ESTIMATION OF SEVERAL CHARACTERISTICS OF DISTRIBUTIONS OF ORDER STATISTICS

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1. Introduction

Suppose that there is a system consisting of m identical components. If failure of a single component causes the system containing it to fail, the failure distribution of the system may correspond to the distribution of the least order statistic of a sample of size m from the failure distribution of components. On the other hand, if failure of all components in a system causes the system to fail, the failure distribution of the system may correspond to the distribution of the largest order statistic. It may, in such a case, be required to estimate some characteristics of failure distributions of these systems based on a sample drawn from the failure distribution of components.

This problem will be formulated as follows. Let $F(x)$ be a continuous cumulative distribution function, let $X_{m,k}$ be the kth least order statistic of a sample of size m from $F(x)$ (we assume, throughout this paper, that $m \leqq n$) and let $F_{m,k}$ be the cdf of $X_{m,k}$. Estimate some characteristics of $F_{m,k}$ on the basis of a sample of size n from F.

On the other hand, it may be required in some cases including life tests to estimate some characteristics of a population based on censored samples. We considered this problem in [1]. The nonparametric estimates of population means proposed there were, however, not effective enough to use without uneasiness. Therefore we shall, in this paper, consider the problems of estimation of population characteristics based on censored samples and a random sample from the population. For example, let $X_{11}, X_{12}, X_{21}, X_{22}, \cdots, X_{l1}, X_{l2}, X_1, X_2, \cdots, X_n$ be a random sample from a population and X_i^* $(i=1, 2, \cdots, l)$ the minimum of X_{i1} and X_{i2}. Then the estimation is based on X_i^* $(i=1, 2, \cdots, l)$ and X_j $(j=1, 2, \cdots, n)$. In such a case it is necessary to consider the problem formulated above.

In Section 2 unbiased estimates of $\mathrm{E}\, X_{m,k}$, $\mathrm{Var}\, X_{m,k}$ etc., on the basis of a sample from F, are given. The theory of U statistics will play an important role there.

In Section 3 we shall discuss the asymptotic properties of the estimates, and in Section 4 the problem of estimation of the mean of F based on both samples from F and $F_{m,k}$.

2. Estimators

Let $F(x)$ be a continuous cdf with density function $f(x)$, and $F_{m,k}(x)$ the cdf of the kth least order statistic of a sample of size m from $F(x)$. Consider the problem of estimating a parameter $\theta(F_{m,k})$ which is a characteristic of $F_{m,k}$, based on a sample $(X_1, X_2, \cdots, X_n)$ of size n from $F(x)$.

Denote by Ω a class of absolutely continuous distributions and by $\Omega_{m,k}$ the class of all $F_{m,k}$ each of which corresponds to a cdf F of Ω. Now, we assume that $\theta(F_{m,k})$ is estimable of degree h with respect to $\Omega_{m,k}$ (cf. Fraser [2], p. 136); that is, there exists a statistic $t(y_1, y_2, \cdots, y_h)$ such that

$$\begin{aligned} \mathrm{E}_{F_{m,k}}\{t(Y_1, Y_2, \cdots, Y_h)\} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t(y_1, \cdots, y_h) \prod_{j=1}^{h} dF_{m,k}(y_j) \\ &= \theta(F_{m,k}) \end{aligned}$$

for all $F_{m,k} \in \Omega_{m,k}$. Note that $(Y_1, Y_2, \cdots, Y_h)$ is a sample of size h from $F_{m,k}$. By $F_{m,k}$ we also represent the mapping which transforms F into $F_{m,k}$. Then the parameter $\theta(F_{m,k})$ can be considered as defined on Ω. Denote by $\theta_{m,k}(F)$ the parameter $\theta(F_{m,k})$ when we consider it as one defined on Ω. We define a function $w_{m,k}(x_1, x_2, \cdots, x_m)$ by

$$w_{m,k}(x_1, x_2, \cdots, x_m) = \text{the } k\text{th least value of } \{x_i; i=1, 2, \cdots, m\}.$$

Since the distribution of $w_{m,k}(X_1, X_2, \cdots, X_m)$ and that of Y_1 are identical, it is easily seen that the parameter $\theta_{m,k}(F)$ is estimable with respect to Ω and has degree less than or equal to mh; that is,

$$E_1 = \{t(w_{m,k}(X_{11}, \cdots, X_{1m}), \cdots, w_{m,k}(X_{h1}, \cdots, X_{hm}))\}$$

for all $F \in \Omega$, where $\{X_{ij}; j=1, \cdots, m, i=1, \cdots, h\}$ is a random sample of size mh from F. We assume that $n \geqq mh$. For a sample $(X_1, \cdots, X_n)$ of size n from F we define a U statistic

$$\begin{aligned} (1) \qquad & U(X_1, X_2, \cdots, X_n) \\ &= \sum_P t(w_{m,k}(X_{i1}, \cdots, X_{im}), \cdots, w_{m,k}(X_{i_{m(h-1)+1}}, \cdots, X_{mh}))/{}_nP_{mh}\,, \end{aligned}$$

where the summation P is over all ${}_nP_{mh}$ permutations $(i_1, \cdots, i_{mh})$ of mh integers chosen from $(1, 2, \cdots, n)$.

Let $X_{(i)}$ $(i=1, 2, \cdots, n)$ be the order statistics of $\{X_i; i=1, \cdots, n\}$.

In the case when $h=1$ the expression of the U statistic given by (1) becomes

$$(2)\qquad U(X_1,\cdots,X_n)=\sum_P t(w_{m,k}(X_{i_1},\cdots,X_{i_m}))/{}_nP_m$$
$$=\sum_C t(w_{m,k}(X_{i_1},\cdots,X_{i_m}))/{}_nC_m$$
$$=\sum_{i=k}^{n-(m-k)}\binom{i-1}{k-1}\binom{n-i}{m-k}t(X_{(i)})/{}_nC_m\,,$$

where the summation C is over all ${}_nC_m$ combinations $(i_1,\cdots,i_m)$ of m integers from $(1,\cdots,n)$. The expression (2) follows from the fact that the number of combinations which satisfy $w_{m,k}(X_{i_1},\cdots,X_{i_m})=X_{(i)}$ is $\binom{i-1}{k-1}\binom{n-i}{m-k}$. In the special cases when $k=1$ and $k=m$, we can further reduce the expression (2) to

$$(3)\qquad U(X_1,\cdots,X_n)=\sum_{i=1}^{n-(m-1)}\binom{n-i}{m-1}t(X_{(i)})\Big/\binom{n}{m}$$

or

$$U(X_1,\cdots,X_n)=\sum_{i=m}^{n}\binom{i-1}{m-1}t(X_{(i)})\Big/\binom{n}{m}\,,$$

respectively.

Example 1. The U statistics based on a sample of size n from F as estimates of the means of $F_{2,1}$ and $F_{2,2}$ are

$$(4)\qquad U(X_1,\cdots,X_n)=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}(n-i)X_{(i)}$$

and

$$(5)\qquad U(X_1,\cdots,X_n)=\frac{2}{n(n-1)}\sum_{i=2}^{n}(i-1)X_{(i)}\,,$$

respectively.

Example 2. The U statistic based on a sample of size n from F as an estimate of the value of $F_{m,k}(x_0)=p$, say, is

$$U(X_1,\cdots,X_n)=\sum_{i=k}^{n-(m-k)}\binom{i-1}{k-1}\binom{n-i}{m-k}\chi_{(-\infty,x_0]}(X_{(i)})\Big/\binom{n}{m}$$
$$=\begin{cases}\displaystyle\sum_{i=k}^{J_2}\binom{i-1}{k-1}\binom{n-i}{m-k}\Big/\binom{n}{m} & \text{if } J_1\geqq k\\ 0 & \text{if } J_1<k\,,\end{cases}$$

where $\chi_{(-\infty,x_0]}$ is the indicator function of the set $(-\infty,x_0]$ and $J_1=\max\{i\mid X_{(i)}\leqq x_0\}$ and $J_2=\min\{n-(m-k),J_1\}$.

Next we consider the case $h=2$. From (1) we have

$$(6)\qquad U(X_1,\cdots,X_n)=\sum_P t(w_{m,k}(X_{i_1},\cdots,X_{i_m}),\ w_{m,k}(X_{i_{m+1}},\cdots,X_{i_{2m}}))\,.$$

The number of permutations which satisfy $w_{m,k}(X_{i_1},\cdots,X_{i_m})=X_{(i)}$ and $w_{m,k}(X_{i_{m+1}},\cdots,X_{i_{2m}})=X_{(j)}$ is given by

$$(7)\qquad \sum_{s=0}^{d}(m!)^2\binom{i-1}{k-1}\binom{n-j}{m-k}\binom{j-1-i}{s}\binom{n-j-(m-k)}{m-k-s}\binom{j-2-(k-1)-s}{k-1}\,,$$

where $d=\min\{j-1-i,\ m-k\}$. Note that in this expression $\binom{a}{b}$ means 0 unless $a\geqq b\geqq 0$. Denote the value of (7) by a_{ij}. Then we have

$$(8)\qquad U(X_1,\cdots,X_n)=\sum_{1\leqq i\leqq j\leqq n}2a_{ij}t(X_{(i)},X_{(j)})/({}_nP_{2m})\,.$$

Example 3. In the cases $k=m$ and $k=1$ we have

$$a_{ij}=(m!)^2\binom{i-1}{m-1}\binom{j-1-m}{m-1}$$

and

$$a_{ij}=(m!)^2\binom{n-j}{m-1}\binom{n-m-i}{m-1}\,,$$

respectively, and therefore

$$U(X_1,\cdots,X_n)=\frac{2(m!)^2}{{}_nP_{2m}}\sum_{m\leqq i<j\leqq n}\binom{i-1}{m-1}\binom{j-1-m}{m-1}t(X_{(i)},X_{(j)})$$

and

$$U(X_1,\cdots,X_n)=\frac{2(m!)^2}{{}_nP_{2m}}\sum_{1\leqq i<j\leqq n-(m-1)}\binom{n-j}{m-1}\binom{n-m-i}{m-1}t(X_{(i)},X_{(j)})\,,$$

respectively. For example, the U statistic corresponding to an estimate of the variance of $F_{2,2}$ is

$$U(X_1,\cdots,X_n)=\frac{8}{n(n-1)(n-2)}\sum_{2\leqq i<j\leqq n}(i-1)(j-3)\frac{(X_{(i)}-X_{(j)})^2}{2}\,.$$

The variances of the U statistics given by (2) may be calculated by using the formula (5.6) in Chapter 6 of [2] or by

$$\operatorname{Var}U=\sum_{i=k}^{n-(m-k)}\sum_{j=k}^{n-(m-k)}\binom{i-1}{k-1}\binom{n-i}{m-k}\binom{j-1}{k-1}\binom{n-j}{m-k}\operatorname{Cov}(t(X_{(i)}),t(X_{(j)}))/({}_nC_m)^2\,.$$

Example 4. Let $m=k=2$ and $f(x)=\chi_{[0,1]}(x)$. In this case it is known that

$$\operatorname{Cov}(X_{(i)},X_{(j)})=\frac{i(n-j+1)}{(n+1)^2(n+2)}\qquad\text{for } i\leqq j\,.$$

Thus the variance of the U statistic in Example 1 is given by

$$\operatorname{Var} U=\left\{\frac{2}{n(n-1)}\right\}^2\left\{\sum_{i<j} 2(i-1)(j-1)\frac{i(n-j+1)}{(n+1)^2(n+2)}\right.$$
$$\left.+\sum_i (i-1)^2\frac{2i(n-i+1)}{(n+1)^2(n+2)}\right\}$$
$$=\frac{4n-3}{45n(n-1)}\,.$$

Denote the variance of $F_{m,k}$ by $\sigma^2_{m,k}$. Since $\sigma^2_{2,2}=1/18$, the variance of the sample mean based on a sample of size n from $F_{2,2}$ is $1/18n$. Therefore, the asymptotic efficiency of the estimate based on a sample from F to the sample mean based on a sample from $F_{2,2}$ is $45/72=0.625$.

3. Asymptotic equivalent estimates and asymptotic variances

The asymptotic variances of the U statistics defined by (2) may be calculated by using the method described in Section 5 of Chapter 6 of [2]; the limiting distribution of $\sqrt{n}(U-\theta(F_{m,k}))$ is normal with mean 0 and variance $m^2\zeta_1$, where $\zeta_1=\operatorname{Cov}\{t(w_{m,k}(X_1,\cdots,X_m)),\ t(w_{m,k}(X_1,X_{m+1},\cdots,X_{2m-1}))\}$.

It is easily seen that

$$\mathrm{P}\{w_{m,k}(x_1,X_2,\cdots,X_m)\leqq x\}=\begin{cases} F_{m-1,k}(x) & \text{if } x<x_1\\ F_{m-1,k-1}(x) & \text{if } x\geqq x_1\,,\end{cases}$$

where $F_{m-1,m}(x)\equiv 0$ and $F_{m-1,0}(x)\equiv 1$. Therefore the random variable $w_{m,k}(x_1,X_2,X_3,\cdots,X_m)$ has the density $f_{m-1,k}(x)$ for $x<x_1$ and $f_{m-1,k-1}(x)$ for $x_1>x$ and the positive probability $\binom{m-1}{k-1}F^{k-1}(x_1)(1-F(x_1))^{m-k}$ at $x=x_1$. Let $g_1^*(x_1)=\mathrm{E}\{w_{m,k}(t(x_1,X_2,\cdots,X_m))\}$. We have

$$g_1^*(x_1)=\int_{-\infty}^{x_1} f_{m-1,k}(x)t(x)dx+\int_{x_1}^{\infty} f_{m-1,k-1}(x)t(x)dx$$
$$+t(x_1)\binom{m-1}{k-1}F^{k-1}(x_1)(1-F(x_1))^{m-k}\,.$$

We define a function $J(u)$ by

$$J(u)=m\binom{m-1}{k-1}u^{k-1}(1-u)^{m-k}\,. \tag{9}$$

By integration by parts we have

$$g_1^*(X_1)-\mathrm{E}[g^*(X_1)]=\left\{\int_{-\infty}^{X_1} J(F(x))t'(x)F(x)dx\right.$$
$$\left.-\int_{X_1}^{\infty} J(F(x))t'(x)(1-F(x))dx\right\}\Big/m\,.$$

From this we obtain

(10) $m^2\zeta_1 = \mathrm{Var}\{g_1^*(X_1)\}$

$$=2\iint_{x<y} J(F(x))J(F(y))F(x)(1-F(y))t'(x)t'(y)dxdy=\tau^2\,, \qquad \text{say.}$$

This expression can also be obtained in another way. Rewrite the U statistic (2) as

$$U=\frac{1}{n}\sum_{i=k}^{n-(m-k)} m\binom{m-1}{k-1}\frac{(i-1)^{[k-1]}(n-i)^{[m-k]}}{(n-1)^{[m-1]}}t(X_{(i)})\,, \tag{11}$$

where $a^{[b]}$ means $a(a-1)(a-2)\cdots(a-(b-1))$, and define another statistic $\tilde{U}$ by

$$\tilde{U}=\frac{1}{n}\sum_{i=1}^{n} m\binom{m-1}{k-1}\left(\frac{i}{n}\right)^{k-1}\left(1-\frac{i}{n}\right)^{m-k}t(X_{(i)})\,. \tag{12}$$

Then it seems that these two statistics are asymptotically equivalent in some sense. In fact, from (11) and (12) we have

$$U-\tilde{U}=\sum_{l=1}^{m-1}\sum_{h=1}^{k-1} d_{l,h}U_{l,h}\,,$$

where $d_{l,h}=O(1/n)$ as $n\to\infty$ and

$$U_{l,h}=\frac{1}{n}\sum_{i=h}^{n-(l-h)} l\binom{l-1}{h-1}\frac{(i-1)^{[h-1]}(n-i)^{[l-h]}}{(n-i)^{[l-1]}}t(X_{(i)})\,.$$

Therefore,

$$\mathrm{E}\{\sqrt{n}(U-\tilde{U})\}\to 0 \tag{13}$$

and

$$\mathrm{Var}\{\sqrt{n}(U-\tilde{U})\}\to 0 \qquad \text{as } n\to\infty\,. \tag{14}$$

Since

$$\sqrt{n}\,\tilde{U}=\sqrt{n}\,U+\sqrt{n}(\tilde{U}-U)$$

and $\sqrt{n}(U-\theta(F_{m,k}))$ is asymptotically normal with mean 0 and variance $m^2\zeta_1$, $\sqrt{n}(\tilde{U}-\theta(F_{m,k}))$ is also asymptotically normal with mean 0 and variance $m^2\zeta_1$. Using the function $J(u)$ defined by (9) $\tilde{U}$ is expressed as

$$\tilde{U}=\frac{1}{n}\sum_{i=1}^{n} J\left(\frac{i}{n}\right)t(X_{(i)})\,. \tag{15}$$

Statistics with this form are treated in many papers (for example, see [3], [4], [5]). When $t(x)=x$, the asymptotic variance (10) of $\sqrt{n}\,U$ coincides with that of $\sqrt{n}\,\tilde{U}$ given by (1.2) of [3].

Example 5. Let $m=k=2$, $t(x)=x$ and $f(x)=e^{-x}\chi_{(0,\infty)}(x)$. Then we have

$$J(u)=2u\ .$$

Therefore

$$\tau^2=8\iint_{0<x<y}(1-e^{-x})^2(1-e^{-y})e^{-y}dxdy=\frac{7}{3}\ .$$

In this case $\sigma^2_{2,2}=5/4$. Similarly for the case $k=1$ we have

$$J(u)=2(1-u)\ ,$$

$$\tau^2=8\iint_{0<x<y}e^{-x}e^{-2y}(1-e^{-x})dxdy=\frac{1}{3}$$

and

$$\sigma^2_{2,1}=1/4\ .$$

4. Estimation of population means

We now consider the problem of estimating the mean μ of F based on a sample of size l from F and a sample of size h from $F_{m,k}$.

Let $X_1,\cdots,X_l$ be independent random variables having the same distribution F, and $Y_1,\cdots,Y_h$ independent random variables having the same distribution $F_{m,k}$. We assume that $(X_1,\cdots,X_l)$ and $(Y_1,\cdots,Y_h)$ are independent. Define

$$\bar{X}=\sum_{i=1}^{l}X_i/l\ ,$$

$$\bar{Y}=\sum_{j=1}^{h}Y_j/h$$

and

$$\bar{X}^*=m\bar{X}-U_{m,k}\ ,$$

where $U_{m,k}$ is the U statistic based on $X_1,\cdots,X_l$ as an estimate of the mean of $F_{m,k}$. Denote the mean of $F_{m,k}$ by $\mu_{m,k}$. Since $\mathrm{E}(\bar{X})=\mu$, $\mathrm{E}(\bar{Y})=\mu_{m,k}$, $\mathrm{E}(\bar{X}^*)=m\mu-\mu_{m,k}=\sum_{\substack{r=1\\ r\neq h}}^{m}\mu_{m,r}$, and $\mu=\sum_{r=1}^{m}\mu_{m,r}/m$, the statistic

$$\tilde{\mu}_p=p\bar{X}+(1-p)\left(\frac{\bar{Y}+\bar{X}^*}{m}\right)$$

is an unbiased estimate of μ for each p, $0\leqq p\leqq 1$. We can write $\tilde{\mu}_p$ as

$$\tilde{\mu}_p = p\bar{X} + (1-p)\frac{\bar{Y} + m\bar{X} - U_{m,k}}{m} \tag{16}$$

$$= \bar{X} + (1-p)\frac{\bar{Y} - U_{m,k}}{m} .$$

We now consider the variance of $\tilde{\mu}_p$. Noting that $\bar{Y}$ is independent of $\bar{X}$ and $U_{m,k}$, we have from (16)

$$\mathrm{Var}\,\tilde{\mu}_p = \mathrm{Var}\,\bar{X} + \left(\frac{1-p}{m}\right)^2 (\mathrm{Var}\,\bar{Y} + \mathrm{Var}\,U_{m,k})$$

$$-\frac{2(1-p)}{m}\,\mathrm{Cov}\,(\bar{X}, U_{m,k}) .$$

Since

$$\frac{1}{m}\sum_{j=1}^{m} U_{m,j} = \frac{1}{m}\sum_{j=1}^{m}\left\{\frac{1}{\binom{l}{m}}\sum_{c} w_{m,j}(X_{i_1}, \cdots, X_{i_m})\right\}$$

$$= \frac{1}{m\binom{l}{m}}\sum_{c}(X_{i_1} + \cdots + X_{i_m})$$

$$= \frac{1}{m\binom{l}{m}}\sum_{j=1}^{l}\binom{l-1}{m-1}X_j$$

$$= \frac{1}{l}\sum_{i=1}^{l} X_i$$

$$= \bar{X} ,$$

we have

$$\mathrm{Var}\,\tilde{\mu}_p = \mathrm{Var}\,\bar{X} + \left(\frac{1-p}{m}\right)^2 \mathrm{Var}\,\bar{Y} - \frac{2(1-p)}{m^2}\sum_{\substack{j=1\\ j\neq k}}^{m}\mathrm{Cov}\,(U_{m,j}, U_{m,k}) \tag{17}$$

$$-\frac{1-p^2}{m^2}\,\mathrm{Var}\,U_{m,k} .$$

Since each $U_{m,j}$ is a linear combination of the order statistics $X_{(i)}$ $(i=1, 2, \cdots, l)$ with nonnegative coefficients and $\mathrm{Cov}\,(X_{(i)}, X_{(j)}) \geqq 0$, we have

$$\mathrm{Cov}\,(U_{m,a}, U_{m,b}) \geqq 0 .$$

Therefore, if

$$(1-p)\,\mathrm{Var}\,\bar{Y} \leqq (1+p)\,\mathrm{Var}\,U_{m,k} \tag{18}$$

then we have

$$\mathrm{Var}\,\tilde{\mu}_p \leqq \mathrm{Var}\,\bar{X} .$$

The value of p which minimizes $\operatorname{Var} \tilde{\mu}_p$ is given by

$$\operatorname{Max}\left\{0, \frac{\operatorname{Var} Y - \sum_{j \neq k} \operatorname{Cov}(U_{m,j}, U_{m,k})}{\operatorname{Var} Y + \operatorname{Var} U_{m,k}}\right\}. \tag{19}$$

If $\operatorname{Var} \bar{Y}$ is negligible in comparison with $\operatorname{Var} U_{m,k}$, then $\operatorname{Var} \tilde{\mu}_p$ will be minimized when $p=0$. In this case (17) becomes

$$\operatorname{Var} \tilde{\mu}_0 = \operatorname{Var} \bar{X} - \frac{1}{m^2}\left\{\operatorname{Var} U_{m,k} + 2 \sum_{j \neq k} \operatorname{Cov}(U_{m,j}, U_{m,k}) - \operatorname{Var} \bar{Y}\right\}. \tag{20}$$

Example 6. Let $m=2$. Then (17) and (20) become

$$\operatorname{Var} \tilde{\mu}_p = \operatorname{Var} \bar{X} + \frac{1-p}{4}\{(1-p) \operatorname{Var} \bar{Y} - 2 \operatorname{Cov}(U_{2,1}, U_{2,2}) - (1+p) \operatorname{Var} U_{2,k}\} \tag{21}$$

and

$$\operatorname{Var} \tilde{\mu}_0 = \operatorname{Var} \bar{X} + \frac{1}{4}\{\operatorname{Var} Y - 2 \operatorname{Cov}(U_{2,1}, U_{2,2}) - \operatorname{Var} U_{2,k}\}, \tag{22}$$

respectively. Further we can write

$$\operatorname{Var} \tilde{\mu}_0 = \frac{1}{4}(\operatorname{Var} \bar{Y} + \operatorname{Var} U_{2,j}), \tag{23}$$

where $j \neq k$.

(i) Let $f(x) = \chi_{(0,1)}(x)$. By Example 4 we have

$$\operatorname{Var} \tilde{\mu}_0 = \frac{1}{4}\left(\frac{1}{18h} + \frac{4l-3}{45l(l-1)}\right)$$

and

$$\operatorname{Var} \bar{X} = \frac{1}{12l}.$$

If l and h are large then we have approximately

$$\frac{\operatorname{Var} \bar{X}}{\operatorname{Var} \tilde{\mu}_0} \doteqdot \frac{30}{5\left(\frac{l}{h}\right)+8}.$$

Thus

$$\lim_{h \to \infty} \frac{\operatorname{Var} \bar{X}}{\operatorname{Var} \tilde{\mu}_0} = \frac{30}{8} = 3.75.$$

(ii) Let $f(x) = e^{-x}\chi_{(0,\infty)}(x)$. Further assume that l is large. Then we have from Example 5 for the case $k=1$

$$\mathrm{Var}\,\tilde{\mu}_0=\frac{1}{4}\left(\frac{1}{4h}+\frac{7}{3l}\right).$$

Thus

$$\frac{\mathrm{Var}\,\bar{X}}{\mathrm{Var}\,\tilde{\mu}_0}\doteqdot\frac{48}{3\left(\frac{l}{h}\right)+28}$$

and

$$\lim_{h\to\infty}\frac{\mathrm{Var}\,\bar{X}}{\mathrm{Var}\,\tilde{\mu}_0}=\frac{12}{7}=1.7143\,.$$

For the case $k=2$ we have similarly

$$\mathrm{Var}\,\tilde{\mu}_0=\frac{1}{4}\left(\frac{5}{4h}+\frac{1}{3l}\right).$$

Thus

$$\frac{\mathrm{Var}\,\bar{X}}{\mathrm{Var}\,\tilde{\mu}_0}\doteqdot\frac{48}{15\left(\frac{l}{h}\right)+4}$$

and

$$\lim_{h\to\infty}\frac{\mathrm{Var}\,\bar{X}}{\mathrm{Var}\,\tilde{\mu}_0}=12\,.$$

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REFERENCES

[1] Takahasi, K. (1970). Some nonparametric consistent estimates from censored samples, *Ann. Inst. Statist. Math.*, 22, 413-419.

[2] Fraser, D. A. S. (1957). *Nonparametric Methods in Statistics*, John Wiley, New York.

[3] Moore, D. S. (1968). An elementary proof of asymptotic normality of linear functions of order statistics, *Ann. Math. Statist.*, **39**, 263-265.

[4] Chernoff, H., Gastwirth, J. L. and Johns, M. V. (1967). Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation, *Ann. Math. Statist.*, **38**, 52-72.

[5] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics, *Ann. Math. Statist.* **29**, 972-994.