ASYMPTOTIC EXPANSION OF THE DISTRIBUTION OF THE GENERALIZED VARIANCE FOR NONCENTRAL WISHART MATRIX, WHEN $\Omega=O(n)$

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1. Introduction

Let $S(=X'X, X: n\times p)$ be a $p\times p$ noncentral Wishart matrix having $W_p\ (\Sigma, n; \Omega)$. Here we shall define the matrix of noncentrality parameters Ω by $\Sigma^{-1}M'M/2$ instead of $\Sigma^{-1}M'M$, for $M=E(X)$. Asymptotic expansion of the distribution of $|S\Sigma^{-1}/n|$ was obtained by Fujikoshi [3], [4], when $\Omega=O(1)$ and $\Omega=O(\sqrt{n})$, respectively, based on the moments expressed by the hypergeometric function of matrix argument in Constantine [2]. Under a natural assumption of $\Omega=O(n)$, however, this approach was not successful and only the limiting distribution was derived. Using the characteristic function of S and the standard procedure explicitly recognized by Nagao [5] in another problem, we can get the asymptotic expansion of the distribution.

2. Preliminary lemmas

So far as we are concerned with the distribution of the statistic given by the characteristic roots of $\Sigma^{-1/2}S\Sigma^{-1/2}$, we may assume that S has $W_p\ (I, n; \Omega)$ and further we may take $\Omega=n\theta=n\,\mathrm{diag}\,(\theta_1, \cdots, \theta_p)$. Putting $T=(1/2\cdot(1+\delta_{ij})t_{ij})$, the characteristic function of S was obtained by Anderson [1] as

$$E[\mathrm{etr}\,(iTS)]=|I-2iT|^{-n/2}\,\mathrm{etr}\,[-\Omega+\Omega(I-2iT)^{-1}]\ , \tag{2.1}$$

by which it is easy to see that when $\Omega=O(n)$, the statistic $U=\sqrt{n}\,\{S/n-(I+2\theta)\}$ converges in law to a $p(p+1)/2$ variate normal distribution as n tends to infinity.

LEMMA 2.1. *Let S have $W_p\ (I, n; \Omega)$ and let $f(\Lambda)$ be an analytic function with respect to positive definite matrix $\Lambda=(\lambda_{ij})$. Put $\partial=(1/2\cdot(1+\delta_{ij})\partial/\partial\lambda_{ij})$. Then for any diagonal matrix A, we have*

(2.2)
$$E[f(S/n)\,\mathrm{etr}\,(itAU)]$$
$$=\mathrm{etr}\,[-t^2(I+4\theta)A^2]\Big[1+n^{-1/2}\Big\{2it\,\mathrm{tr}\,(I+4\theta)A\partial$$
$$+\frac{4}{3}(it)^3\,\mathrm{tr}\,(I+6\theta)A^3\Big\}+n^{-1}\sum_{\alpha=0}^{3}(it)^{2\alpha}g_{2\alpha}(\partial)$$
$$+n^{-3/2}\sum_{\alpha=0}^{4}(it)^{2\alpha+1}h_{2\alpha+1}(\partial)+O(n^{-2})\Big]f(\Lambda)\Big|_{\Lambda=I+2\theta}\,,$$

where

$$g_0(\partial)=\mathrm{tr}\,(I+4\theta)\partial^2\,,$$
$$g_2(\partial)=4\,\mathrm{tr}\,(I+6\theta)A^2\partial+2\{\mathrm{tr}\,(I+4\theta)A\partial\}^2\,,$$
$$g_4(\partial)=2\,\mathrm{tr}\,(I+8\theta)A^4+\frac{8}{3}\,\mathrm{tr}\,(I+4\theta)A\partial\,\mathrm{tr}\,(I+6\theta)A^3\,,$$
$$g_6(\partial)=\frac{8}{9}\{\mathrm{tr}\,(I+6\theta)A^3\}^2\,,$$

(2.3)
$$h_1(\partial)=4\,\mathrm{tr}\,(I+4\theta)A\partial^2+8\,\mathrm{tr}\,\theta\partial A\partial+2\,\mathrm{tr}\,(I+4\theta)A\partial\,\mathrm{tr}\,(I+4\theta)\partial^2\,,$$
$$h_3(\partial)=8\,\mathrm{tr}\,(I+8\theta)A^3\partial+8\,\mathrm{tr}\,(I+4\theta)A\partial\,\mathrm{tr}\,(I+6\theta)A^2\partial$$
$$+\frac{4}{3}\,\mathrm{tr}\,(I+6\theta)A^3\,\mathrm{tr}\,(I+4\theta)\partial^2+\frac{4}{3}\{\mathrm{tr}\,(I+4\theta)A\partial\}^3\,,$$
$$h_5(\partial)=\frac{16}{5}\,\mathrm{tr}\,(I+10\theta)A^5+4\,\mathrm{tr}\,(I+4\theta)A\partial\,\mathrm{tr}\,(I+8\theta)A^4$$
$$+\frac{16}{3}\,\mathrm{tr}\,(I+6\theta)A^2\partial\,\mathrm{tr}\,(I+6\theta)A^3$$
$$+\frac{8}{3}\{\mathrm{tr}\,(I+4\theta)A\partial\}^2\,\mathrm{tr}\,(I+6\theta)A^3\,,$$
$$h_7(\partial)=\frac{8}{3}\,\mathrm{tr}\,(I+6\theta)A^3\,\mathrm{tr}\,(I+8\theta)A^4$$
$$+\frac{16}{9}\,\mathrm{tr}\,(I+4\theta)A\partial\{\mathrm{tr}\,(I+6\theta)A^3\}^2\,,$$
$$h_9(\partial)=\frac{32}{81}\{\mathrm{tr}\,(I+6\theta)A^3\}^3\,.$$

The term of order n^{-2} in (2.2) *contains the fourth order derivatives as the highest and is expressed by*

(2.4)
$$n^{-2}\Big\{\sum_{\alpha=0}^{2}k_{2\alpha}(\partial)(it)^{2\alpha}+\textit{the lower order derivatives}\Big\}+O(n^{-5/2})\,,$$

where

$$k_0(\partial)=\frac{1}{2}\{\text{tr}\,(I+4\theta)\partial^2\}^2\,,$$

$$k_2(\partial)=2\{\text{tr}\,(I+4\theta)A\partial\}^2\,\text{tr}\,(I+4\theta)\partial^2\,, \tag{2.5}$$

$$k_4(\partial)=\frac{2}{3}\{\text{tr}\,(I+4\theta)A\partial\}^4\,.$$

PROOF. Since S/n converges in probability to $I+2\theta$ as n tends to infinity, we shall expand $f(S/n)$ in a Taylor series at $I+2\theta$, giving $f(S/n)=\text{etr}\,[\{S/n-(I+2\theta)\}\partial]f(\Lambda)|_{\Lambda=I+2\theta}$. Then the left-hand side of (2.2) is equal to

$$\text{etr}\,[-it\sqrt{n}(I+2\theta)A]\,\text{etr}\,[-(I+2\theta)\partial] \cdot E\left[\text{etr}\left\{\left(\frac{it}{\sqrt{n}}A+\frac{1}{n}\partial\right)S\right\}\right]f(\Lambda)\Big|_{\Lambda=I+2\theta}\,. \tag{2.6}$$

From (2.1) we can rewrite the above expression as

$$\text{etr}\,[-it\sqrt{n}(I+2\theta)A]\,\text{etr}\,[-(I+2\theta)\partial-\Omega+\Omega(I-2T)^{-1}] \cdot |I-2T|^{-n/2}f(\Lambda)|_{\Lambda=I+2\theta}\,, \tag{2.7}$$

where $T=itA/\sqrt{n}+\partial/n$. Putting $\Omega=n\theta$ and using the well-known asymptotic formulas $(I-T/\sqrt{n})^{-1}=\sum_{\alpha=0}^{6}n^{-\alpha/2}T^\alpha+O(n^{-7/2})$ and $|I-T/\sqrt{n}|=\text{etr}\left[-\sum_{\alpha=1}^{6}n^{-\alpha/2}\cdot T^\alpha/\alpha+O(n^{-7/2})\right]$, we can expand (2.7) into asymptotic series after some computation, which completes the proof.

Put $E_{ij}=(\partial_{ij}\lambda_{\alpha\beta})_{\alpha,\beta=1,2,\cdots,p}$ for $\partial=(\partial_{ij})$. Then E_{ii} is a diagonal matrix having 1 at ith diagonal and 0 at other places. If $i\neq j$, E_{ij} is a symmetric matrix having 1/2 as (i, j) and (j, i) elements and 0 as all other elements. Although the proof is easy, the following lemma will be used frequently in the computation at the next section. Similar formulas are used by Fujikoshi [4] in deriving the asymptotic expansion of the non-null distribution of Pillai's statistic for test of independence under local alternative.

LEMMA 2.2. *Let* $\theta=\text{diag}\,(\theta_1,\cdots,\theta_p)$ *and* $\Gamma=\text{diag}\,(\gamma_1,\cdots,\gamma_p)$. *Then for any diagonal matrices* A, B *of order* p,

$$\sum_{i,j=1}^{p}\theta_i\gamma_j\,\text{tr}\,AE_{ii}BE_{jj}=\text{tr}\,A\theta B\Gamma\,, \tag{2.8}$$

$$\sum_{i,j=1}^{p}\theta_i\gamma_j\,\text{tr}\,AE_{ij}BE_{ij}=\frac{1}{4}(2\,\text{tr}\,AB\theta\Gamma+\text{tr}\,A\theta\,\text{rt}\,B\Gamma+\text{tr}\,A\Gamma\,\text{tr}\,B\theta)\,, \tag{2.9}$$

$$\sum_{i,j,k,l=1}^{p}\theta_i\gamma_k\{\text{tr}\,(AE_{ij}BE_{kl})\}^2 \tag{2.10}$$

$$=\frac{1}{2}\operatorname{tr} A^2B^2\theta\Gamma+\frac{1}{8}(\operatorname{tr} AB\theta\Gamma \operatorname{tr} AB+\operatorname{tr} AB\theta \operatorname{tr} AB\Gamma)$$
$$+\frac{1}{16}(\operatorname{tr} A^2\theta\Gamma \operatorname{tr} B^2+\operatorname{tr} A^2\theta \operatorname{tr} B^2\Gamma+\operatorname{tr} A^2\Gamma \operatorname{tr} B^2\theta$$
$$+\operatorname{tr} A^2 \operatorname{tr} B^2\theta\Gamma)\ ,$$

(2.11) $$\sum_{i,j=1}^{p} \theta_i\gamma_j(\operatorname{tr} AE_{ij})^2=\operatorname{tr} A^2\theta\Gamma\ .$$

3. Derivation of the asymptotic expansion

The generalized variance $|S|$ can be expressed by U as

(3.1) $$\log|S/n|=\log|I+2\theta|+\frac{1}{\sqrt{n}}\operatorname{tr} AU-\frac{1}{2n}\operatorname{tr}(AU)^2$$
$$+\frac{1}{3n\sqrt{n}}\operatorname{tr}(AU)^3+O_p(n^{-2})\ ,$$

where $A=(I+2\theta)^{-1}$. Hence the characteristic function of $\sqrt{n}\log|S/n|-\sqrt{n}\log|I+2\theta|$ is written by

(3.2) $$E\left[\operatorname{etr}(itAU)\left\{1+\frac{1}{\sqrt{n}}l_1(U)+\frac{1}{n}l_2(U)\right\}\right]+O(n^{-3/2})\ ,$$

where

(3.3)
$$l_1(U)=-\frac{1}{2}it\operatorname{tr}(AU)^2\ ,$$
$$l_2(U)=\frac{it}{3}\operatorname{tr}(AU)^3+\frac{1}{8}(it)^2\{\operatorname{tr}(AU)^2\}^2\ .$$

Now we shall compute three expectations in (3.2). Taking $f(\Lambda)=1$ in Lemma 2.1 yields immediately

(3.4) $$E[\operatorname{etr}(itAU)]$$
$$=\operatorname{etr}[-t^2(I+4\theta)A^2]\Big[1+\frac{4}{3}n^{-1/2}(it)^3\operatorname{tr}(I+6\theta)A^3$$
$$+n^{-1}\left\{2(it)^4\operatorname{tr}(I+8\theta)A^4+\frac{8}{9}(it)^6[\operatorname{tr}(I+6\theta)A^3]^2\right\}$$
$$+O(n^{-3/2})\Big]\ .$$

Taking $f(\Lambda)=\operatorname{tr}\{A(\Lambda-A^{-1})\}^2$ for $A=(I+2\theta)^{-1}$ in Lemma 2.1, we can see that the terms of order 1 and $n^{-1/2}$ in (2.2) vanish. Note that $\partial_{ij}^2 f(\Lambda)=2\operatorname{tr}(AE_{ij})^2$ for $\partial=(\partial_{ij})$ and $E_{ij}=(\partial_{ij}\lambda_{\alpha\beta})$. Using Lemma 2.2, we have

(3.5)
$$E[\operatorname{tr}(AU)^2 \operatorname{etr}(itAU)] = \operatorname{etr}[-t^2(I+4\theta)A^2]\Big[\operatorname{tr}(I+4\theta)A^2+\operatorname{tr}(I+4\theta)A \operatorname{tr} A + (it)^2 4 \operatorname{tr}(I+4\theta)^2A^4+n^{-1/2}\sum_{\alpha=0}^{2}(it)^{2\alpha+1}a_{2\alpha+1}+O(n^{-1})\Big] ,$$

where

(3.6)
$$a_1=4\{\operatorname{tr}(I+4\theta)A^3+\operatorname{tr}(I+4\theta)A^2 \operatorname{tr} A\}+8(\operatorname{tr}\theta A^3+\operatorname{tr} A^2 \operatorname{tr}\theta A) ,$$
$$a_3=16 \operatorname{tr}(I+4\theta)(I+6\theta)A^5+\frac{4}{3}\operatorname{tr}(I+6\theta)A^3 \cdot \{\operatorname{tr}(I+4\theta)A^2+\operatorname{tr}(I+4\theta)A \operatorname{tr} A\} ,$$
$$a_5=\frac{16}{3}\operatorname{tr}(I+6\theta)A^3 \operatorname{tr}(I+4\theta)^2A^4 .$$

Similarly putting $f(\Lambda)=\operatorname{tr}\{A(\Lambda-A^{-1})\}^3$ and noting that $\partial_{ii}\partial_{jk}^2 f(\Lambda)=6 \operatorname{tr}(AE_{jk})^2AE_{ii}$ in Lemma 2.1, we can get

(3.7)
$$E[\operatorname{tr}(AU)^3 \operatorname{etr}(itAU)] = \operatorname{etr}[-t^2(I+4\theta)A^2][3it\{2 \operatorname{tr}(I+4\theta)^2A^4+\operatorname{tr}(I+4\theta)^2A^3 \operatorname{tr} A + \operatorname{tr}(I+4\theta)A^3 \operatorname{tr}(I+4\theta)A\}+8(it)^3 \operatorname{tr}(I+4\theta)^3A^6+O(n^{-1/2})] .$$

Finally for $f(\Lambda)=[\operatorname{tr}\{A(\Lambda-A^{-1})\}^2]^2$, all terms generated from (2.3) vanish and the leading term is given by operating (2.5) to $f(\Lambda)$. Since

(3.8)
$$\partial_{ii'}\partial_{jj'}\partial_{kk'}\partial_{ll'}f(\Lambda)=8\{\operatorname{tr} AE_{ii'}AE_{jj'} \operatorname{tr} AE_{kk'}AE_{ll'} + \operatorname{tr} AE_{ii'}AE_{kk'} \operatorname{tr} AE_{jj'}AE_{ll'} + \operatorname{tr} AE_{ii'}AE_{ll'} \operatorname{tr} AE_{jj'}AE_{kk'}\} ,$$

we can get

(3.9)
$$E[\{\operatorname{tr}(AU)^2\}^2 \operatorname{etr}(itAU)] = \operatorname{etr}[-t^2(I+4\theta)A^2]\sum_{\alpha=0}^{2}(it)^{2\alpha}b_{2\alpha}+O(n^{-1/2}) ,$$

where

(3.10)
$$b_0=\{\operatorname{tr}(I+4\theta)A^2+\operatorname{tr}(I+4\theta)A \operatorname{tr} A\}^2+4 \operatorname{tr}(I+4\theta)^2A^4 + 2\{\operatorname{tr}(I+4\theta)A^2\}^2+2 \operatorname{tr}(I+4\theta)^2A^2 \operatorname{tr} A^2 ,$$
$$b_2=8 \operatorname{tr}(I+4\theta)^2A^4\{\operatorname{tr}(I+4\theta)A^2+\operatorname{tr}(I+4\theta)A \operatorname{tr} A\} + 32 \operatorname{tr}(I+4\theta)^3A^6 ,$$
$$b_4=16\{\operatorname{tr}(I+4\theta)^2A^4\}^2 .$$

Combining (3.4), (3.5), (3.7) and (3.9) together, in view of (3.2), we can write the characteristic function of $\sqrt{n}\{\log|S/n|-\log|I+2\theta|\}/\tau$, where

$\tau^2=2\operatorname{tr}(I+4\theta)A^2$, as

$$\exp(-t^2/2)\left[1+n^{-1/2}\sum_{\alpha=0}^{1}(-it/\tau)^{2\alpha+1}g_{2\alpha+1} +n^{-1}\sum_{\alpha=1}^{3}(it/\tau)^{2\alpha}h_{2\alpha}+O(n^{-3/2})\right], \tag{3.11}$$

where

$$g_1=\frac{1}{2}\{\operatorname{tr}(I+4\theta)A^2+\operatorname{tr}(I+4\theta)A\operatorname{tr}A\}, \tag{3.12}$$

$$g_3=2\operatorname{tr}(I+4\theta)^2A^4-\frac{4}{3}\operatorname{tr}(I+6\theta)A^3,$$

$$\begin{aligned}h_2=&-2\operatorname{tr}(I+6\theta)A^3-4\operatorname{tr}A^2\operatorname{tr}A\theta-2\operatorname{tr}(I+4\theta)A^2\operatorname{tr}A\\&+\frac{5}{2}\operatorname{tr}(I+4\theta)^2A^4+\operatorname{tr}(I+4\theta)^2A^3\operatorname{tr}A+\operatorname{tr}(I+4\theta)A^3\operatorname{tr}(I+4\theta)A\\&+\frac{1}{8}\{\operatorname{tr}(I+4\theta)A^2+\operatorname{tr}(I+4\theta)A\operatorname{tr}A\}^2\\&+\frac{1}{4}\{\operatorname{tr}(I+4\theta)A^2\}^2+\frac{1}{4}\operatorname{tr}(I+4\theta)^2A^2\operatorname{tr}A^2,\end{aligned}$$

$$\begin{aligned}h_4=&2\operatorname{tr}(I+8\theta)A^4-8\operatorname{tr}(I+4\theta)(I+6\theta)A^5\\&+\{\operatorname{tr}(I+4\theta)A^2+\operatorname{tr}(I+4\theta)A\operatorname{tr}A\}\\&\cdot\left\{-\frac{2}{3}\operatorname{tr}(I+6\theta)A^3+\operatorname{tr}(I+4\theta)^2A^4\right\}+\frac{20}{3}\operatorname{tr}(I+4\theta)^3A^6,\end{aligned}$$

$$h_6=2\left\{\frac{2}{3}\operatorname{tr}(I+6\theta)A^3-\operatorname{tr}(I+4\theta)^2A^4\right\}^2.$$

Inverting the characteristic function (3.11) yields:

THEOREM 3.1. *Let S be a noncentral Wishart matrix having $W_p(\Sigma, n;\Omega)$. When $\Omega=n\theta$, we have*

$$\begin{aligned}&P((\sqrt{n}/\tau)(\log|S\Sigma^{-1}/n|-\log|I+2\theta|)<x)\\&=\Phi(x)+n^{-1/2}\sum_{\alpha=0}^{1}\Phi^{(2\alpha+1)}(x)g_{2\alpha+1}\tau^{-2\alpha-1}\\&\quad+n^{-1}\sum_{\alpha=1}^{3}\Phi^{(2\alpha)}(x)h_{2\alpha}\tau^{-2\alpha}+O(n^{-3/2}),\end{aligned} \tag{3.13}$$

where $\tau^2=2\operatorname{tr}(I+4\theta)A^2$ and $\Phi^{(\alpha)}(x)$ means the αth derivative of the standard normal distribution function $\Phi(x)$. The coefficients g_α and h_α are given by (3.12) *for $A=(I+2\theta)^{-1}$.*

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