# ESTIMATION OF THE RECIPROCAL OF SCALE PARAMETER OF A GAMMA DENSITY*

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## 1. Introduction and summary

Let $Y$ be a random variable having gamma density

$$f(y;\beta)=\begin{cases}\dfrac{1}{\beta^n\Gamma(n)}e^{-y/\beta}y^{n-1}, & \text{if } y\geqq 0,\\ 0, & \text{elsewhere},\end{cases} \tag{1}$$

where $\beta>0$ is the scale parameter and $n>-1$. In this note we are concerned with the estimation of $1/\beta$. This situation can arise when, for instance, one is interested in the waiting time $X$ to the $n$th event in a series of events happening in accordance with a Poisson probability law at the rate of $1/\beta=\lambda$ events per unit of time. The random variable $X$ has the density

$$f(x;\lambda)=\begin{cases}\dfrac{\lambda^n}{\Gamma(n)}e^{-\lambda x}x^{n-1}, & x\geqq 0,\\ 0, & \text{otherwise},\end{cases} \tag{2}$$

which is the same as (1) except that $\lambda=1/\beta$. From now on it is this form of gamma density which we assume to be given and we shall be interested in the estimation of the parameter $\lambda$.

We prove that for $n\leqq 1$ there does not exist any unbiased estimator for $\lambda$ and for $n\leqq 2$ there does not exist one with finite variance. The uniformly minimum variance unbiased estimator $(n-1)/X$ $(n>2)$ is seen to be inadmissible compared with $(n-2)/X$ when the loss is squared error. In this case $\delta^*(X)=(n-2)/X$ is an admissible estimator of $\lambda$. This estimator is the unique admissible minimax estimator when the loss is squared error divided by $\lambda^2$. The minimaxity of $\delta^*$, for the case $n=3$, is shown to hold even when the parameter space is either $[\lambda_0,\infty)$ or $(0,\lambda_0]$, $\lambda_0$ a fixed number, but it is then no longer admissible.

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Our result on admissibility can be deduced from Theorem 5 of Farrell [1]. However we felt a self contained proof is more desirable. Our approach is similar to that of Karlin [3]. A different proof based on Schwarz inequality was found by Blyth and will be published separately by him.

## 2. Estimation of $\lambda$

Suppose $X$ is a random variable having density (2). The following theorem shows the non-existence of an unbiased estimator for $n \leqq 1$.

THEOREM 2.1. *Let $n \leqq 1$. Then there does not exist any unbiased estimator of $\lambda$.*

Proof of this follows from Theorem 2.2 [2]. We have only to notice that the family of densities (2) is complete, that for $\lambda=1$, $E(1/X)$ does not exist for $n \leqq 1$ and that $\lambda$ is the scale parameter of the distribution of $1/X$.

It is easy to check that for $n>1$, $(n-1)/X$ is an unbiased estimator of $\lambda$. But for $n \leqq 2$, the following result is true.

THEOREM 2.2. *Let $n \leqq 2$. Then there does not exist any unbiased estimator of $\lambda$ with finite variance.*

Proof of this is similar to that of Theorem 2.1.

As remarked earlier, $(n-1)/X$, $n \geqq 2$, is an unbiased estimator of $\lambda$. In fact, for $n>2$, it is a uniformly minimum variance unbiased estimator of $\lambda$. This follows from the fact that $X$ is a sufficient and complete statistic for $\lambda$. But this estimator is inadmissible as the estimator $\delta^*(X) =(n-2)/X$ uniformly improves upon it. In fact the estimator $\delta^*$ is the best one (with respect to squared error loss) among all estimators of the form $\gamma/X$, where $\gamma$ is a constant. We now prove that $\delta^*$ is admissible.

THEOREM 2.3. *If the loss is squared error divided by $\lambda^2$ then the estimator $\delta^*(X)=(n-2)/X$ is the unique admissible minimax estimator of $\lambda$.*

PROOF. We shall show admissibility. The rest of the theorem is an easy consequence of this fact and the constant risk of $\delta^*$.

Let $\delta$ be, if possible, any other estimator better than $\delta^*$. This implies that the inequality

$$\int [\delta(x)-\lambda]^2 \frac{1}{\lambda^2} f(x;\lambda)dx \leqq \int [\delta^*(x)-\lambda]^2 \frac{1}{\lambda^2} f(x;\lambda)dx$$

must be true for all $\lambda$ and strict for at least one $\lambda$. The above inequality simplifies to

$$(3)\qquad \int_0^\infty [\delta(x)-\delta^*(x)]^2\lambda^{n-2}e^{-\lambda x}x^{n-1}dx \leqq 2\int_0^\infty [\delta^*(x)-\delta(x)][\delta^*(x)-\lambda]\lambda^{n-2}x^{n-1}e^{-\lambda x}dx\,.$$

Let $dF(\lambda)=d\lambda/\lambda$ and let $0<a<b<\infty$. Define $T(\lambda)=\int_0^\infty [\delta^*(x)-\delta(x)]^2 \cdot\lambda^n e^{-\lambda x}x^{n-1}dx$. Then using the fact that $\delta$ is better than $\delta^*$ and $\int_0^\infty [\delta^*(x)-\lambda]^2\frac{\lambda^n}{\Gamma(n)}e^{-\lambda x}x^{n-1}dx=\frac{\lambda^2}{n-1}$ we get, by Schwarz's inequality,

$$(4)\qquad \int_0^\infty [\delta^*(x)-\delta(x)][\delta^*(x)-\lambda]\lambda^n e^{-\lambda x}x^{n-1}dx\leqq k'\lambda^2$$

and similarly

$$(5)\qquad T(\lambda)\leqq k\lambda^2\,,$$

where $k$, $k'$ are constants depending on $n$. Now integrating both sides of (3) with respect to $dF$ we get

$$\int_a^b T(\lambda)\frac{1}{\lambda^3}d\lambda\leqq 2\int_a^b\left[\int_0^\infty [\delta^*(x)-\delta(x)][\delta^*(x)-\lambda]\lambda^{n-2}e^{-\lambda x}x^{n-1}dx\right]\frac{1}{\lambda}d\lambda$$

$$=2\int_0^\infty [\delta^*(x)-\delta(x)]\left\{\int_a^b [\delta^*(x)-\lambda]\lambda^{n-3}e^{-\lambda x}d\lambda\right\}x^{n-1}d\lambda^{1)}$$

$$=2\int_0^\infty [\delta^*(x)-\delta(x)][b^{n-2}e^{-bx}-a^{n-2}e^{-ax}]x^{n-2}dx$$

$$\leqq 2\left\{\int_0^\infty |\,\delta^*(x)-\delta(x)\,|\,b^{n-2}e^{-bx}x^{n-2}dx + \int_0^\infty |\,\delta^*(x)-\delta(x)\,|\,a^{n-2}e^{-ax}x^{n-2}dx\right\}$$

$$\leqq 2\left\{\left[\int_0^\infty (\delta^*(x)-\delta(x))^2b^{n-2}e^{-bx}x^{n-1}dx\right]^{1/2}\left[\int_0^\infty b^{n-2}e^{-bx}x^{n-3}dx\right]^{1/2} + \left[\int_0^\infty (\delta^*(x)-\delta(x))^2a^{n-2}e^{-ax}x^{n-1}dx\right]^{1/2}\left[\int_0^\infty a^{n-2}e^{-ax}x^{n-3}dx\right]^{1/2}\right\}$$

and this implies

$$(6)\qquad \int_a^b T(\lambda)\lambda^{-3}d\lambda\leqq 2\left[\frac{1}{b}\sqrt{T(b)}+\frac{1}{a}\sqrt{T(a)}\right]\Gamma(n-2)^{1/2}\,.$$

The rest of the proof consists of showing that the R.H.S. of (6) tends to 0 as $a\to 0$ and $b\to\infty$. From (4) and (6) it follows

1) The change in the order of integration is justified by (4).

$$\int_0^\infty T(\lambda)\lambda^{-3}d\lambda \leqq 4\sqrt{K}\Gamma(n-2)^{1/2}. \tag{7}$$

Now if for $\varepsilon>0$ there is a $\lambda_0$ such that for $\lambda>\lambda_0$, $T(\lambda)/\lambda^2>\varepsilon$ then $\int_0^\infty T(\lambda)\lambda^{-3}d\lambda > \int_{\lambda_0}^\infty \varepsilon\lambda^{-1}d\lambda=\infty$ which is impossible on account of (7). There exists, therefore, a sequence of real numbers $\{b_i\}\to\infty$ such that $T(b_i)/b_i^2 \to 0$. Similarly we can show that there exists a sequence of real numbers $\{a_i\}\to 0$ such that $T(a_i)/a_i^2\to 0$. These two together and (7) imply that $\int_0^\infty T(\lambda)\lambda^{-3}d\lambda=0$ and hence $\delta(x)=\delta^*(x)$ a.e. proving admissibility of $\delta^*$.

COROLLARY. *$\delta^*$ is admissible with respect to squared error loss.*

It may be remarked that the minimaxity of $\delta^*(x)=(n-2)/X$ with respect to the loss squared error divided by $\lambda^2$ also follows from Lehmann's well-known result [4], by choosing the *a priori* distributions on $\lambda$ as

$$g_{\mu,v}(\lambda)=\frac{e^{-\mu\lambda}\lambda^{-1+v}}{\Gamma(v)/\mu^v}, \qquad 0<\lambda<\infty.$$

Minimaxity also follows from invariance.

Incidentally notice that in the proof of Theorem 2.3 we have been guided by the hope that $\delta^*$ is Bayes in the wide sense with respect to $dF(\lambda)=d\lambda/\lambda$, $0<a<\lambda<b$ as $a\to 0$, $b\to\infty$ but we have not made any use of this Bayes property and indeed we do not know how to prove it for all $n$; the case $n=3$ is easy to handle and the wide sense Bayes property among estimators with bounded risk does follow from our proof.

## 3. Case of truncated parameter space

Let $\lambda\in\Omega_1=[\lambda_0,\infty]$ or $\Omega_2=(0,\lambda_0]$ where $\lambda_0$ is a fixed real number. We shall show that the estimator $\delta^*$ remains minimax for these truncated parameter spaces.

THEOREM 3.1. *Let $\lambda\in\Omega_1$ or $\lambda\in\Omega_2$ and let the loss be squared error divided by $\lambda^2$. Then $\delta^*(X)=(n-2)/X$, $n=3$, is a minimax estimator for $\lambda$.*

PROOF. Let $0<a<b$ and let $g_{a,b}(\lambda)=k/\lambda$ if $a<\lambda<b$, and equal to zero otherwise, be an *a priori* distribution over the given parameter space. Here $k=[\log(b/a)]^{-1}$. The Bayes estimator $\delta_{a,b}$ corresponding to $g_{a,b}$ is given by

$$\delta_{a,b}(x)=\left(\int_a^b \lambda\cdot\lambda^3 e^{-\lambda x}x^2\lambda^{-3}d\lambda\right)\Big/\left(\int_a^b \lambda^3 e^{-\lambda x}x^2\lambda^{-3}d\lambda\right)$$

$$=\frac{1}{x}+(ae^{-ax}-be^{-bx})/(e^{-ax}-e^{-bx})\,.$$

If $r(\delta_{a,b})$ is the Bayes risk corresponding to $g_{a,b}$ then

$$r(\delta_{a,b})=\int_a^b\left[\int_0^\infty [\delta_{a,b}(x)-\lambda]^2\lambda^{-2}\frac{\lambda^3}{2}x^2e^{-\lambda x}dx\right]\frac{K}{\lambda}d\lambda\,,$$

which after change of order of integration simplifies to

$$\begin{aligned}r(\delta_{a,b})&=\frac{1}{2}-\frac{1}{2}\left[\log\frac{b}{a}\right]^{-1}\int_0^\infty x[e^{-ax}-e^{-bx}]^{-1}[ae^{-ax}-be^{-bx}]^2dx\\&>\frac{1}{2}-\frac{1}{2}\left[\log\frac{b}{a}\right]^{-1}\left[(b-a)^2\left(\frac{1}{b(b-a)}+\frac{1}{b^2}\right)\right]\\&=\frac{1}{2}-\frac{1}{2}b^{-1}(b-a)\left[\log\frac{b}{a}\right]^{-1}-\frac{1}{2}b^{-2}(b-a)\left[\log\frac{b}{a}\right]^{-1}.\end{aligned}$$

From this it follows that

$$\lim_{a\to 0} r(\delta_{a,b})\geqq\frac{1}{2}\qquad\text{and}\qquad\lim_{b\to\infty} r(\delta_{a,b})\geqq\frac{1}{2}\,.$$

Since $r(\delta_{a,b})\leqq$risk of $\delta^*$ (which is equal to 1/2) for all $a$, $b$, $\lambda$ it follows that $\lim_{a\to 0} r(\delta_{a,b})=1/2=\lim_{b\to\infty} r(\delta_{a,b})$ proving thereby that $\delta^*$ is minimax for the truncated parameter spaces $\Omega_1$ and $\Omega_2$.

That $\delta^*$ is not admissible follows from a comparison with its truncated version. But even the latter is not admissible which can be shown as in Sacks [5].

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