# DISTRIBUTION OF WILKS' LIKELIHOOD-RATIO CRITERION IN THE COMPLEX CASE*

A. K. GUPTA

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## 1. Summary

The null distribution of Wilks' likelihood ratio criterion, $\Lambda$, in the complex case, is obtained, and explicit expressions for the same are given for $p=2$ and 3, where $p$ is the number of variables. It is shown that unlike the real case the distributions derived have closed form representation for all $p$ and for all $f_2$, the hypothesis degrees of freedom. Tables of correction factors for converting chi-square percentiles to exact percentiles of a logarithmic function of $\Lambda$ are provided for fourteen $(p, f_2)$ pairs. Tables for an additional thirteen pairs can be obtained from those tabulated by interchanging $p$ and $f_2$.

## 2. Introduction

Let the joint density function of complex random matrices $X: p\times f_2$ and $S: p\times p$ be (see [2], [3] and [10])

$$CN(X; \mu, \Sigma) \qquad CW(S; p, f_1, \Sigma) \tag{2.1}$$

where $CN(X; \mu, \Sigma)$ is the complex multivariate normal distribution defined by

$$CN(X; \mu, \Sigma)=\Pi^{-pf_2}|\Sigma|^{-f_2}\exp[-\mathrm{tr}\,\Sigma^{-1}(X-\mu)\overline{(X-\mu)}'] . \tag{2.2}$$

$CW(S; p, f_1, \Sigma)$ is the complex Wishart distribution defined by

$$CW(S; p, f_1, \Sigma)=\{\Gamma_p(f_1)\}^{-1}|\Sigma|^{-f_1}|S|^{f_1-p}\exp(-\mathrm{tr}\,\Sigma^{-1}S) , \tag{2.3}$$

where

$$\Gamma_p(f_1)=\Pi^{p(p-1)/2}\left\{\prod_{j=1}^{p}\Gamma(f_1-j+1)\right\} , \tag{2.4}$$

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$\Sigma: p\times p$ and $S: p\times p$ are hermitian positive definite matrices and $\mu: p\times f_2$ is a complex matrix. The likelihood ratio criterion for testing $H_0$: $\mu(p\times f_2)=0$ against $\mu\neq 0$ can be expressed in terms of the following criterion suggested by Wilks [9] and Pearson and Wilks [6]

$$\Lambda=|S|/|S+X\bar{X}'| \tag{2.5}$$

where the critical region is defined by $\Lambda\leqq\lambda_0$ (which is parallel to the likelihood-ratio criterion for real Gaussian variables) and $P(\Lambda\leqq\lambda_0/H_0)=\alpha$.

## 3. Distributional properties of $\Lambda$

For purposes of notational ease, the symbol $\Lambda$ will be replaced by $U_{p,f_2,f_1}$. Let us denote by $\beta(a, b; X)$ the density function

$$\{1/\beta(a, b)\}X^{a-1}(1-X)^{b-1}, \qquad 0\leqq X\leqq 1 \tag{3.1}$$

of a real beta variable $X$. Khatri [5] has proved the following result which we state as

THEOREM 3.1. *In the complex case, when the hypothesis to be tested is true, $U_{p,f_2,f_1}$ is distributed like $X_1\cdots X_p$ where $X_i$ are independently distributed real beta variables with density $\beta(f_1-i+1, f_2; X_i)$.*

Now if we consider (5.2.1) of Khatri [5] when $\mu=0$ (and therefore in Khatri's notation $\alpha=0$), a little algebraic manipulations give us the following theorem.

THEOREM 3.2. *In the complex case, the distribution of $U_{p,f_2,f_1}$ is the same as that of $U_{f_2,p,f_1+f_2-p}$.*

This implies that without loss of generality we need consider only values of $f_2\geqq p$. The method employed in this paper relies on these two theorems.

## 4. The method of derivation

An immediate consequence of Theorem 3.1 is that since

$$-\log U_{p,f_2,f_1}=\sum_{i=1}^{p}(-\log X_i)=\sum_{i=1}^{p}Y_i \tag{4.1}$$

say, the distribution problem in hand can be reduced to that of a sum of independently distributed random variables. The latter distribution can be handled by taking successive convolutions provided that the procedure yields expressions which can be easily integrated at each stage. Schatzoff [8] has proved that this is in fact the case. Gupta [4], Pillai and Gupta [7] and Schatzoff [8] have used this method to obtain the distribution of Wilks' $\Lambda$ in the real case.

Consider the density of $X_i$,

$$\beta(f_1-i+1, f_2; X_i)=K_i X_i^{f_1-i}(1-X_i)^{f_2-1}, \qquad 0 \leqq X_i \leqq 1,\ f_1 \geqq i, \tag{4.2}$$

where

$$K_i=[1/\beta(f_1-i+1, f_2)]=\Gamma(f_1+f_2-i+1)/\{\Gamma(f_1-i+1)\Gamma(f_2)\}. \tag{4.3}$$

Using the binomial theorem and transforming $Y_i=-\log X_i$, the density of $Y_i$ is given by,

$$K_i \sum_{l=0}^{f_2-1}(-1)^l \binom{f_2-1}{l} e^{-(f_1+l-i+1)Y_i}, \qquad Y_i>0. \tag{4.4}$$

Note that in this case, unlike the real case, an application of the binomial theorem gives a finite series for all values of $f_2$.

Now making use of (2.4) of Schatzoff [8], the density of $Y_1+\cdots+Y_p$ can be obtained by means of successive convolutions, as given in the next section.

## 5. Exact distribution of $U_{p,f_2,f_1}$ in the complex case

In this section, we consider the density and the c.d.f. of $U_{p,f_2,f_1}$ for $p=2$ and 3. We also give the general form of the density and the c.d.f. for any $p$ and general $f_1$ and $f_2$.

*Case* (i). $p=2$. We have $U_{2,f_2,f_1}=X_1X_2$, and hence

$$-\log U_{2,f_2,f_1}=Y_1+Y_2. \tag{5.1}$$

Now using the method of Section 4, we get the density of $U_{2,f_2,f_1}$ in the following finite series form:

$$\begin{aligned} K_1K_2U^{f_1-2}\Bigg[&\sum_{m=1}^{f_2-1}\binom{f_2-1}{m}\binom{f_2-1}{m-1}U^m\log U \\ &+\sum_{\substack{l=0\\ l\neq m-1}}^{f_2-1}\sum_{m=0}^{f_2-1}\frac{(-1)^{l+m}}{m-l-1}\binom{f_2-1}{l}\binom{f_2-1}{m}(U^{l+1}-U^m)\Bigg], \\ &\qquad 0<U\leqq 1, \end{aligned} \tag{5.2}$$

where $K_1$ and $K_2$ are given by (4.3) for $i=1$ and 2 respectively. The c.d.f. of $U_{2,f_2,f_1}$ is obtained from (5.2) by integrating between $(0, u)$, $0<u\leqq 1$, as follows

$$\begin{aligned} &P[U_{2,f_2,f_1}\leqq u] \\ &=K_1K_2u^{f_1-1}\Bigg[-\sum_{m=1}^{f_2-1}\binom{f_2-1}{m}\binom{f_2-1}{m-1}\frac{u^m}{(f_1+m-1)^2}(1-(f_1+m-1)\log u) \end{aligned} \tag{5.3}$$

$$+\sum_{\substack{l=0\\ l\neq m-1}}^{f_2-1}\sum_{m=0}^{f_2-1}\frac{(-1)^{l+m}}{m-l-1}\binom{f_2-1}{l}\binom{f_2-1}{m}$$
$$\cdot\left(\frac{u^{l+1}}{f_1+l}-\frac{u^m}{f_1+m-1}\right)\Bigg].$$

It is interesting to note the special cases for $f_2=2$ and 3. For $f_2=2$, we get the density and the c.d.f. of $U_{2,f_2,f_1}$ respectively as follows:

$$f_1^2(f_1^2-1)U^{f_1-1}[\log U+((U^{-1}-U)/2)]\,,\qquad 0\leqq U\leqq 1\,, \tag{5.4}$$

and

$$\begin{aligned}P[U_{2,2,f_1}\leqq u]=u^{f_1}[&-(f_1^2-1)(1-f_1\log u)\\ &+(f_1^2((f_1+1)u^{-1}-(f_1-1)u))/2]\,.\end{aligned} \tag{5.5}$$

Similarly for $f_2=3$, we obtain the density and the c.d.f. of $U_{2,f_2,f_1}$ respectively as follows:

$$\begin{aligned}&\{f_1^2(f_1+1)^2(f_1-1)(f_1+2)U^{f_1-2}/12\}\\ &\cdot\{6U(1+U)\log U+1+9U-9U^2-U^3\}\,,\qquad 0\leqq U\leqq 1\,,\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}&P[U_{2,3,f_1}\leqq u]\\ &=[f_1^2(f_1+1)^2(f_1-1)(f_1+2)u^{f_1-1}/12]\\ &\cdot\left[\frac{1}{f_1-1}+\frac{9u}{f_1}-\frac{9u^2}{f_1+1}-\frac{u^3}{f_1+2}-\frac{6u}{f_1^2}(1-f_1\log u)\right.\\ &\left.-\frac{6u^2\{1-(f_1+1)\log u\}}{(f_1+1)^2}\right].\end{aligned} \tag{5.7}$$

*Case* (ii). $p=3$. For $p=3$, $-\log U_{3,f_2,f_1}=Y_1+Y_2+Y_3$ and the density of $U_{3,f_2,f_1}$ is given by the following finite series:

$$\begin{aligned}&\left(\prod_{i=1}^{3}K_i\right)U^{f_1-2}\left[\frac{1}{2}\sum_{m=1}^{f_2-2}(-U)^m\mathscr{f}(m+1,m,m-1)(\log U)^2\right.\\ &-\sum_{\substack{m=1\\ m\neq n-1}}^{f_2-1}\sum_{n=0}^{f_2-1}\frac{(-1)^nU^m}{(n-m-1)^2}\mathscr{f}(m,m-1,n)\{U^{n-m-1}-(n-m-1)\log U-1\}\\ &-\sum_{\substack{l=0\\ l\neq m-1}}^{f_2-3}\sum_{m=0}^{f_2-1}\frac{(-1)^mU^{l+1}}{(m-l-1)}\mathscr{f}(l+2,l,m)\log U\\ &+\sum_{\substack{l=0\\ l\neq m-1,n-2}}^{f_2-1}\sum_{m=0}^{f_2-1}\sum_{n=0}^{f_2-1}\frac{(-1)^{l+m+n}}{(m-l-1)(n-l-2)}\mathscr{f}(l,m,n)(U^{l+1}-U^{n-1})\\ &-\sum_{\substack{l=0\\ l\neq m-1}}^{f_2-1}\sum_{m=0}^{f_2-2}\frac{(-1)^lU^m}{(m-l-1)}\mathscr{f}(m+1,m,l)\log U\end{aligned} \tag{5.8}$$

$$-\sum_{l=0}^{f_2-1}\sum_{\substack{m=0\\ m\neq l+1,n-1}}^{f_2-1}\sum_{n=0}^{f_2-1}\frac{(-1)^{l+m+n}}{(m-l-1)(n-m-1)}\mathscr{f}(l,m,n)(U^m-U^{n-1})\Big]\,,$$

$$0\leqq U\leqq 1\,,$$

where $\mathscr{f}(l,m,n)=\binom{f_2-1}{l}\binom{f_2-1}{m}\binom{f_2-1}{n}$, and $K_i$ are given by (4.3). The c.d.f. of $U_{3,f_2,f_1}$ can be easily obtained by straightforward integration of (5.8) between $(0, u)$, $0<u\leqq 1$ and is given by

$$\text{(5.9)}\qquad P[U_{3,f_2,f_1}\leqq u]$$

$$=\Big(\prod_{i=1}^{3}K_i\Big)u^{f_1-1}\Big[\sum_{m=1}^{f_2-1}\frac{(-u)^m}{2(f_1+m-1)^3}\mathscr{f}(m+1,m,m-1)$$

$$\cdot\{(f_1+m-1)^2(\log u)^2-(f_1+m-1)\log u+2\}$$

$$-\sum_{\substack{m=1\\ m\neq n-1}}^{f_2-1}\sum_{n=0}^{f_2-1}\frac{(-1)^n u^m}{(f_1+m-1)(n-m-1)^2}\mathscr{f}(n,m,m-1)$$

$$\cdot\Big\{\frac{(f_1+m-1)u^{n-m-1}}{f_1+n-2}+\frac{n-m-1}{f_1+m-1}(1-(f_1+m-1)\log u)-1\Big\}$$

$$+\sum_{\substack{l=0\\ l\neq m-1}}^{f_2-3}\sum_{m=0}^{f_2-1}\frac{(-1)^m u^{l+1}}{(m-l-1)(f_1+l)^2}\mathscr{f}(m,l+2,l)\{1-(f_1+l)\log u\}$$

$$+\sum_{\substack{l=0\\ l\neq m-1,n-2}}^{f_2-1}\sum_{m=0}^{f_2-1}\sum_{n=0}^{f_2-1}\frac{(-1)^{l+m+n}}{(m-l-1)(n-l-2)}\mathscr{f}(l,m,n)\Big(\frac{u^{l+1}}{f_1+l}-\frac{u^{n-1}}{f_1+n-2}\Big)$$

$$+\sum_{\substack{l=0\\ l\neq m-1}}^{f_2-1}\sum_{m=0}^{f_2-2}\frac{(-1)^l u^m}{(m-l-1)(f_1+m-1)^2}\mathscr{f}(l,m,m+1)$$

$$\cdot\{1-(f_1+m-1)\log u\}$$

$$-\sum_{l=0}^{f_2-1}\sum_{\substack{m=0\\ m\neq l+1,n-1}}^{f_2-1}\sum_{n=0}^{f_2-1}\frac{(-1)^{l+m+n}}{(m-l-1)(n-m-1)}\mathscr{f}(l,m,n)$$

$$\cdot\Big(\frac{u^m}{f_1+m-1}-\frac{u^{n-1}}{f_1+n-2}\Big)\Big]\,.$$

*Case* (iii). General $p$. The density and the distribution function of $U_{p,f_2,f_1}$ for $p>3$ becomes too involved for presentation as well as for computational purposes. However, the above analysis forms the basis of the following theorems.

THEOREM 5.1. *The probability density function of* $U_{p,f_2,f_1}$, *in the complex case, is of the form*

$$\text{(5.10)}\qquad \mathscr{f}(U)=\Big(\prod_{i=1}^{p}K_i\Big)\sum_{j=1}^{m}C_jU^{f_1-l_j}(-\log U)^{k_j}\,,$$

*where* $K_i$ *is defined by* (4.3) *and the constants* $C_j$ *and the integers* $m$, $l_j$, $k_j$ *are determined from* $p$, $f_2$ *and* $f_1$.

PROOF. The theorem can be easily proved by induction. The proof parallels the proof of Theorem 2.3 of Schatzoff [8] in the real case and hence the same has been omitted.

When $p=1$, $f(U)$ is the density function of a beta random variable with parameters $f_2$ and $f_1$. This is readily seen to be of the required form where $m=f_2-1$, $l_j=1-j$, $k_j=0$ and $C_j=(-1)^j\binom{f_2-1}{j}$.

The corresponding distribution function is obtained by integrating (5.10) between $(0, u)$, $0<u\leqq 1$, as indicated by

THEOREM 5.2. *The distribution function of $U_{p,f_2,f_1}$, in the complex case, is of the form*

$$P(U_{p,f_2,f_1}\leqq u)=\left(\prod_{i=1}^{p} K_i\right)\sum_{j=1}^{m}\left\{C_j u^{f_1-l_j}\sum_{r_j=1}^{k_j+1}\frac{(-\log u)^{k_j-r_j+1}k_j!}{(f_1-l_j)^{r_j}(k_j-r_j+1)!}\right\}, \tag{5.11}$$

$$=\left(\prod_{i=1}^{p} K_i\right)\sum_{h=1}^{M}\{d_h u^{f_1-l_h}(-\log u)^{s_h}(f_1-l_h)^{-r_h}\},$$

*where $r_j$, $M$, $d_h$ and $s_h$ are also constants and can be determined from $p$, $f_2$ and $f_1$.*

Note that, unlike the real case, here the formulae for the density and the distribution function are in the closed form for all values of $p$ and $f_2$, even or odd. Theorem 5.1 does not indicate explicitly how to find the values of the constants $m$, $C_j$, $l_j$ and $k_j$, a task which is by no means easy for large values of $p$ or $f_2$. However, the theorem provides a basis for a recursive algorithm which can be programmed and run on the computer to obtain the density and the corresponding distribution functions at successive stages of the convolution process (see Schatzoff [8]).

It should be noted that (5.11) is a finite series of weighted chi-square distributions. From a practical standpoint (5.11) makes possible for the first time the calculation of exact percentage points of Wilks' criterion in the complex case for high dimensionality and small sample size. As noted by Goodman [3], that for every distributional result of classical multivariate Gaussian statistical analysis obtainable in closed (explicit) form, the counterpart analysis for complex Gaussian is also obtainable in closed (explicit) form with necessary changes. The present analysis is the counterpart analysis of the real case, obtained by Consul [1], Gupta [4], Pillai and Gupta [7] and Schatzoff [8].

## 6. Computation of percentage points

As per Theorem 3.1 the distribution of $U_{p,f_2,f_1}$, when $p=1$, is $\beta(f_1, f_2; U)$ and hence the percentage points can be obtained by entering a variance ratio table with

$$F=\frac{1-U}{U}\frac{f_1}{f_2}\,, \qquad \nu_1=2f_2,\ \nu_2=2f_1\,,$$

or by entering directly with $U$ the percentage point table of the beta variable (e.g. Table 16 of Biometrika Tables for Statisticians) with the same degrees of freedom. Also, according to Theorem 3.2, when $f_2=1$, $U_{p,1,f_1}$ (which has the same distribution as $U_{1,p,f_1+1-p}$) is distributed as $\beta(f_1+1-p,\,p;\,U)$ and we use the same tables with degrees of freedom $\nu_1=2p$, $\nu_2=2(f_1+1-p)$.

For $p=2$, (5.3) is programmed to compute the values of $U_{2,f_2,f_1}$ on CDC 6400 to a minimum accuracy of five significant digits based on three arguments $(f_2, f_1, \alpha)$ where $\alpha$ is the lower probability level. For large values of $f_1$ ($\geqq 40$) we used the approximation (5.3.2) of Khatri [5] neglecting terms of order $0(m^{-3})$. These values were then used to obtain the correction factor for converting chi-square percentiles with $2pf_2$ degrees of freedom to the exact percentiles of $-(2f_1+f_2-p)\log U_{p,f_2,f_1}$. Finally tabulation of the correction factors,

$$C=[\text{percentiles of } -(2f_1+f_2-p)\log U_{p,f_2,f_1}]/[\text{percentile of } \chi^2_{2pf_2}]$$

was made for $\alpha=0.1$, 0.05, 0.025, 0.01, 0.005; $f_2=2(1)\,10(2)20$ and $f_1=2(1)10(2)20$, 24, 30, 40, 60, 120, $\infty$. These factors are given to three decimal places although they were obtained to an accuracy of four decimals. The correction factors are presented in Table 1. Schatzoff [8] has shown that the linear interpolation in these tables gives reasonably accurate approximations to $C$. Pillai and Gupta [7] have described some of the uses of such tables in the real case. The computations in Table 1 were carried on CDC 6400 of the University of Arizona's Computer Center.

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Table 1. Chi-square adjustments to Wilks' criterion $U$, in the complex case. Factor $C$ for lower percentiles of $U$ (upper percentiles of $\chi^2$), $p=2$.

| | $f_2=2$ | | | | | $f_2=3$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_1$ \ $\alpha$ | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 |
| 2 | 1.200 | 1.223 | 1.244 | 1.271 | 1.290 | 1.261 | 1.289 | 1.315 | 1.349 | 1.374 |
| 3 | 1.074 | 1.081 | 1.088 | 1.097 | 1.104 | 1.105 | 1.115 | 1.124 | 1.135 | 1.143 |
| 4 | 1.039 | 1.043 | 1.046 | 1.051 | 1.054 | 1.059 | 1.064 | 1.068 | 1.074 | 1.078 |
| 5 | 1.024 | 1.027 | 1.029 | 1.032 | 1.034 | 1.038 | 1.041 | 1.044 | 1.047 | 1.050 |
| 6 | 1.017 | 1.018 | 1.020 | 1.022 | 1.023 | 1.026 | 1.029 | 1.031 | 1.033 | 1.035 |
| 7 | 1.012 | 1.013 | 1.014 | 1.016 | 1.017 | 1.020 | 1.021 | 1.023 | 1.024 | 1.026 |
| 8 | 1.009 | 1.010 | 1.011 | 1.012 | 1.013 | 1.015 | 1.016 | 1.017 | 1.019 | 1.020 |
| 9 | 1.007 | 1.008 | 1.009 | 1.009 | 1.010 | 1.012 | 1.013 | 1.014 | 1.015 | 1.016 |
| 10 | 1.006 | 1.006 | 1.007 | 1.008 | 1.008 | 1.010 | 1.011 | 1.011 | 1.012 | 1.013 |
| 12 | 1.004 | 1.004 | 1.005 | 1.005 | 1.006 | 1.007 | 1.007 | 1.008 | 1.008 | 1.009 |
| 14 | 1.003 | 1.003 | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 | 1.006 | 1.006 | 1.007 |
| 16 | 1.002 | 1.002 | 1.003 | 1.003 | 1.003 | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 |
| 18 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 | 1.003 | 1.003 | 1.004 | 1.004 | 1.004 |
| 20 | 1.001 | 1.001 | 1.002 | 1.002 | 1.002 | 1.002 | 1.003 | 1.003 | 1.003 | 1.003 |
| 24 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 |
| 30 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.002 |
| 40 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 |
| 60 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 120 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| ∞ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\chi^2_{2pf_2}$ | 13.3616 | 15.5073 | 17.5346 | 20.0902 | 21.9550 | 18.5494 | 21.0261 | 23.3367 | 26.2170 | 28.2995 |

| | $f_2=4$ | | | | | $f_2=5$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1.319 | 1.351 | 1.382 | 1.422 | 1.450 | 1.373 | 1.409 | 1.444 | 1.488 | 1.521 |
| 3 | 1.138 | 1.149 | 1.160 | 1.173 | 1.183 | 1.170 | 1.183 | 1.195 | 1.211 | 1.222 |
| 4 | 1.080 | 1.086 | 1.092 | 1.099 | 1.104 | 1.102 | 1.109 | 1.116 | 1.124 | 1.130 |
| 5 | 1.053 | 1.057 | 1.060 | 1.065 | 1.068 | 1.069 | 1.074 | 1.078 | 1.083 | 1.087 |
| 6 | 1.038 | 1.040 | 1.043 | 1.046 | 1.048 | 1.050 | 1.053 | 1.056 | 1.060 | 1.063 |
| 7 | 1.029 | 1.030 | 1.032 | 1.035 | 1.036 | 1.038 | 1.041 | 1.043 | 1.046 | 1.048 |
| 8 | 1.022 | 1.024 | 1.025 | 1.027 | 1.028 | 1.030 | 1.032 | 1.034 | 1.036 | 1.038 |
| 9 | 1.018 | 1.019 | 1.020 | 1.022 | 1.023 | 1.024 | 1.026 | 1.027 | 1.029 | 1.030 |
| 10 | 1.015 | 1.016 | 1.017 | 1.018 | 1.019 | 1.020 | 1.021 | 1.023 | 1.024 | 1.025 |
| 12 | 1.010 | 1.011 | 1.012 | 1.013 | 1.013 | 1.014 | 1.015 | 1.016 | 1.017 | 1.018 |
| 14 | 1.008 | 1.008 | 1.009 | 1.009 | 1.010 | 1.011 | 1.012 | 1.012 | 1.013 | 1.014 |
| 16 | 1.006 | 1.006 | 1.007 | 1.007 | 1.008 | 1.008 | 1.009 | 1.010 | 1.010 | 1.011 |
| 18 | 1.005 | 1.005 | 1.005 | 1.006 | 1.006 | 1.007 | 1.007 | 1.008 | 1.008 | 1.008 |
| 20 | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 | 1.006 | 1.006 | 1.006 | 1.007 | 1.007 |
| 24 | 1.003 | 1.003 | 1.003 | 1.003 | 1.003 | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 |
| 30 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 | 1.003 | 1.003 | 1.003 | 1.003 | 1.003 |
| 40 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 |
| 60 | 1.000 | 1.000 | 1.000 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 |
| 120 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| ∞ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\chi^3_{3pf_2}$ | 23.5418 | 26.2962 | 28.8454 | 31.9999 | 34.2672 | 28.4120 | 31.4104 | 34.1696 | 37.5662 | 39.9968 |

$p$=number of variates; $f_1$=error degrees of freedom; $f_2$=hypothesis degrees of freedom; $C$=[percentile for $-(2f_1+f_2-p)\log_e U$]/(percentile for $\chi^2_{2pf_2}$).

Table 1. (Continued)

| | $f_2=6$ | | | | | $f_2=7$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_1 \backslash \alpha$ | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 |
| 2 | 1.423 | 1.462 | 1.500 | 1.549 | 1.585 | 1.468 | 1.511 | 1.552 | 1.605 | 1.644 |
| 3 | 1.201 | 1.216 | 1.229 | 1.247 | 1.259 | 1.231 | 1.247 | 1.262 | 1.281 | 1.295 |
| 4 | 1.124 | 1.132 | 1.140 | 1.149 | 1.156 | 1.145 | 1.154 | 1.162 | 1.173 | 1.181 |
| 5 | 1.085 | 1.091 | 1.096 | 1.102 | 1.106 | 1.102 | 1.108 | 1.113 | 1.120 | 1.125 |
| 6 | 1.063 | 1.067 | 1.070 | 1.074 | 1.078 | 1.076 | 1.080 | 1.084 | 1.089 | 1.093 |
| 7 | 1.048 | 1.051 | 1.054 | 1.057 | 1.060 | 1.059 | 1.062 | 1.065 | 1.069 | 1.072 |
| 8 | 1.039 | 1.041 | 1.043 | 1.045 | 1.047 | 1.047 | 1.050 | 1.052 | 1.055 | 1.057 |
| 9 | 1.032 | 1.033 | 1.035 | 1.037 | 1.038 | 1.039 | 1.041 | 1.043 | 1.045 | 1.047 |
| 10 | 1.026 | 1.028 | 1.029 | 1.031 | 1.032 | 1.033 | 1.034 | 1.036 | 1.038 | 1.039 |
| 12 | 1.019 | 1.020 | 1.021 | 1.022 | 1.023 | 1.024 | 1.025 | 1.026 | 1.028 | 1.029 |
| 14 | 1.014 | 1.015 | 1.016 | 1.017 | 1.018 | 1.018 | 1.019 | 1.020 | 1.021 | 1.022 |
| 16 | 1.011 | 1.012 | 1.012 | 1.013 | 1.014 | 1.014 | 1.015 | 1.016 | 1.017 | 1.017 |
| 18 | 1.009 | 1.010 | 1.010 | 1.011 | 1.011 | 1.012 | 1.012 | 1.013 | 1.014 | 1.014 |
| 20 | 1.008 | 1.008 | 1.008 | 1.009 | 1.009 | 1.010 | 1.010 | 1.011 | 1.011 | 1.012 |
| 24 | 1.005 | 1.006 | 1.006 | 1.006 | 1.007 | 1.007 | 1.007 | 1.008 | 1.008 | 1.008 |
| 30 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 | 1.005 | 1.005 | 1.005 |
| 40 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 | 1.003 | 1.003 | 1.003 | 1.003 | 1.003 |
| 60 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 |
| 120 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\infty$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\chi^2_{2pf_2}$ | 33.1963 | 36.4151 | 39.3641 | 42.9798 | 45.5585 | 37.9159 | 41.3372 | 44.4607 | 48.2782 | 50.9933 |

| | $f_2=8$ | | | | | $f_2=9$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_1 \backslash \alpha$ | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 |
| 2 | 1.511 | 1.557 | 1.600 | 1.657 | 1.698 | 1.551 | 1.599 | 1.645 | 1.704 | 1.748 |
| 3 | 1.259 | 1.277 | 1.293 | 1.314 | 1.328 | 1.286 | 1.305 | 1.323 | 1.345 | 1.361 |
| 4 | 1.166 | 1.176 | 1.186 | 1.197 | 1.205 | 1.187 | 1.198 | 1.208 | 1.220 | 1.229 |
| 5 | 1.118 | 1.125 | 1.131 | 1.138 | 1.144 | 1.134 | 1.141 | 1.148 | 1.156 | 1.162 |
| 6 | 1.089 | 1.094 | 1.098 | 1.104 | 1.108 | 1.102 | 1.107 | 1.112 | 1.118 | 1.122 |
| 7 | 1.070 | 1.074 | 1.077 | 1.081 | 1.084 | 1.081 | 1.085 | 1.088 | 1.093 | 1.096 |
| 8 | 1.056 | 1.059 | 1.062 | 1.065 | 1.068 | 1.066 | 1.069 | 1.072 | 1.075 | 1.078 |
| 9 | 1.047 | 1.049 | 1.051 | 1.054 | 1.058 | 1.054 | 1.057 | 1.060 | 1.062 | 1.064 |
| 10 | 1.039 | 1.041 | 1.043 | 1.045 | 1.047 | 1.046 | 1.048 | 1.050 | 1.053 | 1.054 |
| 12 | 1.029 | 1.030 | 1.032 | 1.033 | 1.034 | 1.034 | 1.036 | 1.037 | 1.039 | 1.040 |
| 14 | 1.022 | 1.023 | 1.024 | 1.026 | 1.026 | 1.026 | 1.028 | 1.029 | 1.030 | 1.031 |
| 16 | 1.018 | 1.018 | 1.019 | 1.020 | 1.021 | 1.021 | 1.022 | 1.023 | 1.024 | 1.025 |
| 18 | 1.014 | 1.015 | 1.016 | 1.016 | 1.017 | 1.017 | 1.018 | 1.019 | 1.020 | 1.020 |
| 20 | 1.012 | 1.012 | 1.013 | 1.014 | 1.014 | 1.014 | 1.015 | 1.016 | 1.016 | 1.017 |
| 24 | 1.009 | 1.009 | 1.009 | 1.010 | 1.010 | 1.010 | 1.011 | 1.011 | 1.012 | 1.012 |
| 30 | 1.006 | 1.006 | 1.006 | 1.007 | 1.007 | 1.007 | 1.007 | 1.008 | 1.008 | 1.008 |
| 40 | 1.003 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 |
| 60 | 1.001 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 | 1.002 |
| 120 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.001 | 1.001 | 1.001 |
| $\infty$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\chi^2_{2pf_2}$ | 42.5847 | 46.1943 | 49.4804 | 53.4858 | 56.3281 | 47.2122 | 50.9985 | 54.4373 | 58.6192 | 61.5812 |

Table 1. (Continued)

| | $f_2=10$ | | | | | $f_2=12$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_1$ \ $\alpha$ | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 |
| 2 | 1.588 | 1.638 | 1.687 | 1.749 | 1.795 | 1.655 | 1.710 | 1.763 | 1.830 | 1.880 |
| 3 | 1.312 | 1.332 | 1.351 | 1.374 | 1.391 | 1.361 | 1.383 | 1.404 | 1.429 | 1.448 |
| 4 | 1.207 | 1.218 | 1.229 | 1.242 | 1.252 | 1.244 | 1.257 | 1.269 | 1.284 | 1.295 |
| 5 | 1.150 | 1.158 | 1.165 | 1.174 | 1.180 | 1.181 | 1.190 | 1.198 | 1.208 | 1.215 |
| 6 | 1.115 | 1.121 | 1.126 | 1.132 | 1.137 | 1.141 | 1.147 | 1.153 | 1.160 | 1.166 |
| 7 | 1.092 | 1.096 | 1.100 | 1.105 | 1.108 | 1.113 | 1.118 | 1.123 | 1.128 | 1.132 |
| 8 | 1.075 | 1.078 | 1.082 | 1.086 | 1.088 | 1.093 | 1.098 | 1.101 | 1.106 | 1.109 |
| 9 | 1.062 | 1.065 | 1.068 | 1.071 | 1.074 | 1.079 | 1.082 | 1.085 | 1.089 | 1.092 |
| 10 | 1.053 | 1.056 | 1.058 | 1.060 | 1.062 | 1.067 | 1.070 | 1.073 | 1.076 | 1.078 |
| 12 | 1.040 | 1.042 | 1.043 | 1.045 | 1.046 | 1.051 | 1.053 | 1.055 | 1.057 | 1.059 |
| 14 | 1.031 | 1.032 | 1.034 | 1.035 | 1.036 | 1.040 | 1.042 | 1.043 | 1.045 | 1.046 |
| 16 | 1.025 | 1.026 | 1.027 | 1.028 | 1.029 | 1.032 | 1.034 | 1.035 | 1.036 | 1.037 |
| 18 | 1.020 | 1.021 | 1.022 | 1.023 | 1.024 | 1.027 | 1.028 | 1.029 | 1.030 | 1.031 |
| 20 | 1.017 | 1.018 | 1.018 | 1.019 | 1.020 | 1.022 | 1.023 | 1.024 | 1.025 | 1.026 |
| 24 | 1.012 | 1.013 | 1.013 | 1.014 | 1.014 | 1.016 | 1.017 | 1.018 | 1.018 | 1.019 |
| 30 | 1.008 | 1.009 | 1.009 | 1.009 | 1.010 | 1.011 | 1.012 | 1.012 | 1.012 | 1.013 |
| 40 | 1.005 | 1.005 | 1.005 | 1.006 | 1.006 | 1.007 | 1.007 | 1.007 | 1.008 | 1.008 |
| 60 | 1.002 | 1.002 | 1.002 | 1.003 | 1.003 | 1.003 | 1.003 | 1.003 | 1.004 | 1.004 |
| 120 | 1.001 | 1.001 | 1.001 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.001 | 1.001 |
| $\infty$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\chi^2_{2pf_2}$ | 51.8050 | 55.7585 | 59.3417 | 63.6907 | 66.7659 | 60.9066 | 65.1708 | 69.0226 | 73.6826 | 76.9688 |

| | $f_2=14$ | | | | | $f_2=16$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1.715 | 1.774 | 1.830 | 1.902 | 1.956 | 1.769 | 1.831 | 1.891 | 1.967 | 2.024 |
| 3 | 1.405 | 1.429 | 1.452 | 1.480 | 1.500 | 1.447 | 1.472 | 1.496 | 1.526 | 1.548 |
| 4 | 1.280 | 1.294 | 1.307 | 1.324 | 1.335 | 1.313 | 1.329 | 1.343 | 1.360 | 1.373 |
| 5 | 1.210 | 1.220 | 1.229 | 1.240 | 1.248 | 1.238 | 1.248 | 1.258 | 1.270 | 1.279 |
| 6 | 1.165 | 1.172 | 1.179 | 1.187 | 1.193 | 1.189 | 1.197 | 1.204 | 1.213 | 1.220 |
| 7 | 1.134 | 1.140 | 1.145 | 1.152 | 1.156 | 1.155 | 1.161 | 1.167 | 1.174 | 1.179 |
| 8 | 1.112 | 1.116 | 1.121 | 1.126 | 1.129 | 1.130 | 1.135 | 1.140 | 1.145 | 1.149 |
| 9 | 1.095 | 1.099 | 1.102 | 1.106 | 1.109 | 1.111 | 1.115 | 1.119 | 1.124 | 1.127 |
| 10 | 1.082 | 1.085 | 1.088 | 1.091 | 1.094 | 1.096 | 1.100 | 1.103 | 1.107 | 1.110 |
| 12 | 1.062 | 1.065 | 1.067 | 1.070 | 1.072 | 1.074 | 1.077 | 1.079 | 1.082 | 1.084 |
| 14 | 1.050 | 1.051 | 1.053 | 1.055 | 1.057 | 1.059 | 1.061 | 1.063 | 1.066 | 1.067 |
| 16 | 1.040 | 1.042 | 1.043 | 1.045 | 1.046 | 1.048 | 1.050 | 1.052 | 1.054 | 1.055 |
| 18 | 1.033 | 1.035 | 1.036 | 1.037 | 1.038 | 1.040 | 1.042 | 1.043 | 1.045 | 1.046 |
| 20 | 1.028 | 1.029 | 1.030 | 1.031 | 1.032 | 1.034 | 1.036 | 1.037 | 1.038 | 1.039 |
| 24 | 1.021 | 1.022 | 1.022 | 1.023 | 1.024 | 1.026 | 1.026 | 1.027 | 1.028 | 1.029 |
| 30 | 1.014 | 1.015 | 1.015 | 1.016 | 1.016 | 1.018 | 1.018 | 1.019 | 1.020 | 1.020 |
| 40 | 1.009 | 1.009 | 1.009 | 1.010 | 1.010 | 1.011 | 1.011 | 1.012 | 1.012 | 1.012 |
| 60 | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 | 1.005 | 1.005 | 1.006 | 1.006 | 1.006 |
| 120 | 1.002 | 1.002 | 1.002 | 1.002 | 1.003 | 1.003 | 1.003 | 1.003 | 1.003 | 1.003 |
| $\infty$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\chi^2_{2pf_2}$ | 69.9185 | 74.4683 | 78.5672 | 83.5134 | 86.9938 | 78.8596 | 83.6753 | 88.0041 | 93.2169 | 96.8781 |

Table 1. (Continued)

| | $f_2=18$ | | | | | $f_2=20$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_1 \backslash \alpha$ | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 |
| 2 | 1.819 | 1.884 | 1.946 | 2.026 | 2.085 | 1.864 | 1.932 | 1.997 | 2.080 | 2.142 |
| 3 | 1.485 | 1.512 | 1.538 | 1.569 | 1.593 | 1.520 | 1.549 | 1.576 | 1.610 | 1.634 |
| 4 | 1.344 | 1.361 | 1.376 | 1.395 | 1.409 | 1.374 | 1.392 | 1.408 | 1.428 | 1.442 |
| 5 | 1.264 | 1.276 | 1.286 | 1.299 | 1.308 | 1.290 | 1.302 | 1.313 | 1.327 | 1.337 |
| 6 | 1.212 | 1.220 | 1.228 | 1.238 | 1.245 | 1.234 | 1.243 | 1.252 | 1.262 | 1.269 |
| 7 | 1.175 | 1.182 | 1.188 | 1.196 | 1.201 | 1.194 | 1.202 | 1.208 | 1.216 | 1.222 |
| 8 | 1.148 | 1.153 | 1.158 | 1.164 | 1.169 | 1.165 | 1.171 | 1.176 | 1.183 | 1.188 |
| 9 | 1.127 | 1.131 | 1.136 | 1.141 | 1.144 | 1.142 | 1.147 | 1.152 | 1.157 | 1.161 |
| 10 | 1.110 | 1.114 | 1.118 | 1.122 | 1.125 | 1.124 | 1.128 | 1.132 | 1.137 | 1.140 |
| 12 | 1.086 | 1.089 | 1.091 | 1.095 | 1.097 | 1.097 | 1.101 | 1.104 | 1.107 | 1.110 |
| 14 | 1.069 | 1.071 | 1.073 | 1.076 | 1.078 | 1.079 | 1.081 | 1.084 | 1.086 | 1.088 |
| 16 | 1.057 | 1.059 | 1.060 | 1.062 | 1.064 | 1.065 | 1.067 | 1.069 | 1.072 | 1.073 |
| 18 | 1.048 | 1.049 | 1.051 | 1.052 | 1.054 | 1.055 | 1.057 | 1.058 | 1.060 | 1.062 |
| 20 | 1.041 | 1.042 | 1.043 | 1.045 | 1.046 | 1.047 | 1.048 | 1.050 | 1.051 | 1.052 |
| 24 | 1.030 | 1.032 | 1.032 | 1.034 | 1.034 | 1.036 | 1.037 | 1.038 | 1.039 | 1.040 |
| 30 | 1.021 | 1.022 | 1.023 | 1.023 | 1.024 | 1.025 | 1.026 | 1.026 | 1.027 | 1.028 |
| 40 | 1.013 | 1.014 | 1.014 | 1.014 | 1.015 | 1.016 | 1.016 | 1.016 | 1.017 | 1.017 |
| 60 | 1.012 | 1.012 | 1.013 | 1.013 | 1.013 | 1.014 | 1.015 | 1.015 | 1.015 | 1.015 |
| 120 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 | 1.005 | 1.005 |
| $\infty$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| $\chi^2_{2pf_2}$ | 87.7433 | 92.8083 | 97.3532 | 102.816 | 106.648 | 96.5782 | 101.879 | 106.629 | 112.329 | 116.321 |

UNIVERSITY OF ARIZONA

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