SUBCLASSES OF GENERALIZED INVERSES OF MATRICES*

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Summary

Structure of all g-inverses of a matrix in a weak sense is shown. Characterizations of main subclasses of g-inverses are investigated thoroughly. The dualities among subclasses and the relation between g-inverses and projections are stressed. The Gauss-Markov theorem reduces to a duality of two types of g-inverses.

1. Introduction and notations

The generalized inverse (g-inverse for short) in a strong sense, the Moore-Penrose inverse, of an $m\times n$ matrix A is an $n\times m$ matrix G such that

(1.1) $$AGA=A ,$$

(1.2) $$(AG)^*=AG ,$$

(1.3) $$(GA)^*=GA ,$$

(1.4) $$GAG=G ,$$

where asterisks denote the conjugate transposes (Penrose [5]). The g-inverse of A in a weak sense is any G which satisfies (1.1) (Rao [6]). According to whether each of (1.2)–(1.4) is satisfied or not, there are eight possible subclasses of g-inverses. We denote these subclasses and their generic elements as follows:

the class of all g-inverses,

$$\mathcal{G}(A)=\{G;\, AGA=A\} \ni A^- ,$$

the class of least squares type,

$$\mathcal{G}_l(A)=\{G;\, (AG)^*=AG;\, G\in\mathcal{G}(A)\} \ni A_l^- ,$$

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the class of minimum norm type,

$$\mathcal{G}_m(A)=\{G;(GA)^*=GA;G\in\mathcal{G}(A)\}\ni A_m^-,$$

the class of reflexive type,

$$\mathcal{G}_r(A)=\{G;GAG=G;G\in\mathcal{G}(A)\}\ni A_r^-,$$

the meets of the above classes,

$$\mathcal{G}_{lm}(A)=\mathcal{G}_l(A)\cap\mathcal{G}_m(A)\ni A_{lm}^-,$$

$$\mathcal{G}_{lr}(A)=\mathcal{G}_l(A)\cap\mathcal{G}_r(A)\ni A_{lr}^-,$$

$$\mathcal{G}_{mr}(A)=\mathcal{G}_m(A)\cap\mathcal{G}_r(A)\ni A_{mr}^-,$$

$$\mathcal{G}_{MP}(A)=\mathcal{G}_l(A)\cap\mathcal{G}_m(A)\cap\mathcal{G}_r(A)\ni A_{MP}^-.$$

Actually $\mathcal{G}_{MP}(A)$ consists of a single element. The classification and the notations owe mainly to C. R. Rao [7], which presented a variety of subclasses of g-inverses of matrices and gave a systematic account on the subject.

In Section 2 we develop the results in J. Z. Hearon and J. W. Evans [3] to clarify the structure of $\mathcal{G}(A)$. In Sections 3 and 4 characterizations of subclasses in C. R. Rao [7] are investigated thoroughly placing emphases on the dualities among subclasses and on the relation between g-inverses and projections. Section 5 contains some results on g-inverses of a product, and Section 6 an aspect of the Gauss-Markov theorem on least squares estimators.

We shall use the following notations.

C^n and $C^{m\times n}$ the linear spaces of complex n-vectors, and of $m\times n$ matrices. In the latter the inner product of two matrices (a_{ij}) and (b_{ij}) is defined by $\Sigma a_{ij}\bar{b}_{ij}$.

$\mathcal{R}(A)$ the range space of a matrix A, or the subspace spanned by the columns of A.

$\mathcal{N}(A)$ the null space of a matrix A, $\{x;Ax=0\}$, or the orthogonal complement of $\mathcal{R}(A^*)$.

$\dim\mathcal{S}$ the dimension of a subspace $\mathcal{S}$.

$\mathcal{H}(A;y)=\{x;Ax=y\}$, where $y\in\mathcal{R}(A)$, the hyperplane of solutions of a consistent equation $Ax=y$. If $Ax_0=y$, this is $x_0+\mathcal{N}(A)$.

$\|x\|$ the Euclidean norm of a vector x.

Π_A the orthogonal projection on $\mathcal{R}(A)$.

2. Structure of $\mathcal{G}(A)$

We use the canonical form of g-inverses which were introduced by Hearon and Evans [3] and Tewarson [10]:

It is well known that for an $m \times n$ matrix A there exist unitary matrices $U(m \times m)$ and $V(n \times n)$ such that

$$A = U^* D V \tag{2.1}$$

and D's elements are zero except for d_{i_i}, $i=1, \cdots, r$ (=rank A). It is not essential that D is diagonal. We need the submatrix of the first r rows and the first r columns of D to be nonsingular.

PROPOSITION 2.1. Let D be an $m \times n$ matrix of the form

$$D = \begin{bmatrix} \Delta & O \\ O & O \end{bmatrix} \tag{2.2}$$

where Δ is an $r \times r$ nonsingular matrix, and O's are null matrices of suitable sizes. Then an $n \times m$ matrix belongs to $\mathcal{J}(D)$, $\mathcal{J}_l(D)$ or $\mathcal{J}_m(D)$, iff it is of the following forms, respectively:

$$\begin{aligned} D^- &= \begin{bmatrix} \Delta^{-1} & R \\ L & S \end{bmatrix}, & D_l^- &= \begin{bmatrix} \Delta^{-1} & O \\ L & S \end{bmatrix} \\ D_m^- &= \begin{bmatrix} \Delta^{-1} & R \\ O & S \end{bmatrix}, & D_r^- &= \begin{bmatrix} \Delta^{-1} & R \\ L & L\Delta R \end{bmatrix}. \end{aligned} \tag{2.3}$$

The forms of D_{lm}^-, etc., are clear from these. Notice that

$$D_{MP}^- = \begin{bmatrix} \Delta^{-1} & O \\ O & O \end{bmatrix}$$

is uniquely determined.

PROOF. Partition D^-, for example, into submatrices and just check the identity $DD^-D = D$. Other forms are also derived by verifying (1.2)–(1.4).

PROPOSITION 2.2. If $A = U^*BV$, U and V are unitary, then G belongs to $\mathcal{J}(A)$, $\mathcal{J}_l(A)$, $\mathcal{J}_m(A)$ or $\mathcal{J}_r(A)$ iff $G = V^*FU$ and F belongs to $\mathcal{J}(B)$, $\mathcal{J}_l(B)$, $\mathcal{J}_m(B)$ or $\mathcal{J}_r(B)$, respectively. The statement can be extended to the meets $\mathcal{J}_{lm}(A)$ etc.

PROOF. $AGA = U^*BVGU^*BV$ is equal to $A = U^*BV$ iff $BVGU^*B = B$, that is $G \in \mathcal{J}(A)$ iff $VGU^* = F \in \mathcal{J}(B)$.

Other propositions are proved similarly.

In expression (2.1) U and V are not always unique. We, however, have a well known result (Penrose [5]) which will be shown very shortly.

PROPOSITION 2.3. A_{MP}^- is unique.

PROOF. Let $A=U_1^*D_1V_1=U_2^*D_2V_2$ be two ways decomposition of the form (2.1). Let $G_1=V_1^*(D_1)_{MP}^-U_1$ and $G_2=V_2^*(D_2)_{MP}^-U_2$ where $(D_1)_{MP}^-$ and $(D_2)_{MP}^-$ are uniquely determined. As $D_2=U_2U_1^*D_1V_1V_2^*$, $G_2=V_2^*(V_1V_2^*)^*\cdot(D_1)_{MP}^-(U_2U_1^*)^*U_2=G_1$ by Prop. 2.2.

Now we define the following subsets of the linear space of all $n\times m$ matrices with complex elements, $C^{n\times m}$. The subscripts correspond to those of $\mathcal{G}_l(A)$, etc.

$$\mathcal{K}(A)=\{G-A_{MP}^-;\, G\in\mathcal{G}(A)\}=\{G_1-G_2;\, G_1,G_2\in\mathcal{G}(A)\},$$

$$\mathcal{K}_l(A)=\{G-A_{MP}^-;\, G\in\mathcal{G}_l(A)\}=\{G_1-G_2;\, G_1,G_2\in\mathcal{G}_l(A)\},$$

$$\mathcal{K}_m(A)=\{G-A_{MP}^-;\, G\in\mathcal{G}_m(A)\}=\{G_1-G_2;\, G_1,G_2\in\mathcal{G}_m(A)\},$$

$$\mathcal{K}_r(A)=\{G-A_{MP}^-;\, G\in\mathcal{G}_r(A)\},$$

$$\mathcal{K}_{lm}(A)=\mathcal{K}_l(A)\cap\mathcal{K}_m(A), \qquad \mathcal{K}_{lr}(A)=\mathcal{K}_l(A)\cap\mathcal{K}_r(A),$$

$$\mathcal{K}_{mr}(A)=\mathcal{K}_m(A)\cap\mathcal{K}_r(A).$$

PROPOSITION 2.4. Except for $\mathcal{K}_r(A)$ these subsets are all linear subspaces of $C^{n\times m}$ and they have the following structures:

$$\mathcal{K}(A)=\mathcal{K}_{lm}(A)\oplus\mathcal{K}_{lr}(A)\oplus\mathcal{K}_{mr}(A),$$

$$\mathcal{K}_l(A)=\mathcal{K}_{lm}(A)\oplus\mathcal{K}_{lr}(A), \qquad \mathcal{K}_m(A)=\mathcal{K}_{lm}(A)\oplus\mathcal{K}_{mr}(A),$$

$\mathcal{K}_{lm}(A)$, $\mathcal{K}_{lr}(A)$ and $\mathcal{K}_{mr}(A)$ are mutually orthogonal (in terms of the inner product mentioned in the last section) and A_{MP}^- is orthogonal to $\mathcal{K}(A)$. Moreover,

$$\dim\mathcal{K}(A)=mn-r, \qquad \dim\mathcal{K}_l(A)=(n-r)m,$$

$$\dim\mathcal{K}_m(A)=(m-r)n, \qquad \dim\mathcal{K}_{lm}(A)=(m-r)(n-r),$$

$$\dim\mathcal{K}_{lr}(A)=(n-r)r, \qquad \dim\mathcal{K}_{mr}(A)=(m-r)r.$$

PROOF. Fix a decomposition (2.1) and consider Prop. 2.2 and the sets of D^- of the form (2.3).

PROPOSITION 2.5. Any g-inverse can be expressed uniquely as

$$A^-=A_{MP}^-+K_{lm}+K_{lr}+K_{mr},$$

where $K_{lm}\in\mathcal{K}_{lm}(A)$ $K_{lr}\in\mathcal{K}_{lr}(A)$ and $K_{mr}\in\mathcal{K}_{mr}(A)$.

PROOF. The first term is the orthogonal projection of A^- on the orthogonal complement of $\mathcal{K}(A)$. The others are the orthogonal projections on $\mathcal{K}_{lm}(A)$, $\mathcal{K}_{lr}(A)$ and $\mathcal{K}_{mr}(A)$.

PROPOSITION 2.6.

$$\mathcal{K}_l(A)=\{K;\ AK=O,\ K\in C^{n\times m}\}\ ,$$

$$\mathcal{K}_m(A)=\{K;\ KA=O,\ K\in C^{n\times m}\}\ .$$

PROOF. If A is decomposed as (2.1) and (2.2) then

$AK=O$ iff

$$K=V^*\begin{bmatrix} O & O \\ L & S \end{bmatrix}U\ ,$$

where L and S are arbitrary, which means $K\in\mathcal{K}_l(A)$. The second expression can be proved similarly.

PROPOSITION 2.7. If rank $A=m<n$ then

$$\mathcal{G}(A)=\mathcal{G}_l(A)=\mathcal{G}_r(A)=\mathcal{G}_{lr}(A)$$

and

$$\mathcal{G}_m(A)=\mathcal{G}_{mr}(A)=\mathcal{G}_{lm}(A)=\mathcal{G}_{MP}(A)\ .$$

If rank $A=n<m$ then

$$\mathcal{G}(A)=\mathcal{G}_m(A)=\mathcal{G}_r(A)=\mathcal{G}_{mr}(A)$$

and

$$\mathcal{G}_l(A)=\mathcal{G}_{lr}(A)=\mathcal{G}_{lm}(A)=\mathcal{G}_{MP}(A)\ .$$

PROOF. If rank $A=m<n$, for example, D in the decomposition (2.1) has the form

$$D=[\Delta\ \ O]$$

where Δ, $m\times m$, is nonsingular, then any

$$D^-=\begin{bmatrix} \Delta^{-1} \\ L \end{bmatrix}$$

belongs to $\mathcal{G}_l(D)$, $\mathcal{G}_r(D)$ and the $\mathcal{G}_{lr}(D)$. And

$$D_m^-=\begin{bmatrix} \Delta^{-1} \\ O \end{bmatrix}$$

is also D_{MP}^-.

Remark. If rank $A=m=n$ then any A^- is the inverse A^{-1}.

Concluding the section we notice that the properties of g-inverses are those as linear mappings (at least so far as discussed in this paper) and free from the coordinate system. Thus, for a given A we choose a suitable coordinate system then we can proceed only considering a matrix of the form (2.2).

3. Characterization of subclasses—(1)

The theory of g-inverses is closely related to that of projection, so we summarize its basic features.

If the linear space of complex m-vectors $\boldsymbol{C}^m$ is a direct sum of subspaces $\mathcal{S}$ and $\mathcal{T}$, any vector is expressed uniquely as $x=x_1+x_2$, $x_1 \in \mathcal{S}$ and $x_2 \in \mathcal{T}$. Then the linear mapping $Px=x_1$ is the (generalized or oblique) projection on $\mathcal{S}$ along $\mathcal{T}$ and P is uniquely determined. The following proposition is well known.

PROPOSITION 3.1. The following conditions are equivalent and characterize an $m\times m$ matrix P to be a projection.

(1) $P^2=P$,
(2) $Px=x$ for all $x \in \mathcal{R}(P)$,
(3) $\mathcal{R}(I-P)=\mathcal{N}(P)$ or $\mathcal{R}(P)=\mathcal{N}(I-P)$,
(4) $\mathcal{R}(P)\cap\mathcal{R}(I-P)=\{0\}$ or $\boldsymbol{C}^m=\mathcal{R}(P)\oplus\mathcal{R}(I-P)$,
(5) $\operatorname{rank}(I-P)=m-\operatorname{rank} P$.

Now, let A be an $m\times n$ matrix. We shall need

PROPOSITION 3.2. $P=AG$ is a projection on $\mathcal{R}(A)$ iff one of the following equivalent conditions is satisfied.

(1) $P^2=P$ and $\operatorname{rank} P=\operatorname{rank} A$,
(2) $PA=A$,
(3) $\mathcal{R}(A)\oplus\mathcal{R}(I-P)=\boldsymbol{C}^m$,
(4) $\operatorname{rank}(I-P)=m-\operatorname{rank} A$.

PROOF. Considering Prop. 3.1 we have just to check a few facts. (1) As $\operatorname{rank} P=\operatorname{rank} A$, $\mathcal{R}(P)=\mathcal{R}(A)$. (2) $\mathcal{R}(P)=\mathcal{R}(A)$ and $Px=x$ for all $x \in \mathcal{R}(A)$. (3) In general $\mathcal{R}(P)+\mathcal{R}(I-P)=\boldsymbol{C}^m$, so $\operatorname{rank} P\geqq m-\operatorname{rank}(I-P)$. Here, $\operatorname{rank} P\leqq\operatorname{rank} A=m-\operatorname{rank}(I-P)$, then $\mathcal{R}(P)=\mathcal{R}(A)$. (4) Almost the same as (3).

We get similar conditions for $I-Q=GA$ to be a projection on $\mathcal{N}(A)$.

Table 1. A projection P and related ones

P	is a projection	on	$\mathcal{R}(P)=$	along	$\mathcal{R}(I-P)$
$I-P$		along	$\mathcal{N}(I-P)$	on	$=\mathcal{N}(P)$
			$\perp$		$\perp$
$I-P^*$	is a projection	on	$\mathcal{R}(I-P^*)$	along	$\mathcal{R}(P^*)=$
P^*		along	$=\mathcal{N}(P^*)$	on	$\mathcal{N}(I-P^*)$

Table 1 summarizes projections related to a projection P. It shows that P is the orthogonal projection (on $\mathcal{R}(P)$ along $\mathcal{N}(P^*)$) iff $P=P^*$.

PROPOSITION 3.2. $P=\Pi_A$, the orthogonal projection on $\mathcal{R}(A)$, iff one of the following equivalent conditions is satisfied.

(1) $P=AG$, $P=PP^*$ and $\operatorname{rank} P=\operatorname{rank} A$,

(2) $P=AG$ and $P^*A=A$,

(3) $PA=P^*A=A$,

(4) $\|(I-P)x\|\leqq\|(I-Q)x\|$ for any x and for any Q, a projection on $\mathcal{R}(A)$,

PROOF. (1) $P=PP^*$ implies $P=P^*=P^2$ (2) $\mathcal{R}(P)\subset\mathcal{R}(A)\subset\mathcal{R}(P^*)$, so all three subspaces are identical. As P^* is a projection on $\mathcal{R}(A)$ it is Π_A. (3) Both P and P^* are projections on $\mathcal{R}(A)$, so they are identical and orthogonal from Table 1. (4) $\|(I-Q)x\|^2=\|(I-\Pi_A)x\|^2+\|(\Pi_A-Q)x\|^2\geqq\|(I-\Pi_A)x\|^2$ since $(I-\Pi_A)(\Pi_A-Q)=O$, and the equality holds iff $Q=\Pi_A$.

Now we characterize the subclasses of g-inverses. Since the purpose of this and the following sections is to show how our new approach unifies the facts and simplifies the proofs we state many well known results trying to extend them. We shall not try to trace the original contributor. Besides Rao [7], Hearon and Evans [3], and their earlier works, Golden and Zelen [2], Rohde [8], Mitra [4], etc. treat the subject.

PROPOSITION 3.3. Any one of the following equivalent conditions characterizes $G\in\mathcal{G}(A)$, where A is an $m\times n$ matrix:

Dg. 1 $Gy\in\mathcal{K}(A;y)$ for all $y\in\mathcal{R}(A)$,
Dg. 2 $x=Gy$ gives a solution for a consistent equation $y=Ax$,
Dg. 3 $AGA=A$,
Dg. 4 AG is a projection (in C^m) on $\mathcal{R}(A)$ along $\mathcal{N}(AG)$,
Dg. 4′ GA is a projection (in C^n) on $\mathcal{R}(GA)$ along $\mathcal{N}(A)$,
Dg. 5 AG is idempotent and $\operatorname{rank} AG=\operatorname{rank} A$,
Dg. 5′ GA is idempotent and $\operatorname{rank} AG=\operatorname{rank} A$,
Dg. 6 $\operatorname{rank}(I-AG)=m-\operatorname{rank} A$,
Dg. 6′ $\operatorname{rank}(I-GA)=n-\operatorname{rank} A$,
Dg. 7 $AGAA^*=AA^*$,
Dg. 7′ $(GA)^*A^*A=A^*A$.

PROOF. Dg. 1–3 are almost the same expression. Dg. 3–6 show that AG is a projection on $\mathcal{R}(A)$ and Dg. 7 a projection on $\mathcal{R}(AA^*)=\mathcal{R}(A)$. In Dg. 4–7 we can replace A and G by A^* and G^* respectively because of the following Prop. 3.4, and rewriting the results we get Dg. 4′–7′.

PROPOSITION 3.4. $G \in \mathcal{J}(A)$ iff $G^* \in \mathcal{J}(A^*)$

PROOF. Take the conjugate transpose of Dg. 3.

PROPOSITION 3.5. Any one of the following equivalent conditions characterizes $G \in \mathcal{J}_l(A)$:

Dl. 1 $Gy \in H(A; \Pi_A y)$ for all $y \in C^m$,
Dl. 2 Gy gives a least squares solution of $Ax=y$ which is not consistent generally,
Dl. 3 $AG=\Pi_A$,
Dl. 4 $G \in \mathcal{J}(A)$ and $(AG)^*=AG$,
Dl. 5 $G \in \mathcal{J}(A)$ and $(AG)^*(I-AG)=O$,
Dl. 6 $A^*AG=A^*$,
Dl. 7 $G \in \mathcal{J}(A)$ and $\mathcal{R}(G^*A^*) \subset \mathcal{R}(A)$.

PROOF. Dl. 1–3. G maps any point of $y_1+\mathcal{N}(A^*)$, $y_1 \in \mathcal{R}(A)$, to a point of $\mathcal{H}(A; y_1)$, a least squares solution of $Ax=y \in y_1+\mathcal{N}(A^*)$. AG maps any point of $y_1+\mathcal{N}(A^*)$ to y_1. The equivalence of Dl. 3–7 comes from Prop. 3.2. Dl. 7 means, from Table 1 and Dg. 4, that $\mathcal{R}(AG)=\mathcal{R}(G^*A^*)=\mathcal{R}(A)$ so $AG=\Pi_A$.

PROPOSITION 3.6. Any one of the following equivalent conditions characterizes $G \in \mathcal{J}_m(A)$.

Dm. 1 $Gy \in \mathcal{R}(A^*) \cap \mathcal{H}(A; y)$ for all $y \in \mathcal{R}(A)$,
Dm. 2 Gy gives the minimum norm solution of a consistent equation $Ax=y$,
Dm. 3 $GA=\Pi_{A^*}$,
Dm. 4 $G \in \mathcal{J}(A)$ and $(GA)^*=GA$,
Dm. 5 $G \in \mathcal{J}(A)$ and $(GA)^*(I-GA)=O$,
Dm. 6 $GAA^*=A^*$,
Dm. 7 $G \in \mathcal{J}(A)$ and $\mathcal{R}(GA) \subset \mathcal{R}(A^*)$.

PROOF. Dm. 1–3. $\mathcal{H}(A; y)$ is a hyperplane which is parallel to $\mathcal{N}(A)$. $\mathcal{R}(A^*) \cap \mathcal{H}(A; y)$ consists of a single point which is the orthogonal projection of the origin on the hyperplane $\mathcal{H}(A; y)$. We get Dm. 4–7 starting from Dm. 3 based on Prop. 3.2 or from Dl. 4–7 just considering the following proposition.

PROPOSITION 3.7. $G \in \mathcal{J}_l(A)$ iff $G^* \in \mathcal{J}_m(A^*)$.

PROOF. Compare Dm. 3 with Dl. 3, or (1.3) with (1.2).

Any pair of conditions, one from Dl.'s the other from Dm.'s for example $A^*AG=GAA^*=A^*$, characterizes $G \in \mathcal{J}_{lm}(A)$. It should be noticed that $A_{lm}^- y$ does not always belong to $\mathcal{R}(A^*) \cap \mathcal{H}(A; y_1)$, or $\| A_{lm}^- y \|$ is not always minimum unless $y \in \mathcal{R}(A)$. It relates to the fact that if rank $A < \min(m, n)$ there exists A^-_{lm} of rank larger than rank A.

PROPOSITION 3.8.

$$\mathcal{K}_{lm}(A) = \{(I-\Pi_{A^*})Z(I-\Pi_A);\ Z \in C^{n\times m}\}$$
$$= \{(I-\Pi_{A^*})G(I-\Pi_A);\ G \in \mathcal{G}(A)\}.$$

PROOF. If A is decomposed as

$$A = U^*\begin{bmatrix} \Delta & O \\ O & O \end{bmatrix} V$$

then

$$I-\Pi_A = U^*\begin{bmatrix} O & O \\ O & I \end{bmatrix} U,$$

and

$$I-\Pi_{A^*} = V^*\begin{bmatrix} O & O \\ O & I \end{bmatrix} V.$$

4. Characterization of subclasses—(2)

PROPOSITION 4.1. Any one of the following equivalent conditions characterizes $G \in \mathcal{G}_r(A)$. We assume $G \in \mathcal{G}(A)$.

Dr. 1 rank G = rank A,
Dr. 2 $G\mathcal{R}(A) = \mathcal{R}(G)$,
Dr. 3 $\mathcal{R}(A) \oplus \mathcal{N}(G) = C^m$,
Dr. 4 $GAG = G$ or $A \in \mathcal{G}(G)$.

PROOF. Usually $\mathcal{R}(G) \supset \mathcal{R}(GA) = G\mathcal{R}(A)$. Under each condition these two subspaces are equal.

PROPOSITION 4.2.

$$\mathcal{G}_r(A) = \{G_1AG_2;\ G_1, G_2 \in \mathcal{G}(A)\}$$
$$= \{GAG;\ G \in \mathcal{G}_r(A)\}$$
$$= \{G_1AG_2;\ G_1 \in \mathcal{G}_{lr}(A),\ G_2 \in \mathcal{G}_{mr}(A)\}.$$

PROOF. It is well known that {1st set} $\subset \mathcal{G}_r(A)$, and it is clear that {2nd set}, {3rd set} $\subset$ {1st set}. To see that any A^-_r belongs to the {2nd set} just remark Dr. 4 in Prop. 4.1. To show that any A^-_r belongs to the {3rd set} put $G_1 = A^-_r\Pi_A$ and $G_2 = \Pi_{A^*}A^-_r$ and refer to the later Prop. 4.4.

Remark. The statements imply that

$$\mathcal{G}_r(A) = \{GAG;\ G \in \mathcal{G}(A)\}$$
$$= \{G_1AG_2;\ G_1, G_2 \in \mathcal{G}_r(A)\}.$$

PROPOSITION 4.3.

$$G \in \mathcal{G}_r(A) \text{ iff } G^* \in \mathcal{G}_r(A^*) ,$$

$$G \in \mathcal{G}_{lr}(A) \text{ iff } G^* \in \mathcal{G}_{mr}(A^*) .$$

PROPOSITION 4.4.

$$\begin{aligned}\mathcal{G}_{lr}(A) &= \{G\Pi_A ; G \in \mathcal{G}(A)\}\\ &= \{G\Pi_A ; G \in \mathcal{G}_{lr}(A)\}\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{lr}(A) &= \{(I-\Pi_{A^*})Z\Pi_A ; Z \in C^{n\times m}\}\\ &= \{(I-\Pi_{A^*})G\Pi_A ; G \in \mathcal{G}(A)\}\end{aligned}$$

$$\begin{aligned}\mathcal{G}_{mr}(A) &= \{\Pi_{A^*}G ; G \in \mathcal{G}(A)\}\\ &= \{\Pi_{A^*}G ; G \in \mathcal{G}_{mr}(A)\}\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{mr}(A) &= \{\Pi_{A^*}Z(I-\Pi_A) ; Z \in C^{n\times m}\}\\ &= \{\Pi_{A^*}G(I-\Pi_A) ; G \in \mathcal{G}(A)\} .\end{aligned}$$

PROOF. The first half is shown from the canonical form representations. The latter half is clear from the duality of Prop. 4.3.

Remark. The first expressions imply

$$\begin{aligned}\mathcal{G}_{lr}(A) &= \{G\Pi_A ; G \in \mathcal{G}_l(A)\}\\ &= \{G\Pi_A ; G \in \mathcal{G}_r(A)\} .\end{aligned}$$

Similar expressions hold for $\mathcal{G}_{mr}(A)$.

PROPOSITION 4.5. We assume $G \in \mathcal{G}(A)$.

$$G \in \mathcal{G}_{lr}(A) \text{ iff } \mathcal{R}(G^*) = \mathcal{R}(A) ,$$

$$G \in \mathcal{G}_{mr}(A) \text{ iff } \mathcal{R}(G) = \mathcal{R}(A^*) .$$

PROOF. By Prop. 4.3 the two statements are equivalent, so we prove the first. If the condition holds AG is a projection on $\mathcal{R}(A) = \mathcal{R}(G^*)$ along $\mathcal{N}(AG) \supset \mathcal{N}(G) = \mathcal{R}(G^*)^\perp$, so $AG = \Pi_A$ and clearly $\operatorname{rank} A = \operatorname{rank} G$. Conversely if $G \in \mathcal{G}_{lr}(A)$, $\mathcal{R}(G^*) \supset \mathcal{R}(G^*A^*) = \mathcal{R}(A)$ by Dl. 7, and the equality holds since $\operatorname{rank} G^* = \operatorname{rank} A$.

PROPOSITION 4.6.

$$\begin{aligned}\mathcal{G}_{lr}(A) &= \{GA^* ; G \in \mathcal{G}(A^*A)\}\\ &= \{GA^* ; G \in \mathcal{G}_{lr}(A^*A)\}\end{aligned}$$

$$\begin{aligned}\mathcal{G}_{mr}(A) &= \{A^*G ; G \in \mathcal{G}(AA^*)\}\\ &= \{A^*G ; G \in \mathcal{G}_{mr}(AA^*)\} .\end{aligned}$$

PROOF. The first half can be proved by using the canonical forms. The latter half is its dual.

PROPOSITION 4.7.

$$A(A^*A)^-A^*=\Pi_A .$$

PROOF. This is a corollary of Prop. 4.6.

PROPOSITION 4.8. Any one of the following equivalent conditions characterizes $G=A^-_{MP}$.

DMP. 1 $G\in\mathcal{J}_l(A)\cap\mathcal{J}_m(A)\cap\mathcal{J}_r(A)$,
DMP. 2 $Gy\in\mathcal{R}(A^*)\cap\mathcal{H}(A;\Pi_A y)$ for all y,
DMP. 3 $G\in\mathcal{J}_l(A)$ and $\|Gy\|\leq\|A_l^-y\|$ for all A_l^- and y,
DMP. 4 $G=A_m^-\Pi_A$,
DMP. 4′ $G=\Pi_{A^*}A_l^-$,
DMP. 5 $G=\Pi_{A^*}A^-\Pi_A$,
DMP. 6 $G=A_m^-AA_l^-$,
DMP. 7 $G\in\mathcal{J}_l(A)$ and $A\in\mathcal{J}_l(G)$,
DMP. 7′ $G\in\mathcal{J}_m(A)$ and $A\in\mathcal{J}_m(G)$,
DMP. 8 $G\in\mathcal{J}_l(A)$ and $\mathcal{R}(G)\subset\mathcal{R}(A^*)$,
DMP. 8′ $G\in\mathcal{J}_m(A)$ and $\mathcal{R}(G^*)\subset\mathcal{R}(A)$,
DMP. 9 $G=A^*(AA^*)_l^-$,
DMP. 9′ $G=(A^*A)_m^-A^*$,
DMP. 10 $G=A^*(AA^*)^-A(A^*A)^-A^*$,
DMP. 11 $G=A^*(A^*AA^*)^-A^*$, (Zlobec [11])*).

PROOF. $G\in\mathcal{J}_{mr}(A)$ means, as stated in Prop. 4.5, that $\mathcal{R}(G)=G\mathcal{R}(A)=\mathcal{R}(G\Pi_A)=\mathcal{R}(A^*)$. This and Dl. 1–3 shows the equivalence of DMP. 1–4. The equivalence of DMP. 4–6 follows from Dl. 3 and Dm. 3. DMP. 7 is equivalent to DMP. 1. The second condition of DMP. 8 means

Table 2 $\mathcal{S}/\mathcal{T}$ means the projection on $\mathcal{S}$ along $\mathcal{T}$.

In C^m

$G\in$	$\mathcal{J}(A)$, $\mathcal{J}_m(A)$	$\mathcal{J}_l(A)$, $\mathcal{J}_{lm}(A)$	$\mathcal{J}_r(A)$, $\mathcal{J}_{mr}(A)$	$\mathcal{J}_{lr}(A)$, $\mathcal{J}_{MP}(A)$
AG	$\mathcal{R}(A)/\mathcal{N}(AG)$	$\}\ \Pi_A$	$\mathcal{R}(A)/\mathcal{N}(G)$	$\}\ \Pi_A=\Pi_{G^*}$
G^*A^*	$\mathcal{R}(G^*A^*)/\mathcal{N}(A^*)$		$\mathcal{R}(G^*)/\mathcal{N}(A^*)$	

In C^n

$G\in$	$\mathcal{J}(A)$, $\mathcal{J}_l(A)$	$\mathcal{J}_m(A)$, $\mathcal{J}_{lm}(A)$	$\mathcal{J}_r(A)$, $\mathcal{J}_{lr}(A)$	$\mathcal{J}_{mr}(A)$, $\mathcal{J}_{MP}(A)$
GA	$\mathcal{R}(GA)/\mathcal{N}(A)$	$\}\ \Pi_{A^*}$	$\mathcal{R}(G)/\mathcal{N}(A)$	$\}\ \Pi_{A^*}=\Pi_G$
A^*G^*	$\mathcal{R}(A^*)/\mathcal{N}(A^*G^*)$		$\mathcal{R}(A^*)/\mathcal{N}(G^*)$	

*) After the paper was accepted the author learned that the condition DMP. 11 of Prop. 4.8 should have been ascribed to Mitra [4] (p. 111, Example (iv)).

$G \in \mathcal{I}_{mr}(A)$. DMP. 9 means $AG = \Pi_A$ and $G \in \mathcal{I}_{mr}(A)$. DMP. 10 is the same as DMP. 6. DMP. 11 can be shown by the canonical form. DMP. 7′–9′ are obtained from DMP. 7–9 by the dualities.

The properties of AG, etc. as projectors are summarized in Table 2. If rank $A = m \leqq n$ then AA^- is an idempotent $m \times m$ matrix of rank m, or the unit matrix, so any g-inverse is a right inverse. Similarly if rank $A = n \leqq m$ then any g-inverse is a left inverse.

5. g-inverses of product

In this section we shall get expressions of g-inverses of a product AB. The following proposition is an extension of a formula by R. E. Cline [1]. The proof is similar to Cline's.

PROPOSITION 5.1. Let $B_1 = A^-AB$ and $A_1 = AB_1G_1$

(1) if $G_1 \in \mathcal{I}(B_1)$ then $A_1B_1 = AB$ and
$$G_1(A_1)^- \in \mathcal{I}(AB),$$
$$G_1(A_1)_l^- \in \mathcal{I}_l(AB),$$
$$G_1(A_1)_r^- \in \mathcal{I}_r(AB).$$
(2) if $G_1 \in \mathcal{I}_l(B_1)$ then $(B_1)_m^-(A_1)_m^- \in \mathcal{I}_m(AB)$,
$$(B_1)_r^-(A_1)_m^- \in \mathcal{I}_r(AB).$$
(3) if $G_1 \in \mathcal{I}_{lm}(B_1)$ then $G_1(A_1)_{lm}^- \in \mathcal{I}_{lm}(AB)$,
$$G_1(A_1)_{MP}^- = (AB)_{MP}^-.$$
(4) if $G_1 = (B_1)_{MP}^-$ then $G_1(A_1)_{lm}^- = (AB)_{MP}^-$.

Similarly let $A_2 = ABB^-$ and $B_2 = G_2A_2B$

(1) if $G_2 \in \mathcal{I}(A_2)$ then $A_2B_2 = AB$ and
$$(B_2)^-G_2 \in \mathcal{I}(AB),$$
$$(B_2)_m^-G_2 \in \mathcal{I}_m(AB),$$
$$(B_2)_r^-G_2 \in \mathcal{I}_r(AB).$$
(2) if $G_2 \in \mathcal{I}_m(A_2)$ then $(B_2)_l^-(A_2)_l^- \in \mathcal{I}_l(AB)$,
$$(B_2)_l^-(A_2)_r^- \in \mathcal{I}_r(AB).$$
(3) if $G_2 \in \mathcal{I}_{lm}(A_2)$ then $(B_2)_{lm}^-G_2 \in \mathcal{I}_{lm}(AB)$,
$$(B_2)_{MP}^-G_2 = (AB)_{MP}^-.$$
(4) if $G_2 = (A_2)_{MP}^-$ then $(B_2)_{lm}^-G_2 = (AB)_{MP}^-$.

Remark. If A and B have sizes $l \times m$ and $m \times n$ respectively then $(AB)^-$ may have rank $\min(l, n)$, while $(B_1)^-(A_1)^-$ has rank at most $\min(l, m, n)$ so the latter form cannot express all $(AB)^-$.

PROPOSITION 5.2.

$$(A^-)^*A_m^- \text{ and } (A_m^-)^*A^- \in \mathcal{I}(AA^*)$$

$$(A_m^-)^*A_l^- \in \mathcal{I}_l(AA^*)$$

$$(A_l^-)^* A_m^- \in \mathcal{G}_m(AA^*)$$
$$\mathcal{G}_r(AA^*) = \{G_1^* G_2 ; G_1 \in \mathcal{G}(A), G_2 \in \mathcal{G}_{mr}(A)\}$$
$$= \{G_1^* G_2 ; G_1 \in \mathcal{G}_r(A), G_2 \in \mathcal{G}_m(A)\} .$$

Other expressions can be obtained by noticing that if $F \in \mathcal{G}_r(AA^*)$ then $F^* \in \mathcal{G}_r(AA^*)$ also, and recalling that $(A_m^-)^* = (A^*)_l^-$.

Combining these, we get other results. For example,

$$(A_{MP}^-)^* A_{lm}^- = (A_{lm}^-)^* A_{MP}^- = (AA^*)_{MP}^- .$$

PROOF. All are shown easily from the canonical forms. As stated in Remark of Prop. 5.1, only $\mathcal{G}_r(AA^*)$ can be expressed exhaustively by products of g-inverses.

PROPOSITION 5.3.

$$\mathcal{G}_{lr}(A) = \{G_1 G_2^* A^* ; G_1 \in \mathcal{G}_l(A), G_2 \in \mathcal{G}(A)\}$$
$$\mathcal{G}_{mr}(A) = \{A^* G_1^* G_2 ; G_1 \in \mathcal{G}(A), G_2 \in \mathcal{G}_m(A)\} .$$

PROOF. Use the canonical forms. The expressions are suggested from Props. 4.6 and 5.2.

6. The Gauss-Markov theorem

Finally we consider the Gauss-Markov theorem on the least squares estimator from our point of view. We consider the simplest linear model: $\mathrm{E}(y) = X\beta$ and $\mathrm{D}(y) = \sigma^2 I$. In this section we consider only real vectors and matrices. By primes we denote transposes.

If $c \in \mathcal{R}(X')$ then $c'\beta$ is estimable and $l'y$ is the best linear unbiased estimator iff $\|l\|$ is minimum under the restriction $X'l = c$. Our solution is $l = (X')_m^- c$, so the best estimator is $c'((X')_m^-)'y$.

On the other hand the least squares estimator of $c'\beta$ is $c'\hat{\beta}$ where $\hat{\beta}$ minimizes $\|X\hat{\beta} - y\|$. Our solution is $c'X_l^- y$, which is the best linear unbiased estimator by Prop. 3.7. So the theorem reduces to the duality between A_l^- and A_m^-.

The least squares principle leads to the normal equation $X'X\hat{\beta} = X'y$. Our solution is $(X'X)^- X'$, which is X_{lr}^-.

The choice of X_l^- does not affect the value of the estimator if $c'\beta$ is estimable, since if $c = X'f$ then the estimator is $f'XX_l^- y = (\Pi_X f)'y$.

The minimized variance is equal to

$$\mathrm{V}(l'y) = \sigma^2 c' X_l^- (X_l^-)' c$$

or from the least squares solution

$$\mathrm{V}(c'\hat{\beta}) = \sigma^2 c'(X'X)^- X'X(X'X)^- c$$
$$= \sigma^2 c'(X'X)_r^- c .$$

In the first expression $X_l^-(X_l^-)' \in \mathcal{G}(X'X)$ but does not always belong to the smaller subsets, $\mathcal{G}_l(X'X)$ etc. (cf. Prop. 5.2). Anyhow if $c=X'f$ then $c'(X'X)^-c=\|\Pi_X f\|^2$.

Concluding the paper we remark that some discussions on $\mathcal{G}(A)$, $\mathcal{G}_r(A)$ and g-inverses of product can be extended to g-inverses of a general mapping (cf. M. Sibuya [9]).

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REFERENCES

[1] Cline, R. E. (1964). Note on the generalized inverse of the product of matrices, *SIAM Review*, **6**, 56-58.

[2] Goldman, A. J. and Zelen, M. (1964). Weak generalized inverses and minimum variance linear unbiased estimation, *Jour. Res. NBS*, **68** B, 151-172.

[3] Hearon, J. Z. and Evans, J. W. (1968). On Spaces and maps of generalized inverses, *Jour. Res. NBC*, **72** B, 103-107.

[4] Mitra, S. K. (1968). On a generalized inverse of a matrix and applications, *Sankhya*, *Ser*. A, **30**, 107-114.

[5] Penrose, R. (1955). A generalized inverses for matrices, *Proc. Camb. Phil. Soc.*, **51**, 406-413.

[6] Rao, C. R. (1962). A note on a generalized inverse of a matrix with applications to problems in mathematical statistics, *J. Roy., Stat. Soc.*, B, **24**, 152-158.

[7] Rao, C. R. (1967). Calculus of generalized inverses of matrices, Part I—General theory. *Sankhya*, *Ser*. A, **29**, 317-342.

[8] Rohde, C. A. (1966). Some results on generalized inverses, *SIAM Rev.*, 8, 201-205.

[9] Sibuya, M. (1970). Generalized inverses of mappings, *Research Memorandum* No. 35, *Inst. Statist. Math.*

[10] Tewarson, R. P. (1969). On some representations of generalized inverses, *SIAM Review*, **11**, 272-276.

[11] Zlobec, S. (1970). An explicit form of the Moore-Penrose inverse of an arbitrary complex matrix, *SIAM Rev.*, **12**, 132-134.