A FUNDAMENTAL RELATION BETWEEN PREDICTOR IDENTIFICATION AND POWER SPECTRUM ESTIMATION

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Summary

An asymptotic linear relation between the FPE (final prediction error) of the predictor obtained by using the least squares estimate of autoreggression coefficients and the integrated relative mean square error of the estimate of power spectrum obtained from the fitted autoregression model is established. This relation provides a link between the predictor identification and the power spectrum estimation and has already been used as a theoretical background of a former paper [3].

1. Introduction

Recently the present author introduced the notion of final prediction error (FPE), the mean square error of one-step prediction, for the evaluation and decision of the order of autoregressive model to be fitted to data for the purpose of prediction [1]. A procedure, which was later called the minimum FPE procedure and is aiming at approximately minimizing FPE [4], was found to be useful for this purpose and the use of the power spectrum of the fitted autoregressive model as an estimate of the power spectrum of the original process was suggested [1], [2]. Though the theoretical background of the use of the minimum FPE procedure for prediction was discussed [4], the theoretical justification of its use for the spectrum estimation was left open.

In the present paper it will be shown that the integral over the whole frequency of relative mean square error of the power spectrum estimate is asymptotically in a linear relation with FPE when the process under observation is a finite order autoregressive process with independently and identically distributed innovations with finite fourth order moments and the order of the fitted model is larger than or equal to the true one. This result, which was first announced in [3], shows that the minimum FPE procedure which was originally developed to avoid the use of the practically controversial notion of infinitely long autoregression for prediction can also be useful for the purpose of power spectrum estimation.

2. Evaluation of the integrated mean square error of autoregressive power spectrum estimate

For the convenience of the reader a heuristic explanation of the asymptotic variance formula of the estimated power spectrum will be given first. A justification of the formula can be found in [2].

For the stationary autoregressive process

$$X(n)=\sum_{m=1}^{M} a_m X(n-m)+\varepsilon(n)\,, \tag{2.1}$$

where $\varepsilon(n)$ is a white noise with zero mean and $E\varepsilon^2(n)=\sigma^2$, the power spectral density function $p_{xx}(f)$ is given by

$$p_{xx}(f)=\sigma^2/|A(f)|^2\,, \tag{2.2}$$

where

$$A(f)=1-\sum_{m=1}^{M} a_m \exp(-i2\pi f m)\,. \tag{2.3}$$

We assume that $\varepsilon(n)$'s are mutually independently identically distributed with finite fourth order moments. When a_m and σ^2 of (2.2) are replaced by their estimates $\hat{a}_m$ and $\hat{\sigma}^2$ respectively, it gives an estimate $\hat{p}_{xx}(f)$ of $p_{xx}(f)$. The total differential $\Delta p_{xx}(f)$ of $p_{xx}(f)$ for the differentials $\Delta a_m=\hat{a}_m-a_m$ and $\Delta\sigma^2=\hat{\sigma}^2-\sigma^2$ is by definition

$$\Delta p_{xx}(f)=\frac{\partial p_{xx}(f)}{\partial\sigma^2}\Delta\sigma^2+\sum_{m=1}^{M}\frac{\partial p_{xx}(f)}{\partial a_m}\Delta a_m\,. \tag{2.4}$$

The total differential $\Delta\log p_{xx}(f)$ of $\log p_{xx}(f)$ is equal to $\Delta p_{xx}(f)/p_{xx}(f)$ and is given by

$$\frac{\Delta p_{xx}(f)}{p_{xx}(f)}=\frac{\Delta\sigma^2}{\sigma^2}-\frac{\Delta|A(f)|^2}{|A(f)|^2}\,, \tag{2.5}$$

where $\Delta|A(f)|^2=A(f)\overline{\Delta A(f)}+\overline{A(f)}\Delta A(f)$, $\Delta A(f)=-\sum_{m=1}^{M}\Delta a_m \exp(-i2\pi f m)$ and $-$ denotes the conjugate complex.

When for a given set observations $\{X(n);\ n=1,2,\cdots,N\}$ the least squares estimates of a_m are adopted as $\hat{a}_m$ and the corresponding sample average of the square of $\hat{\varepsilon}(n)=X(n)-\sum_{m=1}^{M}\hat{a}_m X(n-m)$ as $\hat{\sigma}^2$ it can be shown that in the limit distribution $\sqrt{N}\Delta a_m$ and $\sqrt{N}\Delta\sigma^2$ are mutually independent [2]. Thus for this estimate it follows that

(2.6) $$E_\infty N\left(\frac{\Delta p_{xx}(f)}{p_{xx}(f)}\right)^2 = E_\infty N\left(\frac{\Delta\sigma^2}{\sigma^2}\right)^2 + E_\infty N\left(\frac{\Delta|A(f)|^2}{|A(f)|^2}\right)^2,$$

where E_∞ denotes the mean value in the limit distribution. This $E_\infty N \cdot(\Delta p_{xx}(f)/p_{xx}(f))^2$ is equal to the variance of the limit distribution of $N(\hat{p}_{xx}(f)-p_{xx}(f))/p_{xx}(f)$ as N tends to infinity [2]. From the relation $\Delta|A(f)|^2/|A(f)|^2 = \Delta A(f)/A(f) + \overline{\Delta A(f)}/\overline{A(f)}$, it holds that

(2.7) $$\left(\frac{\Delta|A(f)|^2}{|A(f)|^2}\right)^2 = 2\left(\frac{|\Delta A(f)|^2}{|A(f)|^2} + Re\left(\frac{\Delta A(f)}{A(f)}\right)^2\right).$$

The limit distribution of $\sqrt{N}\Delta a_m$ $(m=1, 2, \cdots, M)$ is Gaussian with zero means and covariance matrix $\sigma^2 R_M^{-1}$, where R_M is an $M\times M$ matrix with the (i, j)th element $R_M(i, j)=EX(n+i)X(n+j)$ [2]. From this fact it follows that

(2.8) $$\begin{aligned}&\int_{-1/2}^{1/2} E_\infty N\left|\frac{\Delta A(f)}{A(f)}\right|^2 df \\ &= \sum_{l=1}^{M}\sum_{m=1}^{M} E_\infty N\Delta a_l \Delta a_m \int_{-1/2}^{1/2} \frac{\exp(i2\pi f(m-l))}{|A(f)|^2} df \\ &= \sum_{l=1}^{M}\sum_{m=1}^{M} R_M^{-1}(l, m)\int_{-1/2}^{1/2} \exp(i2\pi f(m-l)) \frac{\sigma^2}{|A(f)|^2} df.\end{aligned}$$

As $\sigma^2/|A(f)|^2 = p_{xx}(f)$, it can be seen that

(2.9) $$\int_{-1/2}^{1/2} \exp(i2\pi f(m-l)) \frac{\sigma^2}{|A(f)|^2} df = R_m(m, l).$$

Combining these two results, one can get

(2.10) $$\int_{-1/2}^{1/2} E_\infty N\left|\frac{\Delta A(f)}{A(f)}\right|^2 df = M.$$

In the same way as in (2.8) it is seen that

(2.11) $$\begin{aligned}&\int_{-1/2}^{1/2} E_\infty N\left(\frac{\Delta A(f)}{A(f)}\right)^2 df \\ &= \sigma^2 \sum_{l=1}^{M}\sum_{m=1}^{M} R_M^{-1}(l, m)\int_{-1/2}^{1/2} \frac{\exp(-i2\pi f(l+m))}{(A(f))^2} df.\end{aligned}$$

From the stationarity assumption of $X(n)$ characteristic equation $1-\sum_{m=1}^{M} a_m z^m = 0$ has all its roots outside the unit circle and $X(n)$ has a representation

(2.12) $$X(n) = \sum_{k=0}^{\infty} b_k \varepsilon(n-k),$$

where $b_0=1$ and $\sum_{l=1}^{\infty}|b_k|<\infty$, i.e., b_k is the impulse response of the linear system which produces the output $X(n)$ to the input $\varepsilon(n)$ and satisfies the relation

$$b_k-\sum_{m=1}^{M} a_m b_{k-m}=\delta_{k,0}\,, \tag{2.13}$$

where $\delta_{k,0}=1$, $=0$ $(k\neq 0)$ and $b_k=0$ for $k<0$. The frequency response function of the system is given by

$$\begin{aligned}\frac{1}{A(f)}&=\frac{1}{1-\sum_{m=1}^{M} a_m \exp(-i2\pi f m)}\\ &=\sum_{k=-\infty}^{\infty} b_k \exp(-i2\pi f k)\,.\end{aligned} \tag{2.14}$$

By using this relation and the fact that $b_k=0$ for $k<0$ it can be shown that

$$\begin{aligned}&\int_{-1/2}^{1/2}\exp(i2\pi f p)\left(\frac{1}{A(f)}\right)^2 df\\ &=\sum_{k=-\infty}^{\infty} b_k b_{p-k}\\ &=0 \qquad \text{for } p<0\,.\end{aligned} \tag{2.15}$$

Consequently (2.11) gives

$$\int_{-1/2}^{1/2} E_\infty N\left(\frac{\Delta A(f)}{A(f)}\right)^2 df=0\,. \tag{2.16}$$

Summarizing these results, (2.7) and (2.6), one can get

$$\int_{-1/2}^{1/2} E_\infty N\left(\frac{\Delta p_{xx}(f)}{p_{xx}(f)}\right)^2=E_\infty N\left(\frac{\Delta\sigma^2}{\sigma^2}\right)^2+2M\,. \tag{2.17}$$

Incidentally, the factor 2 in front of M was missing in [3].

3. Relation between the integrated relative mean square error and FPE

It can be shown as in [4] that when $\hat{a}_m$ is applied to another independent realization $Y(n)$ of $X(n)$ to give a one step prediction $\hat{Y}(n)=\sum_{m=1}^{M}\hat{a}_m Y(n-m)$ it holds that

$$\begin{aligned}(\mathrm{FPE})_M&=\frac{1}{N}E_\infty N[E_x(Y(n)-\hat{Y}(n))^2]\\ &=\frac{1}{N}\sigma^2(N+M)\,,\end{aligned} \tag{3.1}$$

where E_x denotes the expectation conditional on the realization of $X(n)$ from which $\hat{a}_m$ is obtained. This $(\mathrm{FPE})_M$ was adopted as an asymptotic evaluation of the mean square one-step prediction error or the final prediction error (FPE) [4].

It should be noted here that (2.17) and (3.1) are valid for M greater than or equal to the order K of the process, which is defined by the relations $a_K \neq 0$ and $a_m = 0$ for $m > K$. This fact shows that in this range of M an asymptotically linear relationship holds between the integrated relative mean square error of the autoregressive power spectrum estimate and the FPE. When the correction for the possibly non-zero mean is introduced, M in (3.1) should only be replaced by $M+1$.

The present observation shows that the application to power spectrum estimation of the result of the minimum FPE procedure [1], [2], [3], which was originally developed to avoid the use of unnecessarily long autoregression model for prediction by watching the behavior of an estimate of $(\mathrm{FPE})_M$, is reasonable from the stand point of controlling the relative mean square error of the corresponding estimate of the power spectrum.

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