ON THE DOMAIN OF PARTIAL ATTRACTION OF SEMI-STABLE DISTRIBUTIONS

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A characteristic function $\varphi(t)$ (or the corresponding distribution F) is said to be semi-stable, if for some constants $\gamma>1$ and $c>1$,

$$(1) \qquad \varphi(t)=\varphi^{\gamma}(c^{-1}t)\,, \qquad \text{for all } t$$

holds. Every semi-stable characteristic function is infinitely divisible. A complex valued function $\varphi(t)$ defined on the real line is a non-normal semi-stable characteristic function which satisfies (1) if and only if it admits a representation

$$(2) \qquad \log\varphi(t)=i\beta t+\int_0^{\infty}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x) +\int_{-\infty}^{0}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x)\,,$$

where

i) β is a suitably chosen constant,
ii) $\alpha=\log\gamma/\log c$, $0<\alpha<2$,
iii) $M(x)$ and $N(x)$ are monotone non-decreasing and are expressed as $M(x)=-\lambda(\log x)x^{-\alpha}$, $N(-x)=\mu(\log x)x^{-\alpha}$, $\lambda(t), \mu(t)\in \boldsymbol{P}^{+}(\log c)$ (=set of all periodic functions with the period $\log c$) ([5], [9], [10]).

$\varphi(t)$ is a stable characteristic function if and only if $\lambda(t)=\lambda$ and $\mu(t)=\mu$ are constants (and $\lambda=\mu$ when $\alpha=1$). A distribution G is said to belong to the domain of attraction (or domain of partial attraction) of F, if for some $\{B_n\}$ and $\{A_n\}$

$$(3) \qquad G^{*n}(B_n x+A_n)$$

converges (or contains a subsequence which converges) weakly to F, where asterisk denotes the convolution operator.

Every semi-stable (or stable) distribution belongs to the domain of partial attraction (or the domain of attraction) of itself. It is known that a distribution G belongs to the domain of partial attraction of the semi-stable distribution F corresponding to (2) if and only if for some $\{C_n\}$, $C_n>0$, and for an increasing sequence $I(n)$ of positive integers,

$$\lim_{n\to\infty} I(n)(1-G(C_nx))=-M(x) \qquad x>0\,, \tag{4}$$

$$\lim_{n\to\infty} I(n)G(-C_nx)=N(-x) \qquad x>0 \tag{5}$$

$$\lim_{\varepsilon\to 0}\overline{\lim_{n\to\infty}}\, I(n)\left\{\int_{|x|<\varepsilon} x^2dG(C_nx)-\left(\int_{|x|<\varepsilon} xdG(C_nx)\right)^2\right\}=0 \tag{6}$$

hold (see Theorem 4, Section 25, [3]) and that, when the attraction is "complete", i.e., when $I(n)=n$, (4)–(6) are equivalent to

$$\lim_{y\to\infty}\frac{1-G(yx)}{G_0(y)}=\xi x^{-\alpha} \qquad \xi\geqq 0 \tag{7}$$

and

$$\lim_{y\to\infty}\frac{G_0(yx)}{G_0(y)}=x^{-\alpha} \tag{8}$$

where $G_0(x)=1-G(x)+G(-x)$, $x>0$. This occurs only when $\lambda(t)$ and $\mu(t)$ are constants, i.e., only when $\varphi(t)$ is stable (see Sections 33–35, [3]).

Hereafter we consider "partial" convergences of the type

$$\lim_{n\to\infty}\frac{1-G(a^nx)}{G_0(a^n)}=R(x)\,, \tag{9}$$

$$\lim_{n\to\infty}\frac{G_0(a^nx)}{G_0(a^n)}=S(x) \qquad a>1\,. \tag{10}$$

Since $1-G(x)$ and $G_0(x)$ are monotone non-increasing the same is true of the limiting functions $R(x)$ and $S(x)$. We do not exclude the possibility that $S(\xi)=\infty$ for some positive ξ (necessarily <1). This is the case if and only if $S(a)=0$. Except for this case we always assume without loss of generality that (i) the limiting functions are right continuous, (ii) the convergences take place at every point of continuity of the limiting functions, and that (iii) the limiting functions are continuous at $x=a$.

We now state the theorem to be proved.

THEOREM 1. *Suppose* (9) *and* (10) *hold, and put* $\alpha=-\log S(a)/\log a$ $(\geqq 0)$. *Then we have the following.*

a) *If* $\alpha<\infty$, *then* $R(x)=\xi(\log x)x^{-\alpha}$, $S(x)=\eta(\log x)x^{-\alpha}$, *where* $\xi(t), \eta(t)\in \boldsymbol{P}^+(\log a)$ *and* $\eta(0)=1$.

b) *G has the moments of positive order* $\beta<\alpha$. *If* $\alpha<\infty$, *any moment* *of order* $\beta>\alpha\geqq 0$ *does not exist. Especially if* $\alpha=\infty$, *then G has all the moments, while if* $\alpha=0$, *G has no moment of positive order at all.*

c) *If* $\alpha>2$, *G belongs to the domain of normal attraction of the normal distribution.*

d) *If* $\alpha=2$, *G belongs to the domain of attraction of the normal distribution.*

e) *If* $0<\alpha<2$, *G belongs to the domain of the semi-stable distribution F corresponding to* (2) *with*

$$M(x)=-\xi(\log x)x^{-\alpha},$$

and

$$N(-x)=(\eta(\log x)-\xi(\log x))x^{-\alpha}$$

or more precisely, if $\{X_n\}$ *is a sequence of independent random variables each having the distribution G, then there exists a sequence* $\{A'_n\}$ *of real numbers such that the distribution of the normalized partial sum* $(X_1+\cdots+X_{I(n)})/a^n-A'_n$, *where* $I(n)=[G(a^n)^{-1}]$, *converges to F.*

f) *If* $\alpha=0$, *G belongs to no domain of partial attraction.*

COROLLARY 1. (*See the remark at the end of the proof.*) *If* $M(x)=-N(-x)=-\xi(\log x)x^{-\alpha}$ *is concave, then the semi-stable distribution corresponding to* (2) *is unimodal.*

COROLLARY 2. *The semi-stable distribution F corresponding to* (2) *has the moments of positive order* β *if and only if* $\beta<\alpha$. ([7], [10], [11])

THEOREM 2. *Suppose that for some increasing sequences* $I(n)$ *of positive integers and* C_n *of positive numbers such that* C_{n+1}/C_n *is bounded (by* $d>1$, *say), convergences*

$$\lim_{n\to\infty} I(n)(1-G(C_nx))=\lambda x^{-\alpha} \tag{12}$$

$$\lim_{n\to\infty} I(n)G(-C_nx)=\mu x^{-\alpha} \tag{13}$$

hold, where $0<\alpha<2$, $\lambda\geqq 0$, $\mu\geqq 0$ *and* $\lambda+\mu>0$. *Then G belongs to the domain of attraction of the stable distribution with* $M(x)=-\lambda x^{-\alpha}$ *and* $N(-x)=\mu x^{-\alpha}$.

THEOREM 3. *Suppose that, in addition to* (9) *and* (10), *another convergences*

$$\lim_{n\to\infty}\frac{1-G(b^nx)}{G_0(b^n)}=U(x) \tag{15}$$

and

(16) $$\lim_{n\to\infty}\frac{G_0(b^n x)}{G_0(b^n)}=V(x)\,,\qquad (b>1)$$

occur. If $\rho\equiv\log a/\log b$ *is an irrational number and if* $0<\alpha<2$, *then* $U(x)=R(x)=\xi x^{-\alpha}$, $V(x)=S(x)=x^{-\alpha}$ *and* G *belongs to the domain of attraction of the stable distribution corresponding to* (2) *with*

(17) $$M(x)=-U(x)\quad \textit{and}\quad N(-x)=V(x)-U(x)\,.$$

Remark 1. Convergences (9)–(10) make sense only when $G_0(x)>0$ for all $x>0$. If $G_0(x_0)=0$ for some $x_0<\infty$, the distribution G is concentrated in the finite interval $[-x_0, x_0]$ and G belongs to the domain of normal attraction of the normal distribution.

Remark 2. The tail $G_0(x)=1-G(x)+G(-x)$ is monotone non-increasing and approaches to zero as $x\to\infty$. This fact is used invariably in the following.

Remark 3. The boundedness of the ratio C_{n+1}/C_n is equivalent to that of the ratio $I(n+1)/I(n)$.

Remark 4. Theorems 2 and 3 state that positive function $G_0(x)$ varies regularly if for an increasing (to infinity) sequence $\{C_n\}$ such that C_{n+1}/C_n is bounded, $\lim_{n\to\infty} G_0(C_n x)/G_0(C_n)=x^{-\alpha}$, or if for some $a>1$ and $b>1$ such that $\log a/\log b$ is irrational, $\lim_{n\to\infty} G_0(a^n x)/G_0(a^n)$ and $\lim_{n\to\infty} G_0(b^n x)/G_0(b^n)$ exist.

PROOF OF THEOREM 1.

a) Evident.

b) Suppose first that $\alpha=\infty$ (or $S(a)=0$), and let $\beta\geqq 0$. Let $\varepsilon>0$ be so small and n_0 be so large that $a^\beta\varepsilon<1$ and that for any $n\geqq n_0$, $G_0(a^{n+1})/G_0(a^n)\leqq\varepsilon$.

Putting $x_0=a^{n_0}$, we obtain

(18) $$G_0(a^{p+1}x_0)/G_0(a^p x_0)=\varepsilon_p\,,\qquad 0\leqq\varepsilon_p\leqq\varepsilon\,,\quad p=0, 1, 2,\cdots.$$

Then

$$\int_{a^p x_0\leqq|x|\leqq a^{p+1}x_0}|x|^\beta dG\leqq(ax_0)^\beta a^{\beta p}G_0(a^p x_0)$$

$$=(ax_0)^\beta a^{\beta p}\varepsilon_0\cdots\varepsilon_{p-1}G_0(x_0)$$
$$\leqq(ax_0)^\beta G_0(x_0)(a^\beta\varepsilon)^p\,,\qquad p=0,1,2,\cdots.$$

Hence $E(|X|^\beta)<\infty$. Suppose next that $\alpha<\infty$ (or $S(a)>0$). Let $\varepsilon>0$ be a small number, and let n_0 be so large that

$$G_0(a^{n+1})/G_0(a^n)-S(a)\leqq\varepsilon a^{-\alpha}$$

holds for all $n\geqq n_0$. If $x_0=a^{n_0}$, we have for any nonnegative integer p

$$G_0(a^{p+1}x_0)/G_0(a^px_0)=S(a)+\varepsilon_p a^{-\alpha} =a^{-\alpha}(1+\varepsilon_p)\,, \tag{19}$$

where $|\varepsilon_p|\leqq\varepsilon$, $p=0,1,2,\cdots$. When $0<\beta<\alpha$, let $\varepsilon>0$ be so small that $(1+\varepsilon)/a^{\alpha-\beta}=c<1$.

$$\begin{aligned}(20)\qquad &\sum_{p=1}^{\infty}\int_{a^px_0<|x|\leqq a^{p+1}x_0}|x|^\beta dG(x)\\ &\leqq\sum_{p=1}^{\infty}(a^{p+1}x_0)^\beta G_0(a^px_0)\\ &=(ax_0)^\beta G_0(ax_0)\left(\sum_{p=2}^{\infty}\prod_{j=1}^{p-1}\left(\frac{1+\varepsilon_j}{a^{\alpha-\beta}}\right)+1\right)\\ &\leqq(ax_0)^\beta G_0(ax_0)\sum_{p=0}^{\infty}c^p<\infty\,.\end{aligned}$$

When $\beta>\alpha$, let $\varepsilon>0$ be such that $(1-\varepsilon)a^{\beta-\alpha}\equiv d>1$, and $1-(1+\varepsilon)a^{-\alpha}\equiv\xi>0$. Then,

$$\begin{aligned}(21)\qquad E(|X|^\beta)&\geqq\sum_{p=1}^{\infty}\int_{a^px_0<|x|\leqq a^{p+1}x_0}|x|^\beta dG(x)\\ &\geqq\sum_{p=1}^{\infty}(a^px_0)^\beta(G_0(a^px_0)-G_0(a^{p+1}x_0))\\ &=\sum_{p=1}^{\infty}(a^px_0)^\beta G_0(a^px_0)\left(1-\frac{G_0(a^{p+1}x_0)}{G_0(a^px_0)}\right)\\ &\geqq\xi x_0^\beta G_0(ax_0)\left(\sum_{p=2}^{\infty}\prod_{j=1}^{p-1}((1+\varepsilon_j)a^{\beta-\alpha})+1\right)\\ &\geqq\xi x_0^\beta G_0(ax_0)\sum_{p=0}^{\infty}d^p=\infty\,.\end{aligned}$$

c) A direct consequence of b).

d) Suppose $\alpha=2$. Then (19) holds with $\alpha=2$. If $a^px_0\leqq s<a^{p+1}$, we have

$$\begin{aligned}(22)\qquad \int_{|x|\leqq s}x^2dG(x)&\geqq\int_{|x|\leqq a^px_0}x^2dG(x)\geqq\sum_{k=0}^{p-1}\int_{a^kx_0<|x|\leqq a^{k+1}x_0}x^2dG(x)\\ &\geqq\sum_{k=0}^{p-1}(a^kx_0)^2(G_0(a^kx_0)-G_0(a^{k+1}x_0))\end{aligned}$$

$$=\sum_{k=0}^{p-1}(a^kx_0)^2G_0(a^kx_0)\left(1-\frac{G_0(a^{p+1}x_0)}{G_0(a^px_0)}\right)$$

$$\geqq(1-(1+\varepsilon)a^{-2})\sum_{k=0}^{p-1}(a^kx_0)^2a^{2(p-k)}G_0(a^px_0)\prod_{j=k}^{p}\frac{1}{1+\varepsilon_j}$$

$$\geqq(1-(1+\varepsilon)a^{-2})(a^px_0)^2G_0(a^px_0)\sum_{k=0}^{p-1}\left(\frac{1}{1+\varepsilon}\right)^{p-k+1}$$

$$=(1-(1+\varepsilon)a^{-2})a^{-2}(a^{p+1}x_0)^2G_0(a^px_0)\sum_{k=2}^{p+1}\left(\frac{1}{1+\varepsilon}\right)^{k}$$

$$\geqq(1-(1+\varepsilon)a^{-2})a^{-2}s^2G_0(s)(1-(1+\varepsilon)^{-p})\varepsilon^{-1}(1+\varepsilon)^{-1}.$$

Hence

$$\frac{s^2G_0(s)}{\int_{|x|\leqq s}x^2dG(x)}\leqq\frac{a^4}{a^2-1-\varepsilon}\frac{\varepsilon(1+\varepsilon)}{1-(1+\varepsilon)^{-p}},$$

or

$$0\leqq\overline{\lim_{s\to\infty}}\frac{s^2\int_{|x|>s}dG}{\int_{|x|\leqq s}x^2dG(x)}\leqq\frac{a^4}{a^2-1-\varepsilon}\varepsilon(1+\varepsilon).$$

Since $\varepsilon>0$ is arbitrary,

$$\lim_{s\to\infty}\frac{s^2\int_{|x|>s}dG}{\int_{|x|\leqq s}x^2dG(x)}=0.$$

The desired result follows from Theorem 1 of Section 35, [3].

e) Let $I(n)$ be the integer part of $1/G_0(a^n)$: $I(n)=[1/G_0(a^n)]$, and put $C_n=a^n$. Then by virture of a), (9) and (10) imply (4), (5) and (11). (6) is proved as in the proof of Theorem 2 of Section 35, [3] (pp. 177–178).

f) Suppose finally that $\alpha=0$. Then from a), $R(a^p)=1$, for $p=0, +1, +2,\cdots$.

If $a^p\leqq x<a^{p+1}$, and if $a^n\leqq y<a^{n+1}$, then

$$\frac{G_0(a^na^{p+2})}{G_0(a^n)}\leqq\frac{G_0(yx)}{G_0(y)}\leqq\frac{G_0(a^{n+1}a^{p-1})}{G_0(a^{n+1})}. \tag{23}$$

Both extreme sides of (23) tend to 1 as $n\to\infty$. Hence

$$\lim_{y\to\infty}G_0(yx)/G_0(y)=1. \tag{24}$$

The desired result is obtained from Theorem VII of [1] (see also 7, p. 190, [3]).

PROOF OF COROLLARY 1. Let A be a positive number such that $M(A)>-1/2$. Then the distribution defined by,

$$G(x)=\begin{cases} 1+M(x+A) & x\geqq 0 \\ -M(-x+A) & x<0 \end{cases}$$

is symmetric and unimodal. Put

$$a=\begin{cases} e^c & \text{when } \xi(t)\in \boldsymbol{P}^+(\log c)\,, \\ 2 & \text{when } \xi(t)\equiv\xi \ (=\text{constant})\,. \end{cases}$$

Then we can easily verify that

$$\lim_{n\to\infty}(1-G(a^n x))/G_0(a^n)=-M(x)/2$$

and

$$\lim_{n\to\infty}G_0(a^n x)/G_0(a^n)=-M(x)\,.$$

By Theorem 1 e), G belongs to the domain of partial attraction of F, or more precisely,

$$G^{*I(n)}(a^n x) \qquad (I(n)=[1/G_0(a^n)])$$

converges weakly to F. Since G is symmetric and unimodal, so are $G_n(x)=G^{*I(n)}(a^n x)$, $n=1, 2,\cdots$. F, being the limit of a sequence of unimodal distributions, is itself unimodal. q.e.d.

Remark: The corollary is a special case of the following theorem proved by P. Medgyessy [6].

THEOREM. *Let $\varphi(t)$ be an infinitely divisible ch.f. with the Gaussian component $e^{-\sigma^2t^2/2}$ and the spectral functions $M(x)$ $(x>0)$ and $N(x)$ $(x<0)$. If $M(x)$ is concave and if $M(x)=-N(-x)$, then the corresponding distribution F is symmetric (about some point) and unimodal.*

We shall give below a proof somewhat different from that of P. Medgyessy. It suffices to prove that the distribution $F_0(x)$ corresponding to the ch.f. $\varphi(t)e^{\sigma^2t^2/2}$ is symmetric and unimodal.

Let $\{a_n\}$ be a sequence of positive numbers such that $a_n\to 0$ and $M(a_n)/n\to 0$. Let $\{G_n\}$ be a sequence of symmetric unimodal distributions defined by,

$$G_n(x)=\begin{cases} 1+M(x+a_n)/n & x\geqq 0 \\ -M(-x+a_n)/n & x<0\,. \end{cases}$$

Then we have

(a) $n(1-G_n(x))=nG_n(-x)\to -M(x)$ as $n\to\infty$

(b) $$0\leqq n\left[\int_{|x|<\varepsilon} x^2dG_n(x)-\left(\int_{|x|<\varepsilon} xdG_n(x)\right)^2\right]=2n\int_0^\varepsilon x^2dG_n(x)$$
$$=2\int_0^\varepsilon x^2dM(x+a_n)\leqq 2\int_0^\varepsilon x^2dM(x)\to 0 \quad \text{as } \varepsilon\to 0\,.$$

It follows from Theorem 4, Section 25, [1], that, if $\{X_{n,1},\cdots,X_{n,n}\}$ is an independent system of random variables each having the distribution $G_n(x)$, then the distribution $G_n^{*n}(x)$ of the sum $X_{n,1}+\cdots+X_{n,n}$ converges to $F_0(x)$. Since $F_0(x)$ is the limit of the sequence of distributions obtained by the convolution operation of symmetric (about zero) unimodal distributions, $F_0(x)$ is itself symmetric and unimodal.

PROOF OF COROLLARY 2. We assume without loss of generally that $\varphi(t)$ is symmetric. Then, if $I(n)=[\gamma^n]$,

$$\varphi(t)=\varphi^{\gamma^n}(c^{-n}t)\geqq\varphi^{I(n)}(c^{-n}t)\varphi(c^{-n}t)$$
$$\geqq\varphi^{\gamma^n}(c^{-n}t)\varphi(c^{-n}t)=\varphi(t)\varphi(c^{-n}t)\to\varphi(t)\,.$$

This shows that

$$F^{*I(n)}(c^nx)\to F(x)\,.$$

Hence we have

$$\lim_{n\to\infty} I(n)(1-F(c^nx))=\lim_{n\to\infty} F(-c^nx)=-M(x)\,,$$

which implies

$$\lim_{n\to\infty}(1-F(c^nx))/F_0(c^n)=M(x)/2M(1)$$

and

$$\lim_{n\to\infty} F_0(c^nx)/F_0(c^n)=M(x)/M(1)\,,$$

where $F(x)=1-F_0(x)+F(-x)$.

Now Theorem 1, b) is applicable for $\beta<\alpha$, and for $\beta>\alpha$. Moreover in this case we have

$$F_0(c^n)c^{n\alpha}=F_0(c^n)\gamma^n\sim F_0(c^n)I(n)\to 1\,,$$

and for sufficiently large n,

$$\int_{c^n}^{c^{n+1}} x^\alpha dF(x)\geqq c^{n\alpha}(F(c^{n+1})-F(c^n))$$
$$=c^{n\alpha}F_0(c^n)\left(1-\frac{F_0(c^{n+1})}{F_0(c^n)}\right)\geqq A>0\,,$$

where A is independent of n. Hence $E|X|^\alpha=\infty$. q.e.d.

PROOF OF THEOREM 2. (12) and (13) imply that $\lim_{n\to\infty}(1-G(C_nx))/G_0(C_n)=\xi x^{-\alpha}$, $\lim_{n\to\infty} G_0(C_nx)/G_0(C_n)=x^{-\alpha}$, $\xi=\lambda/(\lambda+\mu)$. Let $\varepsilon>0$ be a given small number, and let

$$x=x_0<x_1<\cdots<x_p=dx \tag{25}$$

and

$$1=u_0<u_1<\cdots<u_p=d \tag{26}$$

be divisions of intervals $[x, dx]$ and $[1, d]$ respectively such that

$$0<x_{i+1}-x_i<\varepsilon\,, \qquad 0<u_{i+1}-u_i\leqq\varepsilon \tag{27}$$

and

$$0<x_i^{-\alpha}-x_{i+1}^{-\alpha}<\varepsilon\,, \qquad 0<u_i^{-\alpha}-u_{i+1}^{-\alpha}\leqq\varepsilon \tag{28}$$

hold. Let n_0 be so large that for any $n\geqq n_0$ and for $i=0, 1,\cdots, p$,

$$G_0(C_nx_i)/G_0(C_n)=x_i^{-\alpha}+\varepsilon_{i,n} \tag{29}$$

and

$$G_0(C_nu_i)/G_0(C_n)=u_i^{-\alpha}+\delta_{i,n}\,, \tag{30}$$

where $|\varepsilon_{i,n}|\leqq\varepsilon$, $|\delta_{i,n}|\leqq\varepsilon$, $i=0, 1, 2,\cdots, p$, $n\geqq n_0$.

If $C_n\leqq y<C_{n+1}$, $n\geqq n_0$, then for some i and j

$$x_i\leqq\frac{y}{C_n}x<x_{i+1}\,, \qquad u_j\leqq\frac{y}{C_n}<u_{j+1} \tag{31}$$

and

$$\begin{aligned}\left(\frac{y}{C_n}x\right)^{-\alpha}-2\varepsilon&\leqq x_{i+1}^{-\alpha}+\varepsilon_{i+1,n}=\frac{G_0(C_nx_{i+1})}{G_0(C_n)}\\ &\leqq\frac{G_0(C_nx_i)}{G_0(C_n)}=x_i^{-\alpha}+\varepsilon_{i,n}\\ &\leqq\left(\frac{y}{C_n}x\right)^{-\alpha}+2\varepsilon\,.\end{aligned} \tag{32}$$

$$\begin{aligned}\left(\frac{y}{C_n}\right)^{-\alpha}-2\varepsilon&\leqq u_{j+1}^{-\alpha}+\delta_{j+1,n}=\frac{G_0(C_nu_{j+1})}{G_0(C_n)}\\ &\leqq\frac{G_0(C_nu_j)}{G_0(C_n)}=u_j^{-\alpha}+\delta_{j,n}\\ &\leqq\left(\frac{y}{C_n}\right)^{-\alpha}+2\varepsilon\,.\end{aligned} \tag{33}$$

Using these inequalities and

$$(34)\qquad \frac{G_0(C_n x_{i+1})}{G_0(C_n)}\Big/\frac{G_0(C_n u_j)}{G_0(C_n)}\leqq G_0\left(C_n\frac{y}{C_n}x\right)\Big/G_0\left(C_n\frac{y}{C_n}\right) =\frac{G_0(yx)}{G_0(y)}\leqq\frac{G_0(C_n x_i)}{G_0(C_n)}\Big/\frac{G_0(C_n u_{j+1})}{G_0(C_n)}\,,$$

we obtain

$$(35)\qquad \frac{((y/C_n)x)^{-\alpha}-2\varepsilon}{(y/C_n)^{-\alpha}+2\varepsilon}\leqq\frac{G_0(yx)}{G_0(y)}\leqq\frac{((y/C_n)x)^{-\alpha}+2\varepsilon}{(y/C_n)^{-\alpha}-2\varepsilon}$$

or

$$(36)\qquad x^{-\alpha}-2\varepsilon\frac{d^\alpha(1+x^\alpha)}{1+2\varepsilon d^\alpha}\leqq\frac{G_0(yx)}{G_0(y)}\leqq x^{-\alpha}+2\varepsilon\frac{d^\alpha(1+x^\alpha)}{1-2\varepsilon d^\alpha}\,.$$

(36) holds for all large y. Passing to the limit ($y\to\infty$ and then $\varepsilon\to 0$) we obtain (8). (7) with $\xi=\lambda/(\lambda+\mu)$ is obtained in the same way. (We make use of monotonicity of $1-G(x)$.) q.e.d.

PROOF OF THEOREM 3. We shall show that (9)–(10) and (15)–(16) imply that $R(x)=U(x)$ and $S(x)=V(x)$. Since by assumption $\rho=\log a/\log b$ is irrational, it follows from Theorem 1 a) that $R(x)=U(x)=\xi x^{-\alpha}$ and $S(x)=V(x)=x^{-\alpha}$, where ξ is a constant. The desired result follows then from Theorem 2.

Let $\{n_k\}$ be a sequence of positive integers such that as $k\to\infty$, $n_k\to\infty$ and $n_k\rho-[n_k\rho]\to 0$. Write $n(k)=[n_k\rho]$ and $c_k=b^{n_k\rho-[n_k\rho]}$.

Let $\varepsilon>0$ be a given small number and let $k_0=k_0(x,\varepsilon)$ be an integer so large that we have for any $k\geqq k_0$

$$|G_0(b^{n(k)}x)/G_0(b^{n(k)})-V(x)|\leqq\varepsilon\,,$$

$$|G_0(b^{n(k)}(1+\varepsilon)x)/G_0(b^{n(k)})-V((1+\varepsilon)x)|\leqq\varepsilon\,,$$

$$|G_0(b^{n(k)}(1+\varepsilon))/G_0(b^{n(k)})-V(1+\varepsilon)|\leqq\varepsilon\,,$$

and

$$0\leqq c_k-1\leqq\varepsilon\,.$$

Then,

$$(37)\qquad V((1+\varepsilon)x)-\varepsilon\leqq G_0(b^{n(k)}(1+\varepsilon))/G_0(b^{n(k)}) \leqq G_0(b^{n(k)}x)/G_0(b^{n(k)})\leqq V(x)+\varepsilon\,,$$

and

$$(38)\qquad V((1+\varepsilon)x)-\varepsilon\leqq G_0(b^{n(k)}(1+\varepsilon))/G_0(b^{n(k)}) \leqq G_0(b^{n(k)}c_k)/G_0(b^{n(k)})\leqq 1\,.$$

On the other hand we have

$$\frac{G_0(a^{n_k}x)}{G_0(a^{n_k})}=\frac{G_0(b^{n_k\rho}x)}{G_0(b^{n_k\rho})}=\frac{G_0(b^{n(k)}c_kx)}{G_0(b^{n(k)})}\Big/\frac{G_0(b^{n(k)}c_k)}{G_0(b^{n(k)})}\,. \tag{39}$$

From (37)-(39), we obtain

$$V((1+\varepsilon)x)-\varepsilon=\frac{G_0(a^{n_k}x)}{G_0(a^{n_k})}\leqq\frac{V(x)+\varepsilon}{V(1+\varepsilon)-\varepsilon}\,,\qquad \text{for all } k\geqq k_0\,. \tag{40}$$

Passing to the limit (first $k\to\infty$, and then $\varepsilon\to 0$), we get

$$V(x)\leqq S(x)\leqq V(x)\,.$$

Equality $R(x)=U(x)$ is obtained in the same way. q.e.d.

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