DISTRIBUTION OF DISCRIMINANT FUNCTION IN CIRCULAR MODELS*

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Summary

Assuming that the covariance matrices are circular, we make an appropriate transformation which reduces the circular matrices to canonical forms. The discriminant function is given when the populations are multivariate normal with different circular matrices and its distribution is derived. An asymptotic expansion for the distribution is obtained when all the parameters are unknown.

1. Introduction

Let X, a $p\times 1$ vector, be an observation which is known to have come from one of two multivariate normal populations. Denote the ith population by π_i which is $N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ for $i=1, 2$. We assume that $\boldsymbol{\Sigma}_i$ is positive definite and circular, i.e. $\boldsymbol{\Sigma}_i$ is of the form

$$(1.1) \qquad \sigma_i^2 \begin{pmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & \rho_3 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \cdots & \rho_4 & \rho_3 \\ \vdots & & & & & & \vdots \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 & \cdots & \rho_1 & 1 \end{pmatrix} .$$

This matrix has been studied by Whittle [9] and Wise [10]. The purpose of this paper is to derive the discriminant function which will be used to classify the observation X into either π_1 or π_2. It is given in Wise [10] that the matrix in (1.1) can be transformed into canonical form. Thus there exists an orthogonal matrix L with (m, n)th element

$$(1.2) \qquad l_{mn} = p^{-1/2}\left[\cos\frac{2\pi}{p}(m-1)(n-1) + \sin\frac{2\pi}{p}(m-1)(n-1)\right]$$

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such that $L'\Sigma_i L = \mathrm{diag}(\sigma_{i1}^2, \sigma_{i2}^2, \cdots, \sigma_{ip}^2)$. Since L is independent of the elements in Σ_1 and Σ_2, the discriminant function is equivalent to that when the covariance matrices are diagonal. This is true because the discriminant function derived by the likelihood ratio procedure is invariant under any linear transformation.

Section 2 will derive the discriminant function and its distribution when the parameters are known. Section 3 discusses the distribution when the covariance matrices are known but the means are unknown. When the parameters are all unknown, an asymptotic expansion for the distribution is made in Section 4. The expansion is obtained by the "studentization" method of Hartley [2] and of Welch [8] which was used by many authors (e.g. Han [1], Ito [3], Okamoto [4] and Siotani [6]) for other multivariate problems.

2. The discriminant function

Since the circular matrix can be transformed into canonical form, we may let $\Sigma_i = \mathrm{diag}(\sigma_{i1}^2, \sigma_{i2}^2, \cdots, \sigma_{ip}^2)$, $i=1, 2$. The discriminant function is easily obtained by using the likelihood ratio procedure. It is proportional to

$$(X-\mu_2)'\Sigma_2^{-1}(X-\mu_2)-(X-\mu_1)'\Sigma_1^{-1}(X-\mu_1)\,.$$

Substituting μ_i and Σ_i we obtain, apart from a constant,

$$V=\sum_{j=1}^{p}\left\{\left(\frac{1}{\sigma_{2j}^2}-\frac{1}{\sigma_{1j}^2}\right)\left(x_j-\frac{\mu_{2j}/\sigma_{2j}^2-\mu_{1j}/\sigma_{1j}^2}{1/\sigma_{2j}^2-1/\sigma_{1j}^2}\right)^2\right\} \tag{2.1}$$

where x_j and μ_{ij} are the jth component of X and μ_i, $i=1, 2$, respectively. We classify X into π_1 if $V>k$ and into π_2 if $V<k$ for some suitable choice of the constant k.

To find the distribution of V, we shall assume that $\sigma_{1j}^2>\sigma_{2j}^2$ for all j, or equivalently $\Sigma_1-\Sigma_2$ is positive definite. Hence $1/\sigma_{2j}^2-1/\sigma_{1j}^2>0$. Let

$$Z_j=\sqrt{\frac{1}{\sigma_{2j}^2}-\frac{1}{\sigma_{1j}^2}}\left(x_j-\frac{\mu_{2j}/\sigma_{2j}^2-\mu_{1j}/\sigma_{1j}^2}{1/\sigma_{2j}^2-1/\sigma_{1j}^2}\right). \tag{2.2}$$

Then $V=\sum_{j=1}^{p} Z_j^2$. When X comes from π_i, $i=1$ or 2, Z_j are independently distributed as $N(\xi_{ij}, \tau_{ij}^2)$ where

$$\begin{aligned}\xi_{ij}&=\sqrt{\frac{1}{\sigma_{2j}^2}-\frac{1}{\sigma_{1j}^2}}\left(\mu_{ij}-\frac{\mu_{2j}/\sigma_{2j}^2-\mu_{1j}/\sigma_{1j}^2}{1/\sigma_{2j}^2-1/\sigma_{1j}^2}\right),\\ \tau_{ij}^2&=\sigma_{ij}^2\left(\frac{1}{\sigma_{2j}^2}-\frac{1}{\sigma_{1j}^2}\right).\end{aligned} \tag{2.3}$$

Therefore V is distributed as the sum of $\tau_{ij}^2 \cdot \chi_1'^2(\delta_{ij}^2)$ where $\chi_1'^2(\delta_{ij}^2)$ is a non-central χ^2 distribution with 1 degree of freedom and non-centrality parameters $\delta_{ij}^2 = \xi_{ij}^2/\tau_{ij}^2$. It is not easy to obtain the distribution in a closed form. Patnaik [5] has considered a χ^2 approximation to the distribution of the sum by fitting the first two moments. Thus the distribution may be approximated by $c\chi_\nu^2$ where

$$c = \frac{\sum_j \tau_{ij}^4 + 2\sum_j \tau_{ij}^2 \xi_{ij}^2}{\sum_j \tau_{ij}^2 + \sum_j \xi_{ij}^2},$$

(2.4)

$$\nu = \frac{1}{c}\left(\sum_j \tau_{ij}^2 + \sum_j \xi_{ij}^2\right).$$

3. Distribution of the discriminant function when the means are unknown

When $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are unknown, we estimate them by the sample means $\bar{\boldsymbol{X}}_1$ and $\bar{\boldsymbol{X}}_2$ based on samples of size N_1 and N_2 from the two populations respectively. $\bar{\boldsymbol{X}}_i$, $i=1, 2$, are independent $N(\boldsymbol{\mu}_i, 1/N_i \cdot \boldsymbol{\Sigma}_i)$ distributions and independent of the distribution of $\boldsymbol{X}$. Furthermore, since $\boldsymbol{\Sigma}_i$ are diagonal, the components of $\bar{\boldsymbol{X}}_i$ are also independently distributed. The discriminant function in (2.1) becomes

$$V = \sum_{j=1}^{p} \left\{\left(\frac{1}{\sigma_{2j}^2} - \frac{1}{\sigma_{1j}^2}\right)\left(x_j - \frac{\bar{x}_{2j}/\sigma_{2j}^2 - \bar{x}_{1j}/\sigma_{1j}^2}{1/\sigma_{2j}^2 - 1/\sigma_{1j}^2}\right)^2\right\} = \sum_{j=1}^{p} Y_j^2 \tag{3.1}$$

where

$$Y_j = \sqrt{\frac{1}{\sigma_{2j}^2} - \frac{1}{\sigma_{1j}^2}}\left(x_j - \frac{\bar{x}_{2j}/\sigma_{2j}^2 - \bar{x}_{1j}/\sigma_{1j}^2}{1/\sigma_{2j}^2 - 1/\sigma_{1j}^2}\right). \tag{3.2}$$

The distribution of V is found in a similar way as in Section 2. When X comes from π_i, $i=1$ or 2, Y_j are independently distributed as $N(\xi_{ij}, \tau_{ij}^{*2})$ where ξ_{ij} is given in (2.3) and

$$\tau_{ij}^{*2} = \sigma_{ij}^2\left(\frac{1}{\sigma_{2j}^2} - \frac{1}{\sigma_{1j}^2}\right) + \frac{N_1\sigma_{1j}^2 + N_2\sigma_{2j}^2}{N_1N_2(\sigma_{1j}^2 - \sigma_{2j}^2)}. \tag{3.3}$$

The second term on the right-hand side is the increase in variance accounted for the unknown means. The distribution of V is the sum of $\tau_{ij}^{*2} \cdot \chi_1^{*2}(\delta_{ij}^{*2})$ where $\delta_{ij}^{*2} = \xi_{ij}^2/\tau_{ij}^{*2}$. Again this can be approximated by the Patnaik's method.

4. Distribution of the discriminant function when all the parameters are unknown

This section will derive an asymptotic expansion for the distribution of the discriminant function when all the parameters are unknown. The technique is the "studentization" method of Hartley [2] and of Welch [8]. The estimators of $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ are the sample means and sample variances, i.e. $\bar{\boldsymbol{X}}_i$ and $\boldsymbol{S}_i^2=(s_{ij}^2)$ respectively where $s_{ij}^2=\sum_{k=1}^{N_i}(x_{ijk}-\bar{x}_{ij})^2/n_i$ and $n_i=N_i-1$, $i=1,2$, $j=1,2,\cdots,p$. $\bar{x}_{ij}$ and s_{ij}^2 are independently distributed. Now the discriminant function in (2.1) becomes

$$V=\sum_{j=1}^{p}\left\{\left(\frac{1}{s_{2j}^2}-\frac{1}{s_{1j}^2}\right)\left(x_j-\frac{\bar{x}_{2j}/s_{2j}^2-\bar{x}_{1j}/s_{1j}^2}{1/s_{2j}^2-1/s_{1j}^2}\right)^2\right\}. \tag{4.1}$$

It is easily seen that V is invariant under any linear transformation. Hence, without loss of generality, we may let $\boldsymbol{\mu}_1=\boldsymbol{0}$, $\boldsymbol{\Sigma}_1=\boldsymbol{I}$, $\boldsymbol{\mu}_2=\boldsymbol{\mu}_0=(\mu_{01}, \mu_{02},\cdots,\mu_{0p})$ and $\boldsymbol{\Sigma}_2=\boldsymbol{\Sigma}_0=\text{diag}(\sigma_{01}^2,\sigma_{02}^2,\cdots,\sigma_{0p}^2)$. We shall derive the cumulative distribution (c.d.f.) of V, $F_i(v)$, given that $\boldsymbol{X}$ comes from π_i, $i=1,2$.

The characteristic function (c.f.) of V when $\boldsymbol{X}$ comes from π_1 is $\varphi(t/\pi_1)=E(e^{itV}/\pi_1)$ which can be written as

$$\begin{aligned}\varphi(t/\pi_1)&=E^{\bar{\boldsymbol{X}}_1,\bar{\boldsymbol{X}}_2,\boldsymbol{S}_1,\boldsymbol{S}_2}\{E(e^{itV}/\bar{\boldsymbol{X}}_1,\bar{\boldsymbol{X}}_2,\boldsymbol{S}_1,\boldsymbol{S}_2;\pi_1\}\\&=E^{\bar{\boldsymbol{X}}_1,\bar{\boldsymbol{X}}_2,\boldsymbol{S}_1,\boldsymbol{S}_2}\{\psi(\bar{\boldsymbol{X}}_1,\bar{\boldsymbol{X}}_2,\boldsymbol{S}_1,\boldsymbol{S}_2)\},\end{aligned}$$

where $\psi(\bar{\boldsymbol{X}}_1,\bar{\boldsymbol{X}}_2,\boldsymbol{S}_1,\boldsymbol{S}_2)$ is the conditional c.f. given $\bar{\boldsymbol{X}}_1$, $\bar{\boldsymbol{X}}_2$, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. It is found that

$$\begin{aligned}\psi(\bar{\boldsymbol{X}}_1,\bar{\boldsymbol{X}}_2,\boldsymbol{S}_1,\boldsymbol{S}_2)=\prod_{j=1}^{p}&\left[1-2it\left(\frac{1}{s_{2j}^2}-\frac{1}{s_{1j}^2}\right)\right]^{-1/2}\\&\times\exp\left\{(it)\left(\frac{\bar{x}_{2j}}{s_{2j}^2}-\frac{\bar{x}_{1j}}{s_{1j}^2}\right)^2\left[\left(\frac{1}{s_{2j}^2}-\frac{1}{s_{1j}^2}\right)\right.\right.\\&\left.\left.\times\left(1-2it\left(\frac{1}{s_{2j}^2}-\frac{1}{s_{1j}^2}\right)\right)\right]^{-1}\right\}.\end{aligned} \tag{4.2}$$

Since the function ψ is analytic about the point $(\bar{\boldsymbol{X}}_1,\bar{\boldsymbol{X}}_2,\boldsymbol{S}_1,\boldsymbol{S}_2)=(\boldsymbol{0},\boldsymbol{\mu}_0,\boldsymbol{I},\boldsymbol{\Sigma}_0)$, expanding ψ into a Taylor's series, we have

$$\begin{aligned}\psi(\bar{\boldsymbol{X}}_1,\bar{\boldsymbol{X}}_2,\boldsymbol{S}_1,\boldsymbol{S}_2)=\Big\{\exp\Big[&\Sigma\bar{x}_{1j}\frac{\partial}{\partial\mu_{1j}}+\Sigma(\bar{x}_{2j}-\mu_{0j})\frac{\partial}{\partial\mu_{2j}}+\Sigma(s_{1j}^2-1)\frac{\partial}{\partial\sigma_{1j}^2}\\&+\Sigma(s_{2j}^2-\sigma_{0j}^2)\frac{\partial}{\partial\sigma_{2j}^2}\Big]\Big\}\psi(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\Sigma}_1,\boldsymbol{\Sigma}_2)|_0\end{aligned}$$

where $|_0$ denotes that the expression is evaluated at $(\mathbf{0}, \boldsymbol{\mu}_0, \boldsymbol{I}, \boldsymbol{\Sigma}_0)$. The c.f. of V is then

$$\begin{aligned}(4.3)\qquad \varphi(t/\pi_1)&=E^{\bar{\boldsymbol{X}}_1, \bar{\boldsymbol{X}}_2, \boldsymbol{S}_1, \boldsymbol{S}_2}\{\psi(\bar{\boldsymbol{X}}_1, \bar{\boldsymbol{X}}_2, \boldsymbol{S}_1, \boldsymbol{S}_2)\}\\ &=\Theta\psi(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)|_0 ,\end{aligned}$$

where Θ is the differential operator

$$\begin{aligned}\Theta=E^{\bar{\boldsymbol{X}}_1, \bar{\boldsymbol{X}}_2, \boldsymbol{S}_1, \boldsymbol{S}_2}\Big\{\exp\Big[\Sigma\bar{x}_{1j}\frac{\partial}{\partial\mu_{1j}}+\Sigma(\bar{x}_{2j}-\mu_{0j})\frac{\partial}{\partial\mu_{2j}}\\ +\Sigma(s_{1j}^2-1)\frac{\partial}{\partial\sigma_{1j}^2}+\Sigma(s_{2j}^2-\sigma_{0j}^2)\frac{\partial}{\partial\sigma_{2j}^2}\Big]\Big\} .\end{aligned}$$

Since $\bar{x}_{ij}$ and s_{ij}^2 are independently distributed as normal and χ^2 respectively, using the moment generating functions, we have

$$\begin{aligned}(4.4)\qquad \Theta=\prod_j \exp\Big\{\frac{1}{2N_1}\frac{\partial^2}{\partial\mu_{1j}^2}+\frac{\sigma_{0j}^2}{2N_2}\frac{\partial^2}{\partial\mu_{2j}^2}-\frac{\partial}{\partial\sigma_{1j}^2}-\sigma_{0j}^2\frac{\partial}{\partial\sigma_{2j}^2}\\ -\frac{n_1}{2}\log\Big(1-\frac{2}{n_1}\frac{\partial}{\partial\sigma_{1j}^2}\Big)-\frac{n_2}{2}\log\Big(1-\frac{2\sigma_{0j}^2}{n_2}\frac{\partial}{\partial\sigma_{2j}^2}\Big)\Big\} .\end{aligned}$$

Substituting the expansion of $\log(1-x)$ and further expanding the exponential function in (4.4), we obtain

$$\begin{aligned}(4.5)\qquad \Theta=1+\frac{1}{2N_1}\Sigma\frac{\partial^2}{\partial\mu_{1j}^2}+\frac{1}{2N_2}\Sigma\sigma_{0j}^2\frac{\partial^2}{\partial\mu_{2j}^2}+\frac{1}{n_1}\Sigma\frac{\partial^2}{\partial(\sigma_{1j}^2)^2}\\ +\frac{1}{n_2}\Sigma\sigma_{0j}^4\frac{\partial^2}{\partial(\sigma_{2j}^2)^2}+O_2 ,\end{aligned}$$

where O_2 stands for the terms of the second order with respect to $(N_1^{-1}, N_2^{-1}, n_1^{-1}, n_2^{-1})$. We can now find the individual terms in (4.3). The principal terms is

$$(4.6)\qquad \psi(\mathbf{0}, \boldsymbol{\mu}_0, \boldsymbol{I}, \boldsymbol{\Sigma}_0)=\prod_j (1-2ita_j)^{-1/2}\exp\{it\mu_0^2[\sigma_{0j}^4a_j(1-2ita_j)]^{-1}\}=\psi_0$$

where $a_j=1/\sigma_{0j}^2-1$. ψ_0 is the c.f. of $\sum_j Z_j^2$ or sum of p non-central chi-square variates with 1 d.f. where Z_j are independently distributed as $N(\mu_{0j}/(\sigma_{0j}^2/a_j), a_j)$.

To find the term associated with N_1^{-1}, we have to differentiate the function $\psi(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_2)$ with respect to μ_{1j}. The second derivative evaluated at the point $(\mathbf{0}, \boldsymbol{\mu}_0, \boldsymbol{I}, \boldsymbol{\Sigma}_0)$ is

$$\frac{1}{N_1}\sum_j\Big\{\frac{it}{(1-2ita_j)a_j}+\frac{2(it)^2\mu_{0j}^2}{(1-2ita_j)^2\sigma_{0j}^4a_j^2}\Big\}\psi_0 .$$

Similarly we can find the terms for N_2^{-1}, n_1^{-1} and n_2^{-1}. They are

$$\frac{1}{N_2}\sum_j\left\{\frac{it}{(1-2ita_j)\sigma_{0j}^2a_j}+\frac{2(it)^2\mu_{0j}^2}{(1-2ita_j)^2\sigma_{0j}^6a_j^2}\right\}\psi_0\,,$$

$$\frac{1}{n_1}\sum_j\left\{\frac{it}{(1-2ita_j)}c'_j+\frac{(it)^2}{(1-2ita_j)^2}[c_j^2+2c_j+b'_j]\right.$$
$$\left.+\frac{(it)^3}{(1-2ita_j)^3}[4b_j+2b_jc_j]+\frac{(it)^4}{(1-2ita_j)^4}b_j^2\right\}\psi_0\,,$$

$$\frac{1}{n_2}\sum_j\left\{\frac{it}{(1-2ita_j)}\sigma_{0j}^4h'_j+\frac{(it)^2}{(1-2ita_j)^2}[\sigma_{0j}^4h_j-2h_j-\sigma_{0j}^4g'_j]\right.$$
$$\left.+\frac{(it)^3}{(1-2ita_j)^3}[4g_j-2\sigma_{0j}^4g_jh_j]+\frac{(it)^4}{(1-2ita_j)^4}\sigma_{0j}^4g_j^2\right\}\psi_0\,,$$

where

$$b_j=\frac{2\mu_{0j}^2}{\sigma_{0j}^4a_j}\,,\qquad b'_j=\frac{-2\mu_{0j}^2}{\sigma_{0j}^4a_j}\left(\frac{1}{a_j}+2\right),$$

$$c_j=\frac{-\mu_{0j}^2}{\sigma_{0j}^4a_j^2}+1\,,\qquad c'_j=2\left(\frac{\mu_{0j}^2}{\sigma_{0j}^6a_j^3}-1\right),$$

$$g_j=\frac{2\mu_{0j}^2}{\sigma_{0j}^8a_j}\,,\qquad g'_j=\frac{2\mu_{0j}^2}{\sigma_{0j}^{10}a_j}\left(\frac{1}{\sigma_{0j}^2a_j}-4\right),$$

$$h_j=\frac{\mu_{0j}^2}{\sigma_{0j}^6a_j}\left(\frac{1}{a_j}-1\right)-\frac{1}{\sigma_{0j}^4}\,,\qquad h'_j=\frac{2}{\sigma_{0j}^6}\left(\frac{\mu_{0j}^2}{\sigma_{0j}^6a_j^3}-\frac{3\mu_{0j}^2}{\sigma_{0j}^2a_j^2}+1\right).$$

The c.f. of V, after collecting terms, is

$$\varphi(t/\pi_1)=\left\{1+\sum_j\frac{it}{(1-2ita_j)}m_{1j}+\sum_j\frac{(it)^2}{(1-2ita_j)^2}m_{2j}\right.$$
$$\left.+\sum_j\frac{(it)^3}{(1-2ita_j)^3}m_{3j}+\sum_j\frac{(it)^4}{(1-2ita_j)^4}m_{4j}\right\}\psi_0+O_2\,,\tag{4.7}$$

where

$$m_{1j}=\frac{1}{N_1a_j}+\frac{1}{N_2\sigma_{0j}^2a_j}+\frac{1}{n_1}c'_j+\frac{1}{n_2}\sigma_{0j}^4h'_j\,,$$

$$m_{2j}=\frac{1}{N_1}\frac{2\mu_{0j}^2}{\sigma_{0j}^4a_j^2}+\frac{1}{N_2}\frac{2\mu_{0j}^2}{\sigma_{0j}^6a_j^2}+\frac{1}{n_1}(c_j^2+2c_j+b'_j)$$
$$+\frac{1}{n_2}(\sigma_{0j}^4h_j-2h_j-\sigma_{0j}^4g'_j)\,.$$

$$m_{3j}=\frac{1}{n_1}(4b_j+2b_jc_j)+\frac{1}{n_2}(4g_j-2\sigma_{0j}^4g_jh_j)\,,$$

$$m_{4j}=\frac{1}{n_1}b_j^2+\frac{1}{n_2}\sigma_{0j}^4g_j^2\,.$$

In order to invert the c.f. to obtain the c.d.f. $F_1(v)$, we use the method given by Wallace [7] which was used by Ito [3] and Han [1] for a similar problem. If $F(x)$ is the c.d.f. of a statistic and $\varphi(t)$ is its c.f., then the c.d.f. corresponding to $(-it)^r\varphi(t)$ is $F^{(r)}(x)$ where $F^{(r)}(x)$ is the rth derivative of $F(x)$. Now let $G_{kj}(x)$ be the c.d.f. of a non-central chi-square variate with k d.f. and non-centrality parameter $\mu_{0j}^2/\sigma_{0j}^4 a_j^2$, then

$$F_1(v)=\sum_j \{G_{1j}(v)-m_{1j}G_{3j}^{(1)}(v)+m_{2j}G_{5j}^{(2)}(v) -m_{3j}G_{7j}^{(3)}(v)+m_{4j}G_{9j}^{(4)}(v)\}+O_2\,. \tag{4.8}$$

To find the c.d.f. $F_2(v)$ when X comes from π_2, a similar procedure is used. The conditional c.f. $\psi(\bar{X}_1, \bar{X}_2, S_1, S_2)$ is

$$\begin{aligned}\psi(\bar{X}_1, \bar{X}_2, S_1, S_2)=&\prod_{j=1}^{p}\left[1-2it\sigma_{0j}^2\left(\frac{1}{s_{2j}^2}-\frac{1}{s_{1j}^2}\right)\right]^{-1/2}\\ &\times\exp\left\{(it)\left[\mu_{0j}\left(\frac{1}{s_{2j}^2}-\frac{1}{s_{1j}^2}\right)-\left(\frac{\bar{x}_{2j}}{s_{2j}^2}-\frac{\bar{x}_{1j}}{s_{1j}^2}\right)\right]^2\right.\\ &\left.\times\left[\left(\frac{1}{s_{2j}^2}-\frac{1}{s_{1j}^2}\right)\left(1-2it\sigma_{0j}^2\left(\frac{1}{s_{2j}^2}-\frac{1}{s_{1j}^2}\right)\right)\right]^{-1}\right\}.\end{aligned} \tag{4.9}$$

Again expanding ψ in a Taylor's series about $(\mathbf{0}, \boldsymbol{\mu}_0, \boldsymbol{I}, \boldsymbol{\Sigma}_0)$, we obtain the principal term to be

$$\psi_0=\prod_j (1-2it\sigma_{0j}^2 a_j)^{-1/2}\exp\{it\mu_{0j}^2[a_j(1-2it\sigma_{0j}^2 a_j)]^{-1}\} \tag{4.10}$$

which is the c.f. of $\sum_j Z_j^2$ where Z_j are independently distributed as $N(\mu_{0j}/\sqrt{a_j}, \sigma_{0j}^2 a_j)$. The linear terms are found to be

$$\frac{1}{N_1}\sum_j\left\{\frac{it}{(1-2it\sigma_{0j}^2 a_j)a_j}+\frac{2(it)^2\mu_{0j}^2}{(1-2it\sigma_{0j}^2 a_j)a_j^2}\right\}\psi_0\,,$$

$$\frac{1}{N_2}\sum_j\left\{\frac{it}{(1-2it\sigma_{0j}^2 a_j)\sigma_{0j}^4 a_j}+\frac{2(it)^2\mu_{0j}^2}{(1-2it\sigma_{0j}^2 a_j)^2\sigma_{0j}^4 a_j^2}\right\}\psi_0\,,$$

$$\begin{aligned}\frac{1}{n_1}\sum_j&\left\{\frac{it}{(1-2it\sigma_{0j}^2 a_j)}C_j'+\frac{(it)^2}{(1-2it\sigma_{0j}^2 a_j)^2}[C_j^2+2\sigma_{0j}^2 C_j+B_j']\right.\\ &\left.+\frac{(it)^3}{(1-2it\sigma_{0j}^2 a_j)^3}[4\sigma_{0j}^2 B_j+2B_jC_j]+\frac{(it)^4}{(1-2it\sigma_{0j}^2 a_j)^4}B_j^2\right\}\psi_0\,,\end{aligned}$$

$$\begin{aligned}\frac{1}{n_2}\sum_j&\left\{\frac{it}{(1-2it\sigma_{0j}^2 a_j)}\sigma_{0j}^4 H_j'+\frac{(it)^2}{(1-2it\sigma_{0j}^2 a_j)^2}[\sigma_{0j}^4 H_j^2-2\sigma_{0j}^2 H_j-\sigma_{0j}^4 G_j']\right.\\ &\left.+\frac{(it)^3}{(1-2it\sigma_{0j}^2 a_j)^3}[4\sigma_{0j}^2 G_j-2\sigma_{0j}^4 G_jH_j]+\frac{(it)^4}{(1-2it\sigma_{0j}^2 a_j)^4}\sigma_{0j}^4 G_j^2\right\}\psi_0\,,\end{aligned}$$

where

$$B_j=\frac{2\mu_{0j}^2\sigma_{0j}^2}{a_j}\,,\qquad B_j'=-\frac{2\mu_{0j}^2\sigma_{0j}^2}{a_j}\left(\frac{1}{a_j}+4\right),$$

$$C_j=-\frac{\mu_{0j}^2}{a_j^2}-\frac{2\mu_{0j}^2}{a_j}+\sigma_{0j}^2\,,\qquad C_j'=2\left[\frac{\mu_{0j}^2}{a_j^3}+\frac{3\mu_{0j}^2}{a_j^2}+\frac{3\mu_{0j}^2}{a_j}-\sigma_{0j}^2\right],$$

$$G_j=\frac{2\mu_{0j}^2}{\sigma_{0j}^2a_j}\,,\qquad G_j'=2\frac{\mu_{0j}^2}{\sigma_{0j}^4a_j}\left[\frac{1}{\sigma_{0j}^2a_j}-2\right],$$

$$H_j=\frac{\mu_{0j}^2}{\sigma_{0j}^4a_j^2}-\frac{1}{\sigma_{0j}^2}\,,\qquad H_j'=\frac{3\mu_{0j}^2}{\sigma_{0j}^6a_j^3}+\frac{2}{\sigma_{0j}^4}\,.$$

The c.f. of V given that X comes from π_2 is

$$\text{(4.11)}\quad \varphi(t/\pi_2)=\Big\{1+\sum_j\frac{it}{(1-2it\sigma_{0j}^2a_j)}M_{1j}+\sum_j\frac{(it)^2}{(1-2it\sigma_{0j}^2a_j)^2}M_{2j}$$
$$+\sum_j\frac{(it)^3}{(1-2it\sigma_{0j}^2a_j)^3}M_{3j}+\sum_j\frac{(it)^4}{(1-2it\sigma_{0j}^2a_j)^4}M_{4j}\Big\}\varphi_0+O_2\,,$$

where

$$M_{1j}=\frac{1}{N_1a_j}+\frac{1}{N_2\sigma_{0j}^4a_j}+\frac{1}{n_1}C_j'+\frac{1}{n_2}\sigma_{0j}^4H_j'\,,$$

$$M_{2j}=\frac{1}{N_1}\frac{2\mu_{0j}^2}{a_j^2}+\frac{1}{N_2}\frac{2\mu_{0j}^2}{\sigma_{0j}^4a_j^2}+\frac{1}{n_1}(C_j^2+2\sigma_{0j}^2C_j+B_j')$$
$$+\frac{1}{n_2}(\sigma_{0j}^4H_j^2-2\sigma_{0j}^2H_j-\sigma_{0j}^4G_j')\,,$$

$$M_{3j}=\frac{1}{n_1}(4\sigma_{0j}^2B_j+2B_jC_j)+\frac{1}{n_2}(4\sigma_{0j}^2G_j-2\sigma_{0j}^4G_jH_j)\,,$$

$$M_{4j}=\frac{1}{n_1}B_j^2+\frac{1}{n_2}\sigma_{0j}^4G_j^2\,.$$

The c.d.f. $F_2(v)$ is obtained by inverting $\varphi(t/\pi_2)$. Let $G_{kj}(x)$ be the c.d.f. of a non-central chi-square variate with k d.f. and non-centrality parameter $\mu_{0j}^2/\sigma_{0j}^2a_j^2$. Using the same method to obtain $F_1(v)$, we have

$$F_2(v)=\sum_j\{G_{1j}(v)-M_{1j}G_{3j}^{(1)}(v)+M_{2j}G_{5j}^{(2)}(v)-M_{3j}G_{7j}^{(3)}(v)+M_{4j}G_{9j}^{(4)}(v)\}+O_2\,.$$

where $G_{kj}^{(r)}(v)$ is the rth derivative of $G_{kj}(v)$.

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