ON THE STRONG LAW

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(Received May 15, 1969)

1. Introduction

Let $\{X_n,\ n=1, 2, \cdots\}$ be a sequence of independent random variables with $\mathrm{E}X_n=0$ and $\mathrm{E}X_n^2=\sigma_n^2$. Write $S_n=\sum_{k=1}^{n} X_k$ and let $b_n\uparrow\infty$ be a sequence of real numbers. If $\sum_{k=1}^{\infty} b_k^{-2}\mathrm{E}X_k^2<\infty$ then with probability one (a.s.), $b_n^{-1}S_n\to 0$ (see [2], p. 238). In [3], Teicher shows that the series $\Sigma_1=\sum_{n=1}^{\infty} n^{-2}\sigma_n^2$ may be considered as the first in an infinite heirarchy of series $\sum_k$ of decreasing severity, the convergence of any one of which, in conjunction with certain other conditions, implies the strong law: $n^{-1}S_n\overset{\text{a.s.}}{\to}0$. In this note we generalize the results of Teicher by replacing the coefficients n by arbitrary coefficients b_n.

2. Results

THEOREM 1. *Let $\{X_n\}$ be a sequence of independent random variable with $\mathrm{E}X_n=0$, $\mathrm{E}X_n^2=\sigma_n^2$ and let $b_n\uparrow\infty$ be a sequence of real numbers. If*

(a) $\Sigma_2=\sum_{n=2}^{\infty} b_n^{-4}\sigma_n^2\sum_{i=1}^{n-1}\sigma_i^2<\infty$,

(b) $b_n^{-2}\sum_{j=1}^{n}\sigma_j^2\to 0$ *as* $n\to\infty$,

(c) $\sum_{n=1}^{\infty} \mathrm{P}\{|X_n|>a_n\}<\infty$ *for some sequence $\{a_n\}$ of positive real numbers with* $\sum_{n=1}^{\infty} b_n^{-4}a_n^2\sigma_n^2<\infty$,

then

$$b_n^{-1}S_n\overset{\text{a.s.}}{\to}0\,.$$

PROOF. We follow the methods of Teicher [3] closely and see that

$$(b_n^{-1}S_n)^2=b_n^{-2}\sum_{k=1}^{n} X_k^2+2b_n^{-2}\sum_{j<k} X_jX_k\,. \tag{1}$$

It therefore suffices to show that the two terms on the right-hand side of (1) converge to zero a.s..

Define $Y_n = X_n^2$ for $|X_n| \leqq a_n$ and 0 otherwise and note that

$$\sum_{n=1}^{\infty} \mathrm{P}(Y_n \neq X_n^2) = \sum_{n=1}^{\infty} \mathrm{P}(|X_n| > a_n) < \infty$$

in view of (c). It follows by Equivalence Lemma ([2], p. 233) that $b_n^{-2} \sum_{k=1}^{n} X_k^2$ and $b_n^{-2} \sum_{k=1}^{n} Y_k$ converge a.s. to the same limit. But

$$\mathrm{E}\{Y_n - \mathrm{E}Y_n\}^2 \leqq \mathrm{E}Y_n^2 = \int_{[|x_n| \leq a_n]} x_n^4 dF_n(x) \leqq a_n^2 \sigma_n^2$$

where F_n is the distribution function of X_n and so from (c)

$$\sum_{n=1}^{\infty} \mathrm{E}\left\{\frac{Y_n - \mathrm{E}Y_n}{b_n^2}\right\}^2 \leqq \sum_{n=1}^{\infty} b_n^{-4} a_n^2 \sigma_n^2 < \infty .$$

Therefore the series $\sum_{n=1}^{\infty} b_n^{-2}\{Y_n - \mathrm{E}Y_n\}$ of independent summands with zero means converges a.s. ([2], p. 236). A simple application of Kronecker's Lemma ([2], p. 238) now yields $b_n^{-2} \sum_{i=1}^{n} (Y_i - \mathrm{E}Y_i) \xrightarrow{\text{a.s.}} 0$. But

$$b_n^{-2} \sum_{i=1}^{n} \mathrm{E}Y_i = b_n^{-2} \sum_{i=1}^{n} \int_{[|x_i| \leq a_i]} x_i^2 dF_i(x) \leqq b_n^{-2} \sum_{i=1}^{n} \sigma_i^2 \to 0$$

as $n \to \infty$ by (b) and so $b_n^{-2} \sum_{i=1}^{n} Y_i \xrightarrow{\text{a.s.}} 0$.

Next we look at the term $b_n^{-2} \sum_{j<k} X_j X_k$. Following the notation of [3] we write

$$U_{2,n} = \sum_{j<k} X_j X_k = \sum_{j=2}^{n} X_j S_{j-1}$$

and for $n \geqq 2$

$$W_{2,n} = \sum_{j=2}^{n} b_j^{-2} X_j S_{j-1} .$$

The sequence $\{W_{2,n}\}$ is easily shown to be a martingale with respect to the σ-field generated by $X_1, \cdots, X_n$. Also

$$\mathrm{E}W_{2,n}^2 = \sum_{j=2}^{n} b_j^{-4} \mathrm{E}X_j^2 \mathrm{E}(S_{j-1})^2$$
$$= \sum_{j=2}^{n} b_j^{-4} \sigma_j^2 \sum_{i=1}^{j-1} \sigma_i^2 < \infty$$

in view of (a) and it follows by the Martingale Convergence Theorem ([1], p. 236) that $W_{2,n}$ converges a.s. to some random variable. An application of Kronecker's Lemma immediately yields the required result that $b_n^{-2}\sum_{j=2}^{n} X_j S_{j-1} \xrightarrow{\text{a.s.}} 0$. This completes the proof of Theorem 1.

Remarks 1. Teicher [3] proved the $b_n = n$ case of this theorem as we remarked earlier.

2. If $\Sigma_1 = \sum_{n=1}^{\infty} b_n^{-2}\sigma_n^2 < \infty$ then all the conditions of Theorem 1 are satisfied and we get Kolmogorov's result. In this case choose $a_n = b_n$.

3. The conclusion of Theorem 1 holds if we replace (c) by

$$\text{(c}'\text{)} \quad \sum_{n=1}^{\infty} \mathrm{P}\left\{ |X_n| > \left(\sum_{i=1}^{n-1} \sigma_i^2 \right)^{1/2} \right\} < \infty$$

or

$$\text{(c}''\text{)} \quad \sum_{n=1}^{\infty} b_n^{-4} \mathrm{E} X_n^4 < \infty \, .$$

To see (c′) we choose $a_n = \left(\sum_{i=1}^{n-1} \sigma_i^2 \right)^{1/2}$ and then

$$\sum_{n=1}^{\infty} b_n^{-4} a_n^2 \sigma_n^2 = \sum_{n=1}^{\infty} b_n^{-4} \sigma_n^2 \sum_{i=1}^{n-1} \sigma_i^2 < \infty$$

by (b). To see (c″) we observe that $\sum_{i=1}^{n} b_i^{-2}(X_i^2 - \sigma_i^2)$ is an $\mathcal{L}_2$-bounded martingale and it follows again by Kronecker's Lemma and (b) that $b_n^{-2} \sum_{i=1}^{n} X_i^2 \xrightarrow{\text{a.s.}} 0$.

In an analogous manner one can follow the methods of [3] and obtain the following generalization of his Theorem 2.

THEOREM 2. *Let* $\{X_n,\ n = 1, 2, \cdots\}$ *be a sequence of independent random variables with* $\mathrm{E}X_n = 0$, $\mathrm{E}X_n^2 = \sigma_n^2$. *Let* $b_n \uparrow \infty$ *be a sequence of real numbers. Suppose* (b), (c) *hold and for any integer* $k \geqq 2$

$$\text{(a}'\text{)} \qquad \Sigma_k = \sum_{i_k = k}^{\infty} b_{i_k}^{-2k} \sum_{i_{k-1} = k-1}^{i_k - 1} \sigma_{i_{k-1}}^2 \cdots \sum_{i_1 = 1}^{i_2 - 1} \sigma_{i_1}^2 < \infty$$

holds. Then $b_n^{-1} S_n \xrightarrow{\text{a.s.}} 0$.

We omit the details of this demonstration. We remark however that (a′) becomes less stringent with increasing values of k so that (a′) is a generalization of (a).

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REFERENCES

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[2] Loève, M. (1963). *Probability Theory*, (New York).

[3] Teicher, H. (1968). Some new conditions for the strong law, *Proc. National Acad. Sciences*, **59**, 705-707.