FURTHER APPLICATIONS OF A DIFFERENTIAL EQUATION FOR HOTELLING'S GENERALIZED T_0^2 *

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(Received Oct. 22, 1969)

1. Summary

The null distribution of the Hotelling-Lawley generalized T_0^2 statistic has been shown [4] to satisfy a homogeneous linear differential equation (d.e.). The latter has been used to tabulate some exact percentage points of T_0^2 by analytic continuation of Constantine's [3] series, and a table for the 5-variate case is presented in this paper. The Ito-Siotani [9], [16] asymptotic expansions for the distribution function and percentage points of T_0^2 are also extended.

2. The differential equation

The T_0^2 statistic of Hotelling [8] and Lawley [11] is defined by

$$T_0^2 = n_2 \operatorname{tr} S_1 S_2^{-1} = n_2 T\,, \tag{2.1}$$

say, where S_1 and S_2 are independent $m\times m$ Wishart matrices based on n_1 and n_2 degrees of freedom, respectively, estimating the same population covariance matrix. In the general case when S_1 has a non-central Wishart distribution, Constantine [3] has given a power-series representation of the density function $f(T)$ of T, which is valid for $0<T<1$. The exact null distribution when $m=2$ was found by Hotelling [8] in terms of the Gaussian hypergeometric function. This result was extended by the present author [4], who showed that $f(T)$ satisfies a d.e. of fuchsian type of order m, with regular singularities at $T=0, -1, -2, \cdots, -m$ and ∞. Constantine's series reduces in the null case to the relevant solution of this d.e. in the unit circle about $T=0$.

When $n_1, n_2 \geqq m$, the d.e. for $f(T)$ is equivalent to a first order matrix d.e. ([4], Section 3):

* This work was partly supported by The National Institutes of Health Grant GM-12868-05.

$$\text{diag}\,\{T, T+1, \cdots, T+m\} d\boldsymbol{M}/dT = \boldsymbol{C}\boldsymbol{M}\,, \tag{2.2}$$

where

$$\boldsymbol{M}(T) = (M_0(T), \cdots, M_m(T))' \tag{2.3}$$

is an $(m+1)$-vector with

$$M_0(T) \equiv f(T)\,, \tag{2.4}$$

$$\sum_{j=0}^{m} (T+j) M_j(T) \equiv 0\,. \tag{2.5}$$

The constant $(m+1)\times(m+1)$ matrix $\boldsymbol{C}$ has the form

$$\boldsymbol{C} = \begin{bmatrix} \alpha_0, & \beta_0, & 0, & 0, & \cdots & \cdots, & 0 \\ \gamma_1, & \alpha_1, & \beta_1, & 0, & \cdots & \cdots, & 0 \\ 0 & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & \gamma_{m-1}, & \alpha_{m-1}, & \beta_{m-1} \\ 0 & \cdots & \cdots & \cdots & 0, & \gamma_m, & \alpha_m \end{bmatrix} \tag{2.6}$$

$$= \{(\beta_0, \beta_1, \cdots, \beta_{m-1}), (\alpha_0, \cdots, \alpha_m), (\gamma_1, \cdots, \gamma_m)\}\,,$$

say, where

$$\begin{aligned} \alpha_i &= [(m-2i)n_1 - in_2 + (2i^2 - mi - i - 2)]/2\,, \\ \beta_i &= (i+1)(n_1+n_2-i)/2\,, \\ \gamma_i &= -(m-i+1)(n_1-i+1)/2\,. \end{aligned} \tag{2.7}$$

Since (2.2) may be rewritten as

$$d\boldsymbol{M}/dT = \left\{ \sum_{r=0}^{m} (T+r)^{-1} \boldsymbol{V}_r \right\} \boldsymbol{M}\,, \tag{2.8}$$

where $\boldsymbol{V}_r$ is obtained by replacing all elements of $\boldsymbol{C}$ by zeros except those in the rth row $(r=0, 1, \cdots, m)$, it follows from the general theory of such systems ([2], Chapter 4) that the d.e. has regular singularities at $T=0, -1, \cdots, -m$ and ∞. The $(m+1)$ linearly independent solutions of (2.2) in the unit circle about $T=0$ correspond to the latent roots of $\boldsymbol{V}_0$, i.e. zero (with multiplicity m) and

$$a_0 = mn_1/2 - 1\,. \tag{2.9}$$

In virtue of Constantine's result [3], the relevant solution is given by the latter root:

$$M(T)=k(m;n_1,n_2)T^{mn_1/2-1}\sum_{j=0}^{\infty}W_jT^j\,,\qquad (W_j=(W_{0j},\cdots,W_{mj})';\ W_{00}=1,\ |T|<1)\,. \tag{2.10}$$

Here the $M_0(T)$ component is the null case of Constantine's series, and ([3], equation (2)),

$$k(m;n_1,n_2)=\Gamma_m((n_1+n_2)/2)/\Gamma(mn_1/2)\Gamma_m(n_2/2)\,, \tag{2.11}$$

where

$$\Gamma_m(z)=\pi^{m(m-1)/4}\prod_{i=0}^{m-1}\Gamma(z-i/2)\,. \tag{2.12}$$

The following recurrence relations for the W_{ij} are obtained from (2.2):

$$\begin{aligned} &W_{i0}=\delta_{i0}\,, \qquad \text{(Kronecker's delta)};\\ &i(j+mn_1/2-1)W_{ij}=\gamma_iW_{i-1,j}+[\alpha_i-(j+mn_1/2-2)]W_{i,j-1}\\ &\qquad\qquad +\beta_iW_{i+1,j-1}\,, \qquad (i=1,\cdots,m;\ j=1,2,\cdots);\\ &jW_{0j}=(n_1+n_2)W_{1j}/2\,. \end{aligned} \tag{2.13}$$

The d.e. (2.2) may be used to carry out an analytic continuation of the solution (2.10) along the positive T axis, noting that at any "ordinary" point $T^*>0$ it is sufficient to know $M(T^*)=(M_0(T^*),\cdots,M_m(T^*))'$ in order to construct the solution in the neighbourhood of T^*. A computer program has been written which effects this analytic continuation, and calculates percentage points of T_0^2/n_1 by the Newton-Raphson method. Table 1 presents results for $m=5$. Brief tables for $m=3$ and 4 will be reported elsewhere [5]. For $n_1<m$, the distribution of T is obtained by the transformations [3]

$$n_1\to m\,,\qquad m\to n_1\,,\qquad n_2\to n_1+n_2-m\,, \tag{2.14}$$

so that Table 1 may also be used for $n_1=5$, and values of m between 6 and 60.

As a check on the accuracy of the program, the range of T was mapped onto the unit interval (0, 1) by $Y=T/(T+1)$, and the percentage points recalculated using the transformed d.e.. For $m=2, 3, 4$ and 5, $n_1\leqq 60$ and $n_2\geqq m$, the results agreed to five significant figures, except for small n_2 and large n_1. However, even for such relatively extreme values of the parameters as $m=5$, $n_1=40$, $n_2=20$, the values arrived at (a) by analytic continuation of Constantine's series and (b) after transformation onto (0, 1), were

Table 1. Upper percentage points for

	$n_2 \backslash n_1$	5	6	8	10
5%	5	81.991 +	83.352 +	85.093 +	86.160 +
	6	3.0093+	3.0142+	3.0204+	3.0241+
	7	93.762	93.042	92.102	91.515
	8	51.339	50.646	49.739	49.170
	10	27.667	27.115	26.387	25.927
	12	20.169	19.701	19.079	18.683
	14	16.643	16.224	15.666	15.309
	16	14.624	14.239	13.722	13.389
	18	13.326	12.963	12.476	12.161
	20	12.424	12.078	11.612	11.310
	25	11.046	10.728	10.297	10.016
	30	10.270	9.9689	9.5592	9.2907
	35	9.7739	9.4836	9.0879	8.8277
	40	9.4292	9.1469	8.7613	8.5070
	50	8.9825	8.7107	8.3385	8.0921
	60	8.7057	8.4406	8.0769	7.8355
	70	8.5174	8.2570	7.8991	7.6612
	80	8.3811	8.1241	7.7705	7.5351
	100	8.1969	7.9446	7.5969	7.3649
	200	7.8505	7.6070	7.2706	7.0451
	∞	7.5305	7.2955	6.9698	6.7505
1%	5	20.495 *	20.834 *	21.267 *	21.53 *
	6	15.014 +	15.019 +	15.025 +	15.029 +
	7	2.7354+	2.7045+	2.6646+	2.6400+
	8	1.1498+	1.1276+	1.0989+	1.0811+
	10	48.048	46.670	44.877	43.758
	12	31.108	30.065	28.701	27.846
	14	24.016	23.145	22.001	21.279
	16	20.240	19.472	18.459	17.817
	18	17.929	17.228	16.302	15.713
	20	16.380	15.727	14.862	14.310
	25	14.107	13.529	12.759	12.265
	30	12.880	12.345	11.629	11.167
	35	12.115	11.607	10.926	10.486
	40	11.593	11.105	10.448	10.022
	50	10.928	10.465	9.8408	9.4336
	60	10.523	10.076	9.4712	9.0758
	70	10.251	9.8142	9.2229	8.8354
	80	10.055	9.6261	9.0446	8.6629
	100	9.7929	9.3742	8.8058	8.4319
	200	9.3055	8.9065	8.3629	8.0036
	∞	8.8628	8.4820	7.9613	7.6154

+ Multiply entry by 100. * Multiply entry by 10^4.

	(a)	(b)
Upper 5% point	10.25171	10.25169
Upper 1% point	12.43142	12.43134.

The entries given in Table 1 are 10.252 and 12.431, respectively. It may be noted that the null distribution of Pillai's trace [13]

$$V = \text{tr}\, S_1(S_1+S_2)^{-1} \tag{2.15}$$

satisfies a d.e. obtained from (2.2) by the transformations

Hotelling's generalized T_0^2/n_1, ($m=5$)

12	15	20	25	40	60
86.88 +	—	—	—	—	—
3.0266+	3.0291+	3.032 +	—	—	—
91.113	90.705	90.29	90.04	—	—
48.780	48.382	47.973	47.723	47.35	—
25.610	25.284	24.947	24.740	24.422	—
18.409	18.124	17.830	17.647	17.365	17.20
15.059	14.800	14.530	14.361	14.100	13.95
13.157	12.914	12.659	12.499	12.250	12.105
11.939	11.708	11.463	11.310	11.068	10.928
11.097	10.874	10.637	10.488	10.252	10.113
9.8168	9.6061	9.3814	9.2386	9.0102	8.8745
9.0995	8.8964	8.6785	8.5389	8.3141	8.1790
8.6419	8.4437	8.2301	8.0926	7.8693	7.7339
8.3250	8.1303	7.9195	7.7833	7.5607	7.4247
7.9150	7.7248	7.5177	7.3829	7.1605	7.0229
7.6615	7.4741	7.2692	7.1351	6.9124	6.7730
7.4894	7.3039	7.1004	6.9667	6.7434	6.6024
7.3648	7.1807	6.9782	6.8448	6.6208	6.4785
7.1968	7.0145	6.8133	6.6801	6.4550	6.3103
6.8811	6.7023	6.5032	6.3702	6.1416	5.9908
6.5902	6.4144	6.2171	6.0838	5.8499	5.6899
—	—	—	—	—	—
15.033 +	15.03 +	15.06 +	—	—	—
2.6232+	2.6064+	2.590 +	2.579 +	—	—
1.0689+	1.0567+	1.0440+	1.0364+	—	—
42.992	42.210	41.408	40.921	—	—
27.257	26.653	26.031	25.648	25.06	24.71
20.781	20.268	19.736	19.408	18.90	18.61
17.373	16.913	16.435	16.138	15.678	15.412
15.304	14.878	14.435	14.159	13.727	13.478
13.925	13.525	13.105	12.843	12.431	12.192
11.918	11.555	11.172	10.930	10.547	10.322
10.842	10.500	10.136	9.9059	9.5378	9.3188
10.174	9.8453	9.4944	9.2706	8.9106	8.6946
9.7204	9.4006	9.0581	8.8387	8.4838	8.2691
9.1441	8.8361	8.5041	8.2901	7.9404	7.7261
8.7938	8.4930	8.1674	7.9563	7.6090	7.3940
8.5586	8.2626	7.9411	7.7319	7.3858	7.1697
8.3899	8.0973	7.7787	7.5708	7.2251	7.0078
8.1638	7.8758	7.5611	7.3547	7.0093	6.7897
7.7448	7.4652	7.1572	6.9532	6.6062	6.3798
7.3650	7.0929	6.7903	6.5878	6.2361	5.9984

$$T \to -V, \qquad n_2 \to m-n_1-n_2+1, \tag{2.16}$$

m and n_1 remaining unchanged. This d.e. therefore has regular singularities at $0, 1, 2, \cdots, m$ and ∞, i.e. within the range $(0, m)$ of V. However, the program written for the present paper may with trivial modifications be used to calculate accurate percentage points of V in the ranges $(0, 1)$ and $(m-1, m)$, and some investigation has been made of approximations to its distribution [6].

3. Extension of the Ito-Siotani asymptotic expansions

As $n_2 \to \infty$, the null distribution of T_0^2 approaches the chi-squared distribution on mn_1 degrees of freedom, and we may show that its cumulant generating function

$$K(\theta)=\log E \exp (i\theta T_0^2) \qquad (\theta \text{ real}) \tag{3.1}$$

may be developed in an asymptotic expansion for large n_2 of the type considered by Box [1]:

$$K(\theta)-(mn_1/2) \log (1-2i\theta)+\sum_{r=1}^{\infty} \omega_r[(1-2i\theta)^{-r}-1] . \tag{3.2}$$

The ω_r are functions of m, n_1 and n_2, and will be given below to order n_2^{-4}. Writing $t=T_0^2$, introduce the vector of functions

$$\begin{aligned} \boldsymbol{N}(t)&=(N_0(t), \cdots, N_m(t))' \\ &=n_2 \operatorname{diag} \{1, n_2, \cdots, n_2^m\} \boldsymbol{M}(t/n_2) , \end{aligned} \tag{3.3}$$

$N_0(t)$ being the density function of t. Then $\boldsymbol{N}$ may be shown [4] to satisfy the d.e.:

$$\operatorname{diag} (n_2^{-1}, n_2^{-1}t+1, \cdots, n_2^{-1}t+m)d\boldsymbol{N}/dt=[\boldsymbol{\Delta}_0+n_2^{-1}\boldsymbol{\Delta}_1+n_2^{-2}\boldsymbol{\Delta}_2]\boldsymbol{N} , \tag{3.4}$$

where

$$\begin{aligned} \boldsymbol{\Delta}_0&= \{(0, \cdots, 0), (0, -1/2, -1, \cdots, -m/2), (\gamma_1, \cdots, \gamma_m)\} , \\ \boldsymbol{\Delta}_1&= \{(1/2, 1, \cdots, m/2), (\bar{\alpha}_0, \cdots, \bar{\alpha}_m), (0, \cdots, 0)\} , \\ \boldsymbol{\Delta}_2&= \{(\bar{\beta}_0, \cdots, \bar{\beta}_{m-1}), (0, \cdots, 0), (0, \cdots, 0)\} , \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \bar{\alpha}_i&=[(m-2i)n_1+2i^2-mi-i-2]/2 , \\ \bar{\beta}_i&=(i+1)(n_1-i)/2 . \end{aligned} \tag{3.6}$$

If we take the Fourier transform of $\boldsymbol{N}$,

$$\boldsymbol{C}(\theta)=\int_0^{\infty} e^{it\theta}\boldsymbol{N}(t)dt=(C_0(\theta), \cdots, C_m(\theta))' , \qquad (\theta \text{ real}) , \tag{3.7}$$

so that $C_0(\theta)$ is the characteristic function of t, then $\boldsymbol{C}$ clearly satisfies a first-order matrix d.e. with respect to $i\theta$ which we omit.

Let

$$\begin{aligned} Q(\theta)&=\log [(1-2i\theta)^{mn_1/2}C_0(\theta)]=(mn_1/2) \log (1-2i\theta)+K(\theta) , \\ R_j(\theta)&=C_j(\theta)/C_0(\theta) , \qquad (j=1, \cdots, m) , \end{aligned} \tag{3.8}$$

and assume expansions of the form

$$Q(\theta)\sim\sum_{r=1}^{\infty}\omega_r[(1-2i\theta)^{-r}-1]$$
(3.9)
$$R_j(\theta)\sim\sum_{r=0}^{\infty}\xi_{j,r}[(1-2i\theta)^{-r}-1]\,,\qquad (j=1,\cdots,m)\,.$$

The following recurrence relations may be obtained:

(3.10) $$2r\omega_r=2(r-1)\omega_{r-1}+mn_1\delta_{1,r}+(1+n_2^{-1}n_1)\xi_{1,r}\,,$$

where the $\xi_{j,r}$ are given recursively by:

(3.11)
$$\begin{aligned}
&\xi_{0,r}=\xi_{r,0}=\delta_{0,r}\,,\\
&j\xi_{j,r}=2\gamma_j\xi_{j-1,r-1}-n_2^{-1}[\lambda_j+2(r-1)]\xi_{j,r-1}\\
&\qquad+[n_2^{-1}(j+1)2n_2^{-2}\bar{\beta}_j]\xi_{j+1,r-1}+n_2^{-1}[mn_1+2(r-2)]\xi_{j,r-2}\\
&\qquad+2n_2^{-1}\sum_{s=1}^{r-2}s\omega_s(\xi_{j,r-s-2}-\xi_{j,r-s-1})\,,\\
&\qquad\qquad\qquad\qquad(j=1,\cdots,m;\ r=1,2,\cdots)\,,\\
&\lambda_j=j(m+2n_1-2j+1)\,,\qquad\qquad (j=1,\cdots,m)\,.
\end{aligned}$$

The first eight ω_r are necessary to give the expansion (3.2) to order n_2^{-4}:

(3.12)
$$\begin{cases}
\omega_1=-mn_1^2/2n_2\,,\\
\omega_2=(1/4)mn_1[n_2^{-1}(m+n_1+1)+n_2^{-2}n_1(m+2n_1+1)]\,,\\
\omega_3=-(1/6)mn_1\{(n_2^{-2}+n_2^{-3}n_1)[m^2+3m(2n_1+1)\\
\qquad\qquad+(4n_1^2+6n_1+4)]+n_2^{-3}n_1^3\}\,,\\
\omega_4=(1/8)mn_1\{n_2^{-2}[2m^2+5m(n_1+1)+(2n_1^2+5n_1+5)]\\
\qquad+n_2^{-3}[m^3+2m^2(7n_1+3)+m(34n_1^2+39n_1+21)\\
\qquad\quad+(15n_1^3+34n_1^2+47n_1+20)]\\
\qquad+n_2^{-4}n_1[m^3+6m^2(2n_1+1)+m(29n_1^2+34n_1+21)\\
\qquad\quad+(14n_1^3+29n_1^2+42n_1+20)]\}\,,\\
\omega_5=-(1/10)mn_1\{5n_2^{-3}[m^3+m^2(7n_1+5)+2m(5n_1^2+9n_1+7)\\
\qquad+(3n_1^3+10n_1^2+19n_1+12)]+n_2^{-4}[m^4+5m^3(5n_1+2)\\
\qquad\quad+5m^2(26n_1^2+27n_1+13)+10m(18n_1^3+35n_1^2+42n_1+16)\\
\qquad\quad+4(14n_1^4+45n_1^3+100n_1^2+95n_1+37)]+0(n_2^{-5})\}\,,\\
\omega_6=(1/12)mn_1\{n_2^{-3}[5m^3+22m^2(n_1+1)+2m(11n_1^2+27n_1+26)\\
\qquad+(5n_1^3+22n_1^2+52n_1+41)]+n_2^{-4}[9m^4+15m^3(8n_1+5)\\
\qquad\quad+m^2(388n_1^2+585n_1+397)+3m(122n_1^3+332n_1^2+531n_1+289)
\end{cases}$$

$$+(84n_1^4+366n_1^3+1048n_1^2+1350n_1+732)]+0(n_2^{-5})\} ,$$

$$\omega_7=-(1/2)mn_1[3m^4+m^3(27n_1+22)+m^2(62n_1^2+122n_1+100)$$
$$+m(44n_1^3+154n_1^2+299n_1+199)$$
$$+4(2n_1^4+11n_1^3+38n_1^2+60n_1+39)]/n_2^4+0(n_2^{-5}) ,$$

$$\omega_8=(1/16)mn_1[14m^4+93m^3(n_1+1)+m^2(164n_1^2+398n_1+374)$$
$$+m(93n_1^3+398n_1^2+899n_1+690)$$
$$+(14n_1^4+93n_1^3+374n_1^2+690n_1+509)]/n_2^4+0(n_2^{-5}) .$$

For $r\geqq 9$, the ω_r do not exceed $0(n_2^{-5})$; apparently, $\omega_r=0(n_2^{-[r/2]})$ where [ ] denotes the integer part. The first six ω_r have been given to $0(n_2^{-3})$ by Muirhead [12], using partial d.e.'s for Constantine's hypergeometric functions of matrix argument. Exponentiation of (3.2), followed by inversion of the resulting linear combination of chi-squared characteristic functions, in principle yields an extension of Ito's and Siotani's expansions of the cumulative distribution function of T_0^2 in the null case. Their formula for T_0^2 percentiles in terms of percentage points, u say, of $\chi^2_{mn_1}$ may be extended using a general inversion of Box-type expansions [7] and the above ω_r. To order n_2^{-3}:

$$\begin{aligned}(3.13)\quad T_0^2\sim\; & u+(1/2n_2)[u(m-n_1+1)+u^2(m+n_1+1)/(mn_1+2)]\\
&+(1/24n_2^2)\{u[7m^2-12m(n_1-1)+(7n_1^2-12n_1+1)]\\
&+u^2[13m^2+24m-11n_1^2+7]/(mn_1+2)\\
&+u^3[4m^3n_1+2m^2(3n_1^2+3n_1+10)+2m(2n_1^3+3n_1^2+17n_1+18)\\
&+4(5n_1^2+9n_1+2)]/(mn_1+2)^2(mn_1+4)\\
&+6u^4(m-1)(m+2)(n_1-1)(n_1+2)/(mn_1+2)^2(mn_1+4)(mn_1+6)\}\\
&+(1/48n_2^3)\{3u[3m^3-7m^2(n_1-1)+m(7n_1^2-12n_1+1)\\
&-(3n_1^3-7n_1^2+n_1+3)]+u^2[23m^3-m^2(19n_1-59)\\
&-m(13n_1^2+36n_1-29)+(17n_1^3-13n_1^2-13n_1-7)]/(mn_1+2)\\
&+2u^3[7m^4n_1+2m^3(2n_1^2+8n_1+17)-m^2(2n_1^3-9n_1^2-29n_1-88)\\
&-m(5n_1^4+2n_1^3+13n_1^2-46n_1-46)-(26n_1^3+20n_1^2-22n_1+8]/\\
&(mn_1+2)^2(mn_1+4)+2u^4[m^5n_1^2+2m^4n_1(7n_1^2+7n_1-6)\\
&-m^3(4n_1^4-21n_1^3-83n_1^2+4)+m^2(n_1^5-4n_1^4-7n_1^3+70n_1^2+196n_1+16)\\
&+4m(6n_1^4+4n_1^2+57n_1+25)+4(17n_1^3+22n_1^2-11n_1+20)]/\\
&(mn_1+2)^3(mn_1+4)(mn_1+6)-4u^5(m-1)(m+2)(n_1-1)(n_1+2)\\
&\times[m^2n_1+m(n_1^2+7n_1-28)-4(7n_1+4)]/(mn_1+2)^3(mn_1+4)\\
&\times(mn_1+6)(mn_1+8)+8u^6(m-1)(m+2)(n_1-1)(n_1+2)\\
&\times[m^2n_1+m(n_1^2+4n_1-10)-(10n_1+4)]/(mn_1+2)^3(mn_1+4)\\
&\times(mn_1+6)(mn_1+8)(mn_1+10)\}+0(n_2^{-4}) .\end{aligned}$$

We note also that, corresponding to (2.16), the cumulant generating function of Pillai's n_2V may be expanded in the form (3.2), with coefficients $\omega_{r,v}$ related to those for T_0^2 by

$$(3.14) \qquad (-n_2)^r\omega_{r,v} = mn_1(n_1-m-1)^r/2r + \sum_{s=1}^{r}\binom{r-1}{s-1}(m-n_1-n_2+1)^s(n_1-m-1)^{r-s}\omega_s^*,$$

where ω_s^* is obtained by replacing n_2 by $m-n_1-n_2+1$ in ω_s. The $\omega_{r,v}$ thus specified agree with those given by Muirhead [12] to order n_2^{-3}. A percentile expansion for n_2V corresponding to (3.13) may also be derived [6].

4. Discussion of approximations

Another approach to approximating the distribution of T_0^2 has been taken by Pillai and his associates ([14], [15], [17]). They have given formulas for the moment-ratios β_1 and β_2 required for fitting a Pearson curve. The rth central moment μ_r of T_0^2 exists if $r<(n_2-m+1)/2$, ([3], Section 5), so that β_1 and β_2 are defined if $n_2>m+7$, independently of the value of n_1. In the present notation,

$$(4.1) \qquad \beta_1 = \mu_3^2/\mu_2^3 = \frac{8(n_2-m-3)(n_2-m)(n_2+m-1)^2(2n_1+n_2-m-1)^2}{mn_1(n_2-1)(n_2-m-5)^2(n_2-m+1)^2(n_1+n_2-m-1)}.$$

Also, writing

$$(4.2) \qquad c = n_2-m-1,$$

the following reduced form of β_2 has been derived by the present author from the recurrence relations for moments given in [4], Section 7.

$$(4.3) \qquad \beta_2 = \mu_4/\mu_2^2 = \frac{3(c-2)(c+1)A}{mn_1(n_2-1)(c-6)(c-4)(c-1)(c+2)(c+3)(c+n_1)},$$

where

$$(4.4) \qquad A = n(c+n_1)[m(n_2-1)(c^3-5c^2+78c-72)+4c^2(5c-6)] + 4c^2[m(n_2-1)(5c-6)+c(c^2-c+2)].$$

The mean and variance of T_0^2 are also required in fitting a Pearson curve:

$$(4.5) \qquad \begin{aligned} \mu_1 &= mn_1n_2/(n_2-m-1), \\ \mu_2 &= 2mn_1n_2^2(n_2-1)(n_1+n_2-m-1)/(n_2-m-3)(n_2-m-1)^2(n_2-m). \end{aligned}$$

It may be noted that the transformation (2.16) converts the above quantities into those required for fitting a Pearson curve to V.

The extended tables of the Pearson curve given in [10] have been used to compute the Pillai approximation to T_0^2, and its accuracy has been compared with that of the Ito-Siotani approximation. The results will be presented in diagram form in [5]. Briefly, the Pillai method gives remarkably accurate percentage points for $m \leqq 5$ provided n_1 and n_2 are not both small. On the other hand, the accuracy of the Ito-Siotani approximations (3.13) falls away as n_1 increases. However, the latter has the advantage of being a direct formula, and its accuracy is considerably improved by the addition of the n_2^{-3} term. For $m \leqq 4$, $n_1 \leqq 30$, formula (3.13) achieves 3-decimal place accuracy for the 1% points of T_0^2/n_1 for smaller values of n_2 than Pillai's approximation.

Acknowledgement

The author's thanks are due to Miss J. Hawkes of CSIRO for writing the computer program and preparing the tables.

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