ASYMPTOTICALLY NEARLY EFFICIENT PROCEDURES FOR BIVARIATE LOCATION PARAMETERS

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1. Introduction

Ogawa [4] obtained the asymptotically minimum variance linear asymptotically unbiased estimator (ABLUE) for location or scale from a chosen set of sample quantiles. It was soon observed (Tischendorf [7]) that the asymptotic variance of Ogawa's estimator is essentially the reciprocal of a Riemann sum for the information integral for the parameter being estimated. Thus under mild regularity conditions the ABLUE approaches asymptotic efficiency as larger sets of more closely spaced quantiles are chosen for use.

In [3] the author examined several analogs of Ogawa's estimators for multivariate location parameters. The basic idea was to use the ABLUE from chosen sets of sample quantiles in each direction and from the observed cell frequencies in the random partition generated by these quantiles. These classes of estimators are *asymptotically nearly efficient* (ANE) in the sense that for every $\varepsilon>0$ there is an estimator in the class with asymptotic efficiency $>1-\varepsilon$. Here efficiency is measured by a comparison of the asymptotic covariance matrix to the inverse of the information matrix.

In Section 2 of the present paper we present another class of ANE estimators for bivariate location parameters. This is the class of ABLUE's from a chosen set of marginal sample quantiles in one coordinate direction and from conditional sample quantiles in the other direction (a precise description is given in Section 2). The resulting estimators are somewhat simpler than those of [3], and are perhaps a more natural generalization of Ogawa's work.

Common estimators of multivariate location parameters have the property that each component of the parameter is estimated using only the corresponding component of the observations. Except in special cases (which, however, include the normal case) such estimators cannot approach asymptotic efficiency, since they use only the information contained in the marginal distributions. The present estimators and those of [3] allow efficiency arbitrarily near 1 to be attained for any smooth location parameter family, but at the cost of computational complexity and probable loss of robustness.

There is a close connection between estimation and testing in location parameter families. Common tests for multivariate location also fail to use information beyond that contained in the marginal distributions. See Bickel [1] and Sen and Puri [6] for typical tests. In Section 3 we show how ANE estimators can be used to obtain ANE tests for location. These have the same advantages and disadvantages relative to more common tests as do the corresponding estimators.

2. Estimation

Let $F(x-\theta_1, y-\theta_2)$ be a bivariate location parameter family with continuous density $f(x-\theta_1, y-\theta_2)$. Choose

$$0=\alpha_0<\alpha_1<\cdots<\alpha_N<\alpha_{N+1}=1$$

and let x_i^* be the population α_i-quantile in the x-direction, with the convention that $x_0^*=-\infty$ and $x_{N+1}^*=\infty$. For each $i=1,\cdots, N+1$ choose

$$0=\beta_{i0}<\beta_{i1}<\cdots<\beta_{i\nu_i}<\beta_{i,\nu_i+1}=1$$

and let y_{ij}^* be the conditional population β_{ij}-quantile, i.e.,

$$P_\theta[Y\leqq y_{ij}^* \mid x_{i-1}^*<X\leqq x_i^*]=\beta_{ij} \qquad j=1\cdots, \nu_i$$

and $y_{i0}^*=-\infty$, $y_{i,\nu_i+1}^*=\infty$. The x_i^* and y_{ij}^* depend on the parameter $\theta=(\theta_1, \theta_2)$ as follows: if x_i and y_{ij} are the corresponding quantiles when $\theta=0$, then $x_i^*=x_i+\theta_1$ and $y_{ij}^*=y_{ij}+\theta_2$.

Let $\xi_1,\cdots, \xi_N$ denote the sample α_i-quantiles from the x-components of a random sample of size n on the population. For each $i=1,\cdots, N+1$ let $\zeta_{i1},\cdots, \zeta_{i\nu_i}$ be the sample β_{ij}-quantiles from the y-components of those observations (x, y) with $\xi_{i-1}<x\leqq\xi_i$. We will find the ABLUE of θ_1 and θ_2 in terms of the ξ_i and ζ_{ij} and show that the resulting class of estimators is ANE. The method of proof is that of Ogawa: we find the joint asymptotic distribution of the ξ_i and ζ_{ij} and hence the least squares estimators of (θ_1, θ_2) from the asymptotic distribution. By standard least squares theory these are the ABLUE's. The only mathematical difficulty is to find the joint asymptotic distribution needed. We handle this by observing that the distribution of the ζ_{ij} conditional on the ξ_i is multinomial.

The lines $x=x_i^*$ and $y=y_{ij}^*$ partition the plane into $\sum_{i=1}^{N+1}(\nu_i+1)$ cells. The probability of an observation on $F(x-\theta_1, y-\theta_2)$ falling into the cell with "northeast corner" (x_i^*, y_{ij}^*) is independent of θ and is given by

$$p_{ij}=F(x_i, y_{ij})-F(x_i, y_{i,j-1})-F(x_{i-1}, y_{ij})+F(x_{i-1}, y_{i,j-1})$$
$$=(\alpha_i-\alpha_{i-1})(\beta_{ij}-\beta_{i,j-1}) .$$

Setting $p_{ij}=\Delta_{ij}F$ defines difference operators Δ_{ij} which we will later apply to other functions. Note that our conventions imply that $F(x_i, y_{i0})=0$ and $F(x_i, y_{i,\nu_i+1})=F_X(x_i)$, where F_X is the marginal cdf in the x-direction. If F_1 and F_2 are the first partial derivatives of F with respect to x and y, respectively, then $F_1(x_0, y)=F(x, y_{i0})=0$, $F_1(x, y_{i,\nu_i+1})=f_X(x)$ and $F_1(x_{N+1}, y)=0$; similar results hold for F_2. Define the 2×2 information matrices I and I^* with entries

$$I_{st}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f_s(x, y)\cdot f_t(x, y)}{f(x, y)}\,dxdy$$

$$I_{st}^*=\sum_{i=1}^{N+1}\sum_{j=1}^{\nu_i+1}\frac{\Delta_{ij}F_s\cdot\Delta_{ij}F_t}{p_{ij}}$$

for $s, t=1, 2$.

Then we will consider the linear estimator θ^* of θ given by $\theta^*=(I^*)^{-1}Q$, where $Q'=(Q_1, Q_2)$ (prime denotes transpose) and

$$Q_1=\sum_{i=1}^{N} m_{1i}(\xi_i-x_i)+\sum_{i=1}^{N+1}\sum_{j=1}^{\nu_i} h_{1ij}(\zeta_{ij}-y_{ij})$$

$$Q_2=\sum_{i=1}^{N} m_{2i}(\xi_i-x_i)+\sum_{i=1}^{N+1}\sum_{j=1}^{\nu_i} h_{2ij}(\zeta_{ij}-y_{ij})$$

$$m_{si}=\sum_{j=1}^{\nu_i+1}\left\{\frac{\Delta_{ij}F_s}{P_{ij}}[F_1(x_i, y_{ij})-F_1(x_i, y_{i,j-1})]\right\}$$
$$-\sum_{j=1}^{\nu_{i+1}+1}\left\{\frac{\Delta_{i+1,j}F_s}{P_{i+1,j}}[F_1(x_i, y_{i+1,j})-F_1(x_i, y_{i+1,j-1})]\right\}$$

$$h_{sij}=\left(\frac{\Delta_{ij}F_s}{P_{ij}}-\frac{\Delta_{i,j+1}F_s}{P_{i,j+1}}\right)[F_2(x_i, y_{ij})-F_2(x_{i-1}, y_{ij})] .$$

The estimator θ^* is translation-invariant, as follows easily from the relations

$$\sum_{i=1}^{N} m_{1i}=I_{11}^*$$

$$\sum_{i=1}^{N} m_{2i}=I_{12}^*$$

$$\sum_{i=1}^{N+1}\sum_{i=1}^{\nu_i} h_{1ij}=I_{12}^*$$

$$\sum_{i=1}^{N+1}\sum_{j=1}^{\nu_i} h_{2ij}=I_{22}^* .$$

The coefficients of θ^* are rather complicated, but are straight forward to compute if F, F_1 and F_2 can be expressed in closed form. In that case it is simple to program a computer to produce the various difference operators required and hence the coefficients of θ^*.

THEOREM. *Let $F(x-\theta_1, y-\theta_2)$ be a location parameter family with continuous density $f(x-\theta_1, y-\theta_2)$. Then the components of θ^* are the ABLUE's for θ_1 and θ_2 in terms of the ξ_i and ζ_{ij}. When θ* is true, *$\sqrt{n}(\theta^*-\theta)$ converges in law to the bivariate normal distribution with means 0 and covariance matrix $(I^*)^{-1}$. I^* is the information matrix for θ from the asymptotic distribution of*

$$\{\sqrt{n}(\xi_i-x_i^*), \sqrt{n}(\zeta_{ij}-y_{ij}^*): \text{ all } i \text{ and } j\} .$$

If the integrals I_{st} are finite and the derivatives f_1 and f_2 are continuous, each I_{st}^* can be made as close as desired to I_{st} by appropriately choosing N, ν_i, α_i and β_{ij}. The class of estimators θ^* is therefore ANE.

PROOF. We first compute the asymptotic distribution of the $\sqrt{n}(\zeta_{ij}-y_{ij}^*)$ conditional on the $\sqrt{n}(\xi_i-x_i^*)$. Let

$$P_n=P_\theta[\sqrt{n}(\zeta_{ij}-y_{ij}^*)\leqq v_{ij}: \text{ all } i \text{ and } j \mid \sqrt{n}(\xi_i-x_i^*)=u_i: i=1,\cdots, N] .$$

Partition the plane by the lines $x=x_i^*+u_i/\sqrt{n}$ and $y=y_{ij}^*+v_{ij}/\sqrt{n}$, and denote by N_{ij} the number of observations falling in the cell with "northeast corner" $(x_i^*+u_i/\sqrt{2}, y_{ij}^*+v_{ij}/\sqrt{n})$. Then if we understand the sample α-quantile from n observations to be the $[n\alpha]+1$ order statistic, events involving the ξ_i and ζ_{ij} can be described in terms of the N_{ij}. In what follows we will neglect the difference between $[n\alpha]$ and $n\alpha$. It may be easily verified that the asymptotic results are not affected, and ignoring the greatest integer notation will greatly simplify our notation. With this understanding, we have

$$P_n=P\Big[\sum_{j=1}^{k} N_{ij}\geqq(\alpha_i-\alpha_{i-1})\beta_{ik}n: i=1,\cdots, N+1; k=1,\cdots, \nu_i \Big| \sum_{j=1}^{\nu_i+1} N_{ij}=n(\alpha_i-\alpha_{i-1}): i=1,\cdots, N\Big] .$$

The key to the proof is the observation that under the stated conditions, the sets of r.v.'s $\{N_{ij}: j=1,\cdots, \nu_i+1\}$ for $i=1,\cdots, N+1$ have independent multinomial distributions. Specifically, for each $i=1,\cdots, N+1$ let $\{n_{ij}: j=1,\cdots, \nu_i+1\}$ be multinomial r.v.'s based on $n_i=n(\alpha_i-\alpha_{i-1})$ observations with cell probabilities $p_{ij}^n \Big/ \sum_{j=1}^{\nu_i+1} p_{ij}^n$, where

$$p_{ij}^n = F(x_i + u_i/\sqrt{n}\,,\ y_{ij} + v_{ij}/\sqrt{n}) - F(x_i + u_i/\sqrt{n}\,,\ y_{i,j-1} + v_{i,j-1}/\sqrt{n})$$
$$- F(x_{i-1} + u_{i-1}/\sqrt{n}\,,\ y_{ij} + v_{ij}/\sqrt{n})$$
$$- F(x_{i-1} + u_{i-1}/\sqrt{n}\,,\ y_{i,j-1} + v_{i,j-1}/\sqrt{n}) \,.$$

Then

$$P_n = \prod_{i=1}^{N+1} P\left[\sum_{i=1}^{k} n_{ij} \geqq n_i \beta_{ik} :\ k = 1, \cdots, \nu_i\right] = \prod_{i=1}^{N+1} P_{in} \,.$$

Clearly p_{ij}^n do not depend on θ and $p_{ij}^n \to p_{ij}$ as $n \to \infty$.

We now find the limit of P_{in} for each fixed i. Set $p_j^n = p_{ij}^n \Big/ \sum_{j=1}^{\nu_i+1} p_{ij}^n$ and define

$$Q_{jn} = \sqrt{n_i}(n_{ij}/n_i - p_j^n) \qquad j = 1, \cdots, \nu_i \,.$$

Since $p_j^n \to p_{ij}/(\alpha_i - \alpha_{i-1}) = \beta_{ij} - \beta_{i,j-1}$ the r.v.'s $\{Q_{jn}: j = 1, \cdots, \nu_i\}$ are asymptotically $N(0, V_i)$, where V_i has entries

$$V_{st} = -(\beta_{is} - \beta_{i,s-1})(\beta_{it} - \beta_{i,t-1}) \qquad s \neq t$$
$$V_{tt} = (\beta_{it} - \beta_{i,t-1})(1 - \beta_{it} + \beta_{i,t-1}) \,.$$

This follows from the usual characteristic function proof of asymptotic normality for multinomial r.v.'s, which is not affected by convergent sequences of cell probabilities.

In terms of the Q_{jn},

$$P_{in} = P\left[\sum_{j=1}^{k} Q_{jn} \geqq \sqrt{n_i}\left(\beta_{ik} - \sum_{j=1}^{k} p_j^n\right) :\ k = 1, \cdots, \nu_i\right]$$
$$= P\left[\sum_{j=1}^{k} Q_{jn} \geqq \sqrt{n_i} \sum_{j=1}^{k} (p_j - p_j^n) :\ k = 1, \cdots, \nu_i\right]$$

where $p_j = p_{ij}/(\alpha_i - \alpha_{i-1})$. Applying Taylor's theorem to the difference $p_j - p_j^n$ gives, after some arithmetic,

$$P_{in} = P\left[\sum_{j=1}^{k} Q_{jn} \geqq -(\alpha_i - \alpha_{i-1})^{-1/2}\right.$$
$$\left.\cdot\ (b_{ik} v_{ik} + c_{ik} u_i - d_{ik} u_{i-1}) + o(1) :\ k = 1, \cdots, \nu_i\right]$$

where

$$b_{ik} = F_2(x_i\,, y_{ik}) - F_2(x_{i-1}\,, y_{ik})$$
$$c_{ik} = F_1(x_i\,, y_{ik}) - \beta_{ik} f_X(x_i)$$
$$d_{ik} = F_1(x_{i-1}\,, y_{ik}) - \beta_{ik} f_X(x_{i-1})$$

with the conventions

$$c_{i,\nu_i+1}=0\,;\quad c_{N+1,k}=0 \qquad \text{for all } k\,;$$

$$d_{i1}=d_{i,\nu_i+1}=0\,;\quad d_{1k}=0 \qquad \text{for all } k\,.$$

Finally, replacing Q_{jn} by $-Q_{jn}$ (which does not change the asymptotic distribution),

$$\begin{aligned}P_{in}=P\Big[\,b_{ik}^{-1}\Big\{(\alpha_i-\alpha_{i-1})^{1/2}\sum_{j=1}^{k}Q_{jn}-c_{ik}u_i+d_{ik}u_{i-1}\Big\}\\ \leqq v_{ik}+o(1)\,;\ k=1,\cdots,\nu_i\Big]\\ \to P\Big[\,b_{ik}^{-1}\Big\{(\alpha_i-\alpha_{i-1})^{1/2}\sum_{j=1}^{k}Z_j-c_{ik}u_i+d_{ik}u_{i-1}\Big\}\\ \leqq v_{ik}\,;\ k=1,\cdots,\nu_i\Big]\end{aligned}$$

where the Z_j have the $N(0, V_i)$ joint distribution.

It is now routine to compute that the r.v.'s $\left\{\sum_{j=1}^{k}Z_j\colon k=1,\cdots,\nu_i\right\}$ are $N(0, \bar{V}_i)$, where $\bar{V}_i$ has entries

$$\bar{V}_{st}=\beta_{is}(1-\beta_{it}) \qquad s\leqq t\,,$$

and that the joint asymptotic distribution of the $\sqrt{n}(\zeta_{ij}-y_{ij}^*)$ conditional on $\sqrt{n}(\xi_i-x_i^*)=u_i$ is $N(Au, \Sigma)$. Here $u'=(u_1,\cdots,u_N)$ (prime denotes transpose) and A is the $\left(\sum_{i=1}^{N+1}\nu_i\right)\times N$ matrix with all entries 0 except

$$\begin{aligned}A_{st}&=-c_{tj}/b_{tj}\,,\ s=\sum_{i=0}^{t-1}\nu_i+j,\ 1\leqq j\leqq\nu_t\,, &&1\leqq t\leqq N+1\\ &=d_{t,j+1}/b_{t,j+1}\,,\ s=\sum_{i=1}^{t}\nu_i+j,\ 1\leqq j\leqq\nu_{t+1}\,, &&1\leqq t\leqq N\end{aligned}$$

(recall that $\nu_0=0$). Setting $r=\sum_{i=1}^{N+1}\nu_i$, Σ is the $r\times r$ matrix with all entries 0 except for $\nu_i\times\nu_i$ submatrices Σ_i $(i=1,\cdots,N+1)$ arranged in order of increasing i down the diagonal. Σ_i is trivially related to $\bar{V}_i$ and has entries

$$\Sigma_{st}=(\alpha_i-\alpha_{i-1})\frac{\beta_{is}(1-\beta_{it})}{b_{is}b_{it}}\,, \qquad s\leqq t\,.$$

The asymptotic distribution of the $\sqrt{n}(\xi_i-x_i^*)$, $i=1,\cdots,N$ is well known to be $N(0, C)$, where

$$C_{st}=\frac{\alpha_s(1-\alpha_t)}{f_X(x_s^*)f_X(x_t^*)}\,, \qquad s\leqq t\,.$$

From this it follows routinely that the asymptotic distribution of the

$N+r$ random variables

$$\{\sqrt{n}(\xi_i - x_i^*),\ i=1,\cdots,N;\ \sqrt{n}(\zeta_{ij}-y_{ij}^*),\ i=1,\cdots,N+1;\ j=1,\cdots,v_i\}$$

is $N(0,\Sigma_*)$, where

$$\Sigma_*^{-1}=\begin{vmatrix} C^{-1}+A'\Sigma^{-1}A & -A'\Sigma^{-1} \\ -\Sigma^{-1}A & \Sigma^{-1}\end{vmatrix}.$$

Σ^{-1} is easily calculated; its entries are 0 except for $\nu_i\times\nu_i$ blocks Σ_i^{-1} along the diagonal. Σ_i^{-1} in turn has all entries 0 except

$$(\Sigma_i^{-1})_{tt}=b_{it}^2(1/p_{it}+1/p_{i,t+1}), \qquad t=1,\cdots,\nu_i$$

$$(\Sigma_i^{-1})_{t,t-1}=(\Sigma_i^{-1})_{t-1,t}=-b_{it}b_{i,t-1}/p_{it}, \qquad t=2,\cdots,\nu_i.$$

Setting $x_i^*=x_i+\theta_1$ and $y_{ij}^*=y_i+\theta_2$, we see that asymptotically the $(N+r)$-dimentional r.v.

$$Z=(\xi_1-x_1,\cdots,\xi_N-x_N,\zeta_{11}-y_{11},\cdots,\zeta_{N+1,\nu_{N+1}}-y_{N+1,\nu_{N+1}})$$

is $N(B\theta, n^{-1}\Sigma_*)$, where $\theta'=(\theta_1,\theta_2)$ and B is the $(N+r)\times 2$ matrix with

$$B_{s1}=\begin{cases} 1 & s=1,\cdots,N \\ 0 & s=N+1,\cdots,N+r \end{cases}$$

$$B_{s2}=\begin{cases} 0 & s=1,\cdots,N \\ 1 & s=N+1,\cdots,N+r. \end{cases}$$

By least squares theory the ABLUE for θ is

$$\theta^*=(B'\Sigma_*^{-1}B)^{-1}B'\Sigma_*^{-1}Z$$

and $\sqrt{n}(\theta^*-\theta)$ has asymptotic distribution $N(0,V(\theta^*))$, where

$$V(\theta^*)=(B'\Sigma_*^{-1}B)^{-1}.$$

It is now straight forward (although quite tedious) to verify that θ^* is as given in the statement of the theorem and that $B'\Sigma_*^{-1}B=I^*$. The form of B quickly yields that $(B'\Sigma_*^{-1}B)_{11}$ is the sum of the entries of $C^{-1}+A'\Sigma^{-1}A$, $(B'\Sigma_*^{-1}B)_{12}$ is the sum of the entries of $-A'\Sigma^{-1}$, and so on. Similarly, the entries of the $2\times(N+r)$ matrix $B'\Sigma_*^{-1}$ are sums of the columns of the 4 submatrices comprising Σ_*^{-1}. We omit the details of the computations.

If f has continuous derivatives, we may represent F, F_1 and F_2 as integrals of their derivatives and observe that each I_{st}^* is essentially a Riemann sum for I_{st}. Each I_{st}^* can be made arbitrarily close to I_{st} by choosing α_1 and β_{i1} sufficiently near 0, α_N and $\beta_{i\nu_i}$ sufficiently near 1, and the remaining α_i and β_{ij} so that the norm of the partition formed by the lines $x=x_i$, $y=y_{ij}$ is sufficiently small. The routine details of the proof will be omitted. A similar proof is given in detail in [3].

3. Testing

There is a close connection between estimation and testing in multivariate location parameter problems. Suppose that $\hat{\theta}_n(X)$ is an estimator for the k-variate location parameter $\theta'=(\theta_1,\cdots,\theta_k)$ such that

$$\mathcal{L}\{\sqrt{n}\,\hat{\theta}_n(X)\,|\,\theta=0\}\to N(0,\beta)\,.$$

Here we have used a standard notation for convergence in distribution and $X'=(X_1,\cdots,X_n)$ is a random sample. Then we can test the hypothesis $H_0\colon\theta=0$ by using the critical region

$$n\hat{\theta}_n\beta^{-1}\hat{\theta}_n'\geqq c$$

or

$$n\hat{\theta}_n\hat{\beta}_n^{-1}\hat{\theta}_n'\geqq c\,.$$

where $\hat{\beta}_n$ is a consistent estimator of the asymptotic covariance matrix β. Clearly

$$\mathcal{L}\{n\hat{\theta}_n\beta^{-1}\hat{\theta}_n'\,|\,\theta=0\}\to\chi_k^2\,,$$

the chi-square distribution with k degrees of freedom.

If $\hat{\theta}_n$ is translation-invariant, we can find the asymptotic relative Pitman efficiency (ARE) of two such tests by an elegant standard argument. Considering the sequence of alternatives $\theta_n=\delta/\sqrt{n}$ (δ a k-vector), we have

$$\mathcal{L}\{\sqrt{n}\,\hat{\theta}_n(X)\,|\,\theta=\delta/\sqrt{n}\}=\mathcal{L}\{\sqrt{n}\,\hat{\theta}_n(X+\delta/\sqrt{n})\,|\,\theta=0\}$$

since X has location parameter θ. Translation-invariance of $\hat{\theta}_n$ implies that

$$\mathcal{L}\{\sqrt{n}\,\hat{\theta}_n(X+\delta/\sqrt{n})\,|\,\theta=0\}\to N(\delta,\beta)\,. \tag{2.1}$$

therefore, if $T_n=n\hat{\theta}_n\beta^{-1}\hat{\theta}_n'$,

$$\mathcal{L}\{T_n\,|\,\theta=\delta/\sqrt{n}\}\to\chi_k^2(\delta'\beta^{-1}\delta)\,, \tag{2.2}$$

where $\delta'\beta^{-1}\delta$ is the non-centrality parameter. It is well known ([2]) that the ARE of two test statistics which are asymptotically non-central chi-square under Pitman alternatives is the ratio of their non-centrality parameters. Most test statistics for multivariate location, in particular those of [1] and [6], satisfy (2.2).

We have already remarked that our estimators θ^* are translation invariant; they therefore satisfy (2.1) and (2.2) with $\beta^{-1}=I^*$. If T_n satisfies (2.2), the ARE of the test based on θ^* to that based on T_n is therefore $\delta' I^* \delta / \delta' \beta^{-1} \delta$. This of course depends on the direction of approach to the null hypothesis. But I^* can be made arbitrarily close to the information matrix I, and it is well known (Satz 1.13 on p. 356 of [5], for example) that for any regular unbiased estimator with covariance matrix β

$$\delta' I \delta \geqq \delta' \beta^{-1} \delta \qquad \text{for all } \delta.$$

The quantity $\delta' I \delta$ therefore plays a role in the theory of ARE for tests based on estimators analogous to that of the Cramer-Rao lower bound in the theory of ARE for estimators. We can say that tests based on the ANE class of estimators θ^* form an ANE class of tests among tests satisfying (2.2).

The remarks of this section apply also to the three classes of ANE estimators discussed in [3]. The first two classes, which are ABLUE's from sample quantiles and cell frequencies in certain random partitions of k-space, can be seen on inspection to be translation-invariant. The third class, based on RBAN estimators for multinomial problems, need not be invariant. In this case (2.1) can be shown to hold by direct computation.

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