THE VARIANCE FUNCTION OF THE ERLANG PROCESS*

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Summary

A relatively simple exact expression of closed form is obtained for the variance $\sigma^2(t)$ of the asynchronous counting distribution for a counting period of length t, $t>0$, in an Erlang process. Useful bounds are placed upon the error of the linear approximation to $\sigma^2(t)$. Implications of these results are examined. In particular, a new exact expression and related bounds are obtained for the mean function of the synchronous counts (also known as the renewal function of the process). All bounds given are sharp in asymptotic order of magnitude as the length of the counting period is allowed to increase.

1. Introduction and preliminaries

Consider a series of events in which the gaps between events are independently distributed with a gamma density

$$f(x)=\theta^m e^{-\theta x}x^{m-1}/(m-1)! \qquad 0<x<\infty , \tag{1.1}$$

where θ is a positive constant and m a positive integer. If $m=1$, this is a Poisson process. However, we shall assume $m>1$ and refer to such a process as *Erlang of order m*.

Erlang processes arise for consideration in traffic flow theory (see Haight [3], [4] and Whittlesey and Haight [11]), in queueing theory and operations research (see Morse [8] and Jewell [5]), and in reliability theory (see Mercer [7] and Cox and Lewis [1]).

Following Whittlesey and Haight [11], we mean by a counting distribution for the process a probability distribution which gives the probability of occurrence of n events in a given counting period $n=0, 1, \cdots$. We may define such a distribution for each choice of length $t>0$ for the

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counting period. Two types of counting distribution are distinguished, the *synchronous* and the *asynchronous*, depending upon whether the counting period begins just after an event, or at an arbitrary point in time. In the synchronous case, with regard to the Erlang process, the probability of 0 events in an interval of length t is the sum of the first m terms of a Poisson (θt) distribution, the probability of 1 event is the sum of the next m terms, etc. In the asynchronous case, the probability $v_n(t)$ of n events in a counting period of length t is ([8], [11]) given by

$$v_0(t)=\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)e^{-\theta t}(\theta t)^k/k! \tag{1.2a}$$

and, for $n\geqq 1$,

$$v_n(t)=\sum_{k=-m+1}^{m-1}\left(1-\frac{|k|}{m}\right)e^{-\theta t}(\theta t)^{nm+k}/(nm+k)!\ . \tag{1.2b}$$

Discussion in Whittlesey and Haight [11] indicates that, especially for applications such as traffic flow theory, it is the asynchronous case which is more apropos to the count data typically collected in actual observation of a series of events (vehicle arrivals).

The theoretical analysis of the Erlang process has been carried out by Nabeya [9], Goodman [2], Morse [8], Haight [3], Jewell [5] and Whittlesey and Haight [11]. (Further discussion is available also in Haight [4] and Cox and Lewis [1]). The present paper continues this line of investigation, especially the work in [3] and [11], by resolving some problems concerning the *variance function* of the asynchronous counting distributions:

$$\sigma^2(t)=\sum_{n=1}^{\infty}n^2v_n(t)-\left(\frac{\theta t}{m}\right)^2, \qquad t>0\,, \tag{1.3}$$

namely the variance of the random count in an arbitrary interval of length t, for each fixed $t>0$. In (1.3) use is made of the fact that the mean of such a count is simply $\theta t/m$.

For the case $m=2$ we have the simple formula

$$\sigma^2(t)=\frac{\theta t}{4}+\frac{1}{8}-\frac{1}{8}e^{-2\theta t}\,. \tag{1.4}$$

However, for an order $m>2$ there has not been available any such simple formula for $\sigma^2(t)$. Numerical approximations have been determined by Whittlesey and Haight [11] using as a starting point the exact formula

$$\sigma^2(t)=\frac{\theta t}{m}-\left(\frac{\theta t}{m}\right)^2+\frac{1}{m}e^{-\theta t}\sum_{j=1}^{\infty}\sum_{k=1}^{m}[j(j-1)m+2jk]\frac{(\theta t)^{jm+k}}{(jm+k)!}\,. \tag{1.5}$$

Here it should be noted that the leading term of (1.5) is not the asymptotically correct linear approximation to $\sigma^2(t)$ as $t\to\infty$, so that the remaining terms should not be construed as an "error of approximation" in any sense. Below we shall obtain a new exact formula for $\sigma^2(t)$ and also obtain useful upper and lower bounds for $\sigma^2(t)$. These results are given in the theorem of Section 2. In Section 3 we prove certain lemmas needed in obtaining the theorem. In particular, a closed-form expression is given for the probability that a Poisson variate takes a value equal to a multiple of m. In Section 4 some implications of the theorem of Section 2 toward the further analysis of the Erlang model are examined. For example, the results obtained for the variance function $\sigma^2(t)$ yield analogous conclusions regarding the mean function of the synchronous counts (otherwise known as the renewal function of the process).

2. The variance function $\sigma^2(t)$

A simple exact formula for $\sigma^2(t)$ is obtained, yielding in addition useful upper and lower bounds for $\sigma^2(t)$. The derivation of these results employs two lemmas which are given in Section 3.

Define

$$A_m(z)=\sum_{k=1}^{m-1}\frac{k}{m}\left(1-\frac{k}{m}\right)\sum_{j=0}^{\infty}e^{-z}z^{mj+k}/(mj+k)! \tag{2.1}$$

and

$$f_m(z)=\sum_{j=0}^{\infty}z^{mj}/(mj)!\,, \tag{2.2}$$

for real-valued $z>0$. It is seen easily that the kth derivative $f_m^{(k)}(z)$ of the function $f_m(z)$ is given by

$$f_m^{(k)}(z)=\sum_{j=0}^{\infty}z^{mj+m-k}/(mj+m-k)! \tag{2.3}$$

for each $k=1,2,\cdots,m$. On the other hand, by Lemma 3.2, the function $f_m(z)$ has the expression

$$f_m(z)=\frac{1}{m}\sum_{j=0}^{m-1}e^{zR^j}\,. \tag{2.4}$$

where $R=\exp(2\pi i/m)$, one of the mth roots of unity. This gives the following alternative to (2.3):

$$f_m^{(k)}(z)=\frac{1}{m}\sum_{j=0}^{m-1}R^{jk}e^{zR^j}\,. \tag{2.5}$$

Combining (2.1), (2.3) and (2.5), and using the formula

$$\sum_{k=1}^{m-1} \frac{k}{m}\left(1-\frac{k}{m}\right)=\frac{m^2-1}{6m}, \tag{2.6}$$

we have

$$A_m(z)=\frac{m^2-1}{6m^2}+\frac{1}{m}e^{-z}B_m(z), \tag{2.7}$$

where

$$B_m(z)=\sum_{j=1}^{m-1} e^{zR^j} \sum_{k=1}^{m-1} \frac{k}{m}\left(1-\frac{k}{m}\right)R^{-jk}. \tag{2.8}$$

We now proceed to reduce $B_m(z)$. Because of the relation

$$R^{-(m-k)j}=R^{jk} \tag{2.9}$$

we have

$$\sum_{k=1}^{m-1} \frac{k}{m}\left(1-\frac{k}{m}\right)R^{-jk}=\sum_{k=1}^{m-1} \frac{k}{m}\left(1-\frac{k}{m}\right) \cos(jkw), \tag{2.10}$$

where $w=2\pi/m$. Further, denoting the right-hand side of (2.10) by C_j, it follows from the relation $C_j=C_{m-j}$ that

$$C_j e^{zR^j}+C_{m-j}e^{zR^{m-j}}=C_j e^{z\cos(jw)}[2\cos(z\sin(jw))].$$

Thus (2.8) becomes reduced to

$$B_m(z)\doteq\sum_{j=1}^{m-1} e^{z\cos(jw)} \cos(z\sin(jw)) \sum_{k=1}^{m-1} \frac{k}{m}\left(1-\frac{k}{m}\right) \cos(jkw), \tag{2.11}$$

an expression purely real-valued in form.

Applying (2.11) in (2.7), we thus have, by contrast with (2.1), an expression for $A_m(z)$ as a finite sum of (real) terms, namely

$$A_m(z)=\frac{m^2-1}{6m^2}+\frac{1}{m}e^{-z}\sum_{j=1}^{m-1} e^{z\cos(jw)}d_j(z), \tag{2.12}$$

where

$$d_j(z)=\cos(z\sin(jw))\sum_{k=1}^{m-1} \frac{k}{m}\left(1-\frac{k}{m}\right) \cos(jkw). \tag{2.13}$$

Moreover, again utilizing (2.6), we have

$$\left|A_m(z)-\frac{m^2-1}{6m^2}\right| \leqq \frac{(m-1)(m^2-1)}{6m^2}e^{-(1-\cos w)z}, \qquad z>0, \tag{2.14}$$

and we note that the right-hand side of (2.14) gives the sharp asymptotic order of magnitude, $O(\exp\{-(1-\cos w)z\})$, for $z\to\infty$.

The foregoing results shall now be applied to the function $\sigma^2(t)$. Lemma 3.1 states that

$$\sigma^2(t)=\frac{\theta t}{m^2}+A_m(\theta t)\,, \tag{2.15}$$

and therefore by (2.12), (2.13) and (2.14) we conclude

THEOREM. *For* $t>0$,

$$\sigma^2(t)=\frac{\theta t}{m^2}+\frac{m^2-1}{6m^2}+E_m(\theta t)\,, \tag{2.16}$$

where, with $w=2\pi/m$,

$$E_m(\theta t)=\frac{1}{m}\sum_{j=1}^{m-1} e^{-(1-\cos jw)\theta t}\cos(\theta t\sin(jw))\sum_{k=1}^{m-1}\frac{k}{m}\left(1-\frac{k}{m}\right)\cos(jkw)\,. \tag{2.17}$$

Further,

$$|E_m(\theta t)|\leq\frac{(m-1)(m^2-1)}{6m^2}e^{-2\{\sin^2(\pi/m)\}\theta t}\,, \tag{2.18}$$

and the bound in (2.18) *is sharp in asymptotic order of magnitude as* $t\to\infty$.

The theorem may be used to obtain an exact expression of closed form for $\sigma^2(t)$, or alternatively it may be used to place simple bounds on $\sigma^2(t)$. In this regard, the following two corollaries are easily deduced.

COROLLARY 1. *Exact expressions for* $\sigma^2(t)$ *in the cases* $m=2, 3$ *and* 4 *are, respectively,*

$$\sigma^2(t)=\frac{1}{4}\theta t+\frac{1}{8}-\frac{1}{8}e^{-2\theta t}\,, \tag{2.19}$$

$$\sigma^2(t)=\frac{1}{9}\theta t+\frac{4}{27}-\frac{4}{27}\cos\left(\frac{\sqrt{3}}{2}\theta t\right)e^{-(3/2)\theta t}\,, \tag{2.20}$$

$$\sigma^2(t)=\frac{1}{16}\theta t+\frac{5}{32}-\frac{1}{8}\cos(\theta t)e^{-\theta t}-\frac{1}{32}e^{-2\theta t}\,. \tag{2.21}$$

Similar results for higher values of m may be derived routinely by the use of (2.16) and (2.17). However, observing that the nonlinear part of $\sigma^2(t)$ tends to zero exponentially fast as $t\to\infty$, a convenient and practical simplification is to replace the complicated nonlinear terms by a simple bound appropriately tending to zero at the same exponential rate as its exact counterpart. Thus

COROLLARY 2. *Upper and lower bounds for $\sigma^2(t)$ are given by*

$$\sigma^2(t)=\frac{\theta t}{m^2}+\frac{m^2-1}{6m^2}\pm\frac{(m-1)(m^2-1)}{6m^2}e^{-2\sin^2(\pi/m)\theta t}\,. \tag{2.22}$$

In particular, bounds for the cases $m=2$, 3 and 4 are, respectively,

$$\sigma^2(t)=\frac{1}{4}\theta t+\frac{1}{8}\pm\frac{1}{8}e^{-2\theta t}\,, \tag{2.23}$$

$$\sigma^2(t)=\frac{1}{9}\theta t+\frac{4}{27}\pm\frac{8}{27}e^{-(3/2)\theta t}\,, \tag{2.24}$$

$$\sigma^2(t)=\frac{1}{16}\theta t+\frac{5}{32}\pm\frac{15}{32}e^{-\theta t}\,, \tag{2.25}$$

which may be compared with (2.19), (2.20) *and* (2.21).

For given values of θ and m, formula (2.22) indicates the appropriate linear approximation to $\sigma^2(t)$ and provides a good evaluation of its accuracy. Previously, this linear function has been given by Jewell [5] (his formula (76), with a correction) as an asymptotic expression for $\sigma^2(t)$ and the accuracy has been investigated via electronic computer by Whittlesey and Haight [11].

In contrast to (1.5), which is the expression given by Whittlesey and Haight for numerical calculation of $\sigma^2(t)$, formula (2.16) involves only a finite number of terms and, moreover, the leading terms are asymptotically equivalent to $\sigma^2(t)$, as $t\to\infty$.

The theorem given above provides a useful tool in the theoretical analysis of the Erlang model. Topics in this regard are examined in Section 4.

3. Lemmas

Here we prove two lemmas required in the previous section.

LEMMA 3.1. *For $t>0$,*

$$\sigma^2(t)=\frac{\theta t}{m^2}+A_m(\theta t)\,. \tag{3.1}$$

PROOF. By means of the identity

$$mj\{(j-1)m+2k\}=(mj+k)(mj+k-1)-(m-1)(mj+k)+k(m-k)\,,$$

the third term of the right-hand side of (1.5) is calculated as follows;

$$\frac{1}{m}e^{-\theta t}\sum_{j=1}^{\infty}\sum_{k=1}^{m}j[(j-1)m+2k]\frac{(\theta t)^{mj+k}}{(mj+k)!}$$

$$
\begin{aligned}
&= \frac{1}{m^2} e^{-\theta t}\left[(\theta t)^2 \sum_{j=m-1}^{\infty} \frac{(\theta t)^j}{j!} - (m-1)\theta t \sum_{j=m}^{\infty} \frac{(\theta t)^j}{j!}\right] \\
(3.2) \qquad &+ \sum_{j=1}^{\infty} \sum_{k=1}^{m-1} \frac{k}{m}\left(1-\frac{k}{m}\right) \frac{e^{-\theta t}(\theta t)^{mj+k}}{(mj+k)!} \\
&= \frac{1}{m^2} e^{-\theta t}\left[(\theta t)^2 \sum_{j=m-2}^{\infty} \frac{(\theta t)^j}{j!} - (m-1)\theta t \sum_{j=m-1}^{\infty} \frac{(\theta t)^j}{j!}\right] \\
&+ A_m(\theta t) - \frac{1}{m^2} e^{-\theta t} \sum_{k=1}^{m-1} k(m-k) \frac{(\theta t)^k}{k!} .
\end{aligned}
$$

But as

$$
\sum_{k=1}^{m-1} k(m-k) \frac{(\theta t)^k}{k!} = (m-1)\theta t \sum_{j=0}^{m-2} \frac{(\theta t)^j}{j!} - (\theta t)^2 \sum_{j=0}^{m-3} \frac{(\theta t)^j}{j!} ,
$$

(3.2) is equal to

$$
\frac{1}{m^2}[(\theta t)^2 - (m-1)\theta t] + A_m(\theta t) .
$$

Hence from (1.5) we have (3.1).

The next lemma provides an expression of closed form for the function $f_m(z)$ defined by (2.2). An expression of similar nature thus follows for the quantity

$$
e^{-z} f_m(z) ,
$$

which is the probability of a multiple of m in a Poisson distribution with parameter z. From a conversation with my colleague I. R. Savage, it is clear that this lemma is well known, but a proof is included for completeness as no reference seems available.

LEMMA 3.2. *For* $z>0$,

$$
f_m(z) = \frac{1}{m} \sum_{j=0}^{m-1} e^{zR^j} , \tag{3.3}
$$

where R denotes the mth root of unity $\exp(2\pi i/m)$.

PROOF. It is seen easily that $f_m(z)$ satisfies the linear differential equation of order $m-1$

$$
f_m^{(m-1)}(z) + f_m^{(m-2)}(z) + \cdots + f_m'(z) + f_m(z) = e^z , \tag{3.4}
$$

where $f_m^{(k)}(z)$ denotes the kth derivative of $f_m(z)$. The general solution of the homogeneous equation is ([6], p. 142)

$$
y(z) = \sum_{k=1}^{m-1} c_k e^{zR^k}
$$

and a particular solution of the nonhomogeneous equation is clearly

$$y^*(z)=\frac{1}{m}e^z ,$$

so that the general solution of (3.12) has the form

$$f_m(z)=\frac{1}{m}e^z+\sum_{k=1}^{m-1} c_k e^{zR^k} .$$

From the boundary conditions $f(0)=1$, $f'(0)=0$, $f^{(2)}(0)=0,\cdots$, the coefficients c_k are found to equal $1/m$ for $k=1,2,\cdots$. Hence $f_m(z)$ has the expression (3.3).

4. Further analysis of the Erlang model

The theorem in Section 2 has interesting implications for the theoretical study of the Erlang process. Several such aspects will now be considered.

(i) *The mean function of the synchronous counting distribution*

Denote the mean functions of the synchronous and asynchronous counting distributions, respectively, by $\alpha(t)$ and $\beta(t)$. While $\beta(t)$ is simply $\theta t/m$, a simple exact formula for $\alpha(t)$ has evaded discovery (see [3]). Indeed, the problem is equivalent to that of giving a simple expression for $\sigma^2(t)$, in view of the relation

$$\alpha(t)=\frac{m}{2\theta}\frac{d}{dt}\sigma^2(t)+\frac{\theta t}{m}-\frac{1}{2} , \tag{4.1}$$

which follows from formula (9) of [11]. Therefore, by the theorem of the present paper, we may conclude a new exact expression for $\alpha(t)$ and bounds on the error of the corresponding linear approximation to $\alpha(t)$. This result is

COROLLARY 3. *For $t>0$,*

$$\alpha(t)=\frac{\theta t}{m}-\frac{m-1}{2m}+F_m(\theta t) \tag{4.2}$$

where, with $w=2\pi/m$,

$$F_m(\theta t)=\sum_{j=1}^{m-1} e^{-(1-\cos jw)\theta t}\sin\left(z\sin(jw)-\frac{1}{2}jw\right) \cdot\sum_{k=1}^{m-1}\frac{k}{m}\left(1-\frac{k}{m}\right)\sin\left(\frac{1}{2}jw\right)\cos(jkw) . \tag{4.3}$$

Further,

$$(4.4)\qquad |F_m(\theta t)| \leqq \frac{(m-1)(m^2-1)}{6m} e^{-2\{\sin^2(\pi/m)\}\theta t}$$

and the bound in (4.4) *is sharp in asymptotic order of magnitude as* $t \to \infty$.

The proof is routine and will be omitted. The result that $\alpha(t)$ has the asymptotic linear approximation $\theta t/m-(m-1)/2m$ as $t\to\infty$ is not new, following by a standard result for a renewal process $\{X_i\}$ having $EX_1^2<\infty$ (see Smith [10], p. 248). Nor is it novel to have an exact expression for $\alpha(t)$, as there is a complicated power series given by Haight [3]. However, in the present contribution we achieve a relatively simple exact expression for $\alpha(t)$ and useful bounds on the error of the asymptotic approximation as $t\to\infty$.

(ii) *A relationship between the parameters* m *and* θ

Suppose that t is fixed, say at the value $t=t_0$, and suppose that the mean $\alpha(t_0)$ has a certain fixed value α_0. Under the latter restriction, the parameters m and θ become functionally dependent. Let θ_m denote the value of θ_m which is thus associated with the integer m (for $m=2, 3, \cdots$). Investigating this relationship, Haight [3] has verified empirically that θ_m satisfies approximately the linear relationship given by

$$(4.5)\qquad \theta_m t_0 \doteq \alpha_0 m + \frac{1}{2}(m-1),$$

but also shows that the exact relationship cannot be linear in form. Making use of Corollary 3, we corroborate these findings and evaluate explicitly the accuracy of (4.5). From (4.2) and (4.4) it follows that

$$(4.6)\qquad \theta_m t_0 = \alpha_0 m + \frac{1}{2}(m-1) - mF_m(\theta_m t_0)$$

with

$$(4.7)\qquad |mF_m(\theta_m t_0)| \leqq \frac{1}{6}(m-1)(m^2-1)e^{-2\sin^2(\pi/m)\theta_m t_0}.$$

In view of the sharpness property of (4.4), the approximation (4.5) is valid only when the right-hand side of (4.7) is sufficiently small. Since this quantity becomes small as t_0 increases but increases as $m\to\infty$ for fixed t_0, we conclude that for any fixed t_0 the relation (4.5) is valid for a limited set of values of m but not for m indefinitely large.

(iii) *Correlation between the counts in two intervals*

Let $r(t, d)$ denote the correlation between the counts within two intervals of length t separated by an interval of length d, reference here being to the asynchronous process. It is easily verified that

$$r(t, d)=[\sigma^2(2t+d)-2\sigma^2(t+d)+\sigma^2(d)]/2\sigma^2(t) .$$

At this point the use of the linear approximation to $\sigma^2(\cdot)$ would yield the banal approximation $r(t, d) \doteq 0$. Hence, in order to obtain a useful expression for $r(t, d)$, it is necessary to utilize the exact result (2.16). We obtain

$$r(t, d)=[E_m(2\theta t+\theta d)-2E_m(\theta t+\theta d)+E_m(\theta d)]/2\sigma^2(t)$$

and it then follows by (2.18) that

$$|r(t, d)| \leq K(t) e^{-2\sin^2(\pi/m)\theta d} , \tag{4.8}$$

where $K(t)$ is a constant not depending upon d. Thus the correlation $r(t, d) \to 0$ at an exponential rate as the distance d increases between the two intervals.

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