively. The non-central* χ^2*-variate with* f *degrees of freedom and non-centrality parameter* δ^2 *is denoted by* $\chi_f^2(\delta^2)$.

COROLLARY 1. *The null distribution of* V *can be approximated asymptotically up to order* m^{-2} *by the following distribution*:

$$(4.11)\quad \begin{aligned} P(V<x)=&P(\chi_f^2<x)-\frac{pq}{2m}(p+q)\{P(\chi_f^2<x)-2P(\chi_{f+2}^2<x)\\ &+P(\chi_{f+4}^2<x)\}+\frac{pq}{24m^2}\left\{\sum_{\alpha=0}^{4}(-1)^\alpha h_\alpha P(\chi_{f+2\alpha}^2<x)\right\}\\ &+O(m^{-3})\ . \end{aligned}$$

The formula (4.11) is immediately obtained by putting $\Omega=0$ and $\delta=0$ in (4.10).

5. Asymptotic expansion of the non-null distribution of Hotelling's criterion

The Hotelling's criterion with an appropriate constant multiplier for the testing problem (4.1) is expressed as

$$(5.1)\qquad T_0^2=2n\ \mathrm{tr}\ S_hS_e^{-1}\ ,$$

where $n=N-s$. For the definition of S_h and S_e, see Section 4. Under the alternative K with $p\leq q$, the Laplace transform $g(t)$ of a density function of T_0^2 can be expressed asymptotically as (5.2).

$$(5.2)\qquad g(t)=\{\operatorname{etr}(-\Omega)\}\left\{\frac{\tilde{\Gamma}_p(n+q)}{\tilde{\Gamma}_p(n)n^{pq}}\right\}(2t)^{-pq}\sum_{k=0}^{\infty}\sum_{\kappa}\left(-\frac{1}{2t}\right)^k\frac{\tilde{L}_\kappa^{q-p}(\Omega)}{k!}$$
$$\cdot\left[1+\frac{1}{2n}\{\tilde{a}_1(\kappa)+(2q+1)k\}+\frac{1}{24n^2}\{3\tilde{a}_1(\kappa)^2-2\tilde{a}_2(\kappa)\right.$$
$$+6(2q+1)(k-1)\tilde{a}_1(\kappa)+3(4q^2+4q+1)k(k-1)+k\}$$
$$\left.+O(n^{-3})\right],$$

for $|t|>1/2$ and large n. The formula (5.2) is obtained by the same method as (6.6) in Fujikoshi [3] with the help of (2.10), (2.11), (2.13) and (16) in Hayakawa [7]. Therefore, by the completely same way as Fujikoshi [3] with the help of Lemma 5 we obtain the following Theorem 2 and Corollary 2 (for the inversion, see e.g. Constantine [1963, A.M.S. p. 1274~1275]).

THEOREM 2. *Under alternative K the distribution of Hotelling's criterion* (5.1) *for the multivariate linear hypothesis* (4.1) *can be expressed asymptotically as follows*:

$$(5.3)\qquad P(T_0^2<x)=P(\chi_f^2(\delta^2)<x)+\frac{1}{2n}\{-pq(p-q)P(\chi_f^2(\delta^2)<x)$$
$$-2q(pq-\omega_1)P(\chi_{f+2}^2(\delta^2)<x)+(pq(p+q)$$
$$-2(p+2q)\omega_1+\omega_2)P(\chi_{f+4}^2(\delta^2)<x)$$
$$+2((p+q)\omega_1-\omega_2)P(\chi_{f+6}^2(\delta^2)<x)+\omega_2P(\chi_{f+8}^2(\delta^2)<x)\}$$
$$+\frac{1}{24n^2}\left\{\sum_{\alpha=0}^{8}B_\alpha P(\chi_{f+2\alpha}^2(\delta^2)<x)\right\}+O(n^{-3}),$$

where $n=N-s$, $f=2pq$, $\delta^2=\omega_1=\operatorname{tr}\Omega$ and the coefficients B_α $(\alpha=0,1,\cdots,8)$ are given by

$$(5.4)\qquad B_0=pql_0,\qquad B_1=-l_1(pq-\omega_1),$$
$$B_2=pql_2-(l_1+2l_2)\omega_1+12q^2\omega_1^2-6q(p^2-pq-2)\omega_2,$$
$$B_3=-pql_3+(2l_2+3l_3)\omega_1-24(pq+2q^2+1)\omega_1^2$$
$$+12\{qp^2-2(q^2+1)p-8q\}\omega_2+12q\omega_1\omega_2+8\omega_3,$$
$$B_4=pql_4-(3l_3+4l_4)\omega_1+12(p^2+6pq+6q^2+7)\omega_1^2$$
$$+36\{(q^2+3)p+6q\}\omega_2-12(p+4q)\omega_1\omega_2-48\omega_3+3\omega_2^2,$$
$$B_5=4l_4\omega_1-24\{p^2+3pq+2(q^2+2)\}\omega_1^2-12\{qp^2+2(q^2+6)p+16q\}\omega_2$$
$$+36(p+2q)\omega_1\omega_2+96\omega_3-12\omega_2^2,$$

$$B_6=12(p^2+2pq+q^2+3)\omega_1^2+6\{qp^2+(q^2+10)p+10q\}\omega_2$$
$$-12(3p+4q)\omega_1\omega_2-80\omega_3+18\omega_2^2\ ,$$

$$B_7=12(p+q)\omega_1\omega_2+24\omega_3-12\omega_2^2\ , \qquad B_8=3\omega_2^2\ ,$$

with l_α $(\alpha=0, 1,\cdots, 4)$ *given by*

(5.5)
$$l_0=3qp^3-2(3q^2+2)p^2+3q(q^2+2)p-2(2q^2-1)\ ,$$
$$l_1=-12pq^2(p-q)\ ,$$
$$l_2=-6q\{p^3-(3q^2+2)p-4q\}\ ,$$
$$l_3=4\{(3q^2+2)p^2+3q(q^2+4)p+2(4q^2+1)\}\ ,$$
$$l_4=3\{qp^3+2(q^2+2)p^2+q(q^2+10)p+2(2q^2+1)\}\ .$$

COROLLARY 2. *Under hypothesis H, the distribution of* T_0^2 *can be expressed asymptotically as follows*:

(5.6)
$$P(T_0^2<x)=P(\chi_f^2<x)+\frac{pq}{2n}\{-(p-q)P(\chi_f^2<x)$$
$$-2qP(\chi_{f+2}^2<x)+(p+q)P(\chi_{f+4}^2<x)\}$$
$$+\frac{pq}{24n^2}\left\{\sum_{\alpha=0}^{4}(-1)^\alpha l_\alpha P(\chi_{f+2\alpha}^2<x)\right\}+O(n^{-3})\ .$$

Nete: Inverting the expanded series of Laplace transform, we need to check that the series converges in a uniform sense. However the verification of the problem is left. To examine that the remainder terms in (4.10) and (5.3) converge in what regions is important. It may be pointed that the bound of the convergence for T_0^2 is $0<T_0^2<n$ as in real case (c.f. T. Hayakawa. On the derivation of the asymptotic distribution of the generalized Hotelling's T_0^2, to appear in this journal.).

Appendix 1. Table of zonal polynomials of a hermitian matrix Z up to order 4

It is well known (see e.g. James [3]) that

$$\chi_{[\kappa]}(1)=k!\prod_{i<j}^{m}(k_i-k_j-i+j)\Big/\prod_{i=1}^{m}(k_i+m-i)!$$

for $\kappa=\{k_1, k_2,\cdots, k_m\}$ such that $k=k_1+k_2+\cdots+k_m$ and $k_1\geqq k_2\geqq\cdots\geqq k_m>0$. An explicit usable formula for $\chi_{[\kappa]}(Z)$ is only available in special cases (see e.g. Littlewood [12], p. 86, p. 265). Therefore, we get the following,

Degree k	Partition κ	Zonal polynomial $k!\tilde{C}_\kappa(Z)$
1	(1)	z_1
2	(2)	$z_1^2+z_2$
	(1^2)	$z_1^2-z_2$
3	(3)	$z_1^3+3z_1z_2+2z_3$
	(21)	$4z_1^3 \quad -4z_3$
	(1^3)	$z_1^3-3z_1z_2+2z_3$
4	(4)	$z_1^4+6z_1^2z_2+3z_2^2+8z_1z_3+6z_4$
	(31)	$9z_1^4+18z_1^2z_2-9z_2^2 \quad -18z_4$
	(2^2)	$4z_1^4 \quad +12z_2^2-16z_1z_3$
	(21^2)	$9z_1^4-18z_1^2z_2-9z_2^2 \quad +18z_4$
	(1^4)	$z_1^4-6z_1^2z_2+3z_2^2+8z_1z_3-6z_4$

(z_j is the sum of the jth power of the characteristic roots of Z.)

The referee kindly pointed out that the table of zonal polynomials of hermitian matrix up to 5 order appeared in the paper of Khatri [11].

Appendix 2. Formulas for $\mathrm{tr}\,\partial^2|Z-\phi\Omega^{1/2}\Sigma\Omega^{1/2}|^{-q}|_{\Sigma=I}$, etc.

In this appendix we use the notations appeared in Section 4. Put $g(\Sigma)=Z-\phi\Omega^{1/2}\Sigma\Omega^{1/2}$ and $\mathrm{tr}_j=\mathrm{tr}\,\{\Omega(Z-\phi\Omega)^{-1}\}^j$ $(j=1, 2, 3, 4)$. Then, the following formulas are obtained by the same method as in the case of real differential operators of Fujikoshi [3]:

(A.1) $\mathrm{tr}\,\partial^2|g(\Sigma)|^{-q}|_{\Sigma=I}=q\phi^2|Z-\phi\Omega|^{-q}\{(\mathrm{tr}_1)^2+q\,\mathrm{tr}_2\}$,

(A.2) $\mathrm{tr}\,\partial^3|g(\Sigma)|^{-q}|_{\Sigma=I}=q\phi^3|Z-\phi\Omega|^{-q}\{(\mathrm{tr}_1)^3+3q\,\mathrm{tr}_1\,\mathrm{tr}_2+(q^2+1)\,\mathrm{tr}_3\}$,

(A.3) $(\mathrm{tr}\,\partial^2)^2|g(\Sigma)|^{-q}|_{\Sigma=I}$
$$=q\phi^4|Z-\phi\Omega|^{-q}\{q(\mathrm{tr}_1)^4+2(q^2+2)(\mathrm{tr}_1)^2\,\mathrm{tr}_2+q(q^2+2)(\mathrm{tr}_2)^2+8q\,\mathrm{tr}_1\,\mathrm{tr}_3+2(2q^2+1)\,\mathrm{tr}_4\}\ ,$$

(A.4) $\mathrm{tr}\,\partial^2|g(\Sigma)|^{-q}U_1(\Sigma, Z)|_{\Sigma=I}$
$$=\phi^2|Z-\phi\Omega|^{-q}[f_2(\mathrm{tr}_1)^2+f_3\,\mathrm{tr}_2+f_4(\mathrm{tr}_1)^3+f_5\,\mathrm{tr}_1\,\mathrm{tr}_2+f_6\,\mathrm{tr}_3+\phi^4\{q(\mathrm{tr}_1)^4+2(q^2+2)(\mathrm{tr}_1)^2\,\mathrm{tr}_2+q(q^2+2)(\mathrm{tr}_2)^2+8q\,\mathrm{tr}_1\,\mathrm{tr}_3+2(2q^2+1)\,\mathrm{tr}_4\}]\ ,$$

where f_α $(\alpha=2, 3,\cdots, 6)$ are given by

(A.5) $f_2=pq(p+q)-2(p+q)(pq+2)\phi+\{qp^2+(q^2+4)p+6q\}\phi^2$,

$f_3=pq^2(p+q)-2q(p+q)(pq+2)\phi+\{q^2p^2+q(q^2+4)p+4q^2+2\}\phi^2$,

$$f_4=2\phi^2\{-q(p+q)+(pq+q^2+2)\phi\},$$

$$f_5=2\phi^2[-(q^2+2)(p+q)+\{(q^2+2)p+q(q^2+8)\}\phi],$$

$$f_6=4\phi^2\{-q(p+q)+(pq+2q^2+1)\phi\}.$$

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