BAYES EQUIVARIANT ESTIMATORS OF VARIANCE COMPONENTS*

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(Received Nov. 11, 1968)

Introduction

Consider the results of J independent replica I different treatments. Let Y_{ij} $(i=1,\cdots,I;\ j=1,\cdots,J)$ designate the observed random variables. We assume that the distributions of Y_{ij} follow Model II of the analysis of variance, namely,

$$(1)\qquad Y_{ij}=\mu+a_i+e_{ij}\,,\qquad i=1,\cdots,I,\ j=1,\cdots,J\,,$$

where e_{ij} are mutually independent, having a normal distribution with zero mean and variance σ_e^2. a_i are normally distributed random variables, mutually independent of each other and of the e_{ij}, having zero means and equal variances, σ_a^2. μ is an unknown constant. Furthermore, $-\infty<\mu<\infty$, $0<\sigma_e^2<\infty$, $0<\sigma_a^2<\infty$. The parameters, σ_e^2 and σ_a^2 are called the components of variance (' within ' and ' between ', respectively).

In the present paper we consider the problem of characterizing all Bayes estimators of σ_e^2 and σ_a^2 which are translation invariant and scale preserving. That is, if $\psi(Y_{11},\cdots,Y_{IJ})$ is an estimator of either σ_e^2 or of σ_a^2, and if we subject the obserbations to any transformation of the group,

$$\mathcal{G}=\{Y_{ij}\to\alpha(Y_{ij}+\beta)\,,\quad \alpha>0,\ -\infty<\beta<\infty\}\,,$$

then $\psi(Y_{11},\cdots,Y_{IJ})\to\alpha^2\psi(Y_{11},\cdots,Y_{IJ})$.

Each of the transformations in $\mathcal{G}$ subject all Y_{ij} first to a translation and then to a change of scale. Following the terminology of Berk [1] and of Wijsman [10], we call such estimators equivariant with respect to $\mathcal{G}$.

The estimation problem is studied here with a squared-error loss function. Thus, following the Blackwell-Rao Lehmann-Scheffe's theorem [5], we consider equivariant estimators which are functions of the minimal sufficient statistic $(Y_{\cdot\cdot}, S_e^2, S_a^2)$, where

* Supported by a National Science Foundation Grant GP-9007.

$$Y..=\sum_{i=1}^{I}\sum_{j=1}^{J}Y_{ij}/IJ\,,\qquad S_e^2=\sum_{i=1}^{I}\sum_{j=1}^{J}(Y_{ij}-\bar{Y}_i)^2,$$

$$\bar{Y}_i=\sum_{j=1}^{J}Y_{ij}/J\ (i=1,\cdots,I)\,,\quad\text{and}\quad S_a^2=J\sum_{i=1}^{I}(\bar{Y}_i-Y..)^2\,.$$

These estimators are called *sufficiently-equivariant*. Subjecting Y_{ij} to a transformation in $\mathcal{G}$, the minimal sufficient statistic is transformed to $(\alpha(Y..+\beta), \alpha^2S_e^2, \alpha^2S_a^2)$. In particular, all sufficiently translation invariant estimators of either σ_e^2 or of σ_a^2 are functions (properly measurable) only of (S_e^2, S_a^2). Furthermore, all sufficiently-equivariant estimators of σ_e^2 can be written in the form

$$(2)\qquad \hat{\sigma}_e^2 = S_e^2\varphi\left(\frac{S_a^2}{S_e^2}\right),$$

and those of σ_a^2 can be expressed as

$$(3)\qquad \hat{\sigma}_a^2 = S_a^2\psi\left(\frac{S_e^2}{S_a^2}\right).$$

$R=S_a^2/S_e^2$ is a maximal invariant statistic. The objective of the present paper is to characterize all the Bayes estimators of σ_e^2 and of σ_a^2 which are equivariant with respect to transformations in $\mathcal{G}$. For this purpose we present in Section 1 the required distribution theory. The Bayes-equivariant estimators are derived in Section 2. In Section 3 we illustrate Bayes-equivariant estimators for a particular prior distribution.

The class of all estimators of σ_e^2 and of σ_a^2 which are given in Section 2, together with all estimators obtainable as limits of these estimators is essentially complete with respect to all equivariant estimators. In other words, given any equivariant estimator of σ_e^2 or of σ_a^2, one can find one in the above class which is at least as good with respect to the mean-square-error risk function. However, as we show in Section 4, all Bayes-equivariant estimators of σ_e^2 are inadmissible in the wider class of all possible estimators, with respect to the mean-square-error risk function. This in itself is an interesting theoretical result, the establishment of which requires a judicious application of a method introduced previously by Stein [7]. As shown by Klotz, Milton and Zacks in [4], the performance of certain equivariant estimators of σ_e^2 is very close to optimal, although the estimators are inadmissible in the general class. Roughly speaking, the inadmissibility of the equivariant estimators is due to the fact that they are all independent of the sample grand mean $Y..$. This statistic contains some information concerning σ_e^2 and σ_a^2, since $E\{Y..^2\}=\mu^2+\sigma_e^2/IJ+\sigma_a^2/I$. An example of a *non-equivariant* estimator of σ_e^2, which is admissible, is

$$\hat{\sigma}_e^2(\rho_0)=[S_e^2+(S_a^2+IJY..^2)/(1+\rho_0J)]/(IJ+2)\,.$$

If one knows that $\mu=0$ and $\sigma_a^2/\sigma_e^2=\rho_0$ then $\tilde{\sigma}_e^2(\rho_0)$ is the unique admissible estimator of σ_e^2 (see proof in Hodges and Lehmann [3]). Hence, since the mean-square-error risk function of $\tilde{\sigma}_e^2(\rho_0)$ is a continuous function of $\theta=(\mu, \sigma_e^2, \rho)$, $\tilde{\sigma}_e^2(\rho_0)$ is admissible where $\rho=\sigma_a^2/\sigma_e^2$. However, $\tilde{\sigma}_e^2(\rho_0)$ is a very inefficient estimator when ρ is unknown, and one can exhibit equivariant estimators of σ_e^2 which have a considerably smaller mean-square-error than that of $\tilde{\sigma}_e^2(\rho_0)$, when $|\rho-\rho_0|$ is large. Furthermore, equivariant estimators of σ_e^2 and σ_a^2 are the only ones to consider if we adopt the invariance principle. Thus, the characterization of the essentially complete classes of equivariant estimators seems to attain an important objective in the study of the analysis of variance Model II.

1. Distribution theory

In the present section we present the distributions of the various random variables, and in particular we derive certain conditional distributions required for Sections 2 and 4.

Starting with the sufficient statistics $(Y_{..}, S_e^2, S_a^2)$, it is a straightforward matter to verify their independence (see, Graybille [2], p. 88) and that

$$S_e^2 \sim \sigma_e^2 \chi^2[I(J-1)]$$
$$S_a^2 \sim \sigma_e^2(1+J\rho)\chi^2[I-1] \tag{1.1}$$

and

$$IJY_{..}^2 \sim \sigma_e^2(1+J\rho)\chi^2\left[1 ; \frac{IJ\mu^2}{2\sigma_e^2(1+J\rho)}\right],$$

where $\chi^2[\nu]$ designates a central chi-square random variable, with ν degrees of freedom (d.f.), and $\chi^2[\nu; \lambda]$ designates a non-central chi-square random variable with ν d.f., and parameter of non-centrality λ. The non-central chi-square distribution law is a mixture of central chi-squares, with mixing probabilities given by the Poisson distribution with mean λ. This well known property (see, Graybill [2], p. 74) can be schematically written as

$$\chi^2[\nu; \lambda] \sim \chi^2[\nu+2M], \qquad M \sim P(\lambda), \tag{1.2}$$

where $P(\lambda)$ is a Poisson random variable with mean λ. Let $X|Y \sim Z$ state that the conditional distribution law of X, given Y, is like the distribution law of Z. Thus, we obtain from the model (1) that

$$S_a^2 \,|\, (a_1, \cdots, a_I) \sim \sigma_e^2\chi^2[I-1; \lambda(\boldsymbol{a})] \sim \sigma_e^2\chi^2[I-1+2M_1]; \qquad M_1 \sim P(\lambda(\boldsymbol{a})), \tag{1.3}$$

where

$$\lambda(\boldsymbol{a})=\frac{J}{2\sigma_e^2}\sum_{i=1}^{I}(a_i-\bar{a})^2\,, \qquad \bar{a}=\frac{1}{I}\sum_{i=1}^{I}a_i\,. \tag{1.4}$$

Similarly,

$$IJY_{..}^2\,|\,(a_1,\cdots,a_I)\sim\sigma_e^2\chi^2[1\,;\lambda^*(\boldsymbol{a})]\sim\sigma_e^2\chi^2[1+2M_2]\,, \qquad M_2\sim P(\lambda^*(\boldsymbol{a}))\,, \tag{1.5}$$

where M_2 is independent of M_1, and

$$\lambda^*(\boldsymbol{a})=\frac{IJ}{2\sigma_e^2}(\mu+\bar{a})^2\,. \tag{1.6}$$

In the next section we need expressions for the following conditional expectations: $E\{S_e^2S_a^2\,|\,R\}$, $E\{S_a^2\,|\,R\}$ and $E\{S_a^4\,|\,R\}$, where $R=S_a^2/S_e^2$. To obtain these conditional expectations we prove that

$$S_a^2\,|\,R\sim\frac{\sigma_e^2R(1+\rho J)}{1+R+\rho J}\chi^2[IJ-1] \tag{1.7}$$

and

$$S_eS_a\,|\,R\sim\frac{\sigma_e^2R^{1/2}(1+\rho J)}{1+R+\rho J}\chi^2[IJ-1]\,. \tag{1.8}$$

Indeed, if $g_{S_e^2,S_a^2}(x,y)$ designate the joint density of S_e^2, S_a^2 we have

$$g_{S_e^2,S_a^2}(x,y)\propto x^{\nu/2-1}y^{(I-1)/2-1}\exp\left\{-\frac{x}{2\sigma_e^2}-\frac{y}{2\sigma_e^2(1+\rho J)}\right\}, \tag{1.9}$$

for $0\leqq x,\ y\leqq\infty$, where $\nu=I(J-1)$. Making the transformation $R=S_a^2/S_e^2$ we obtain that, the joint density of S_a^2 and R is

$$g_{S_a^2,R}(y,r)=\psi(r)y^{\nu/2+(I-1)/2-1}\exp\left\{-\frac{y}{2\sigma_e^2}\left(\frac{1}{r}+\frac{1}{1+\rho J}\right)\right\}, \tag{1.10}$$

$0\leqq y,\ r\leqq\infty$, where $\psi(r)$ is a function depending only on r. (1.7) is obtained immediately from (1.10). In a similar manner, making the transformations R and $W=S_e^2S_a^2$ we obtain from (1.9) that the joint density of R and W is

$$g_{W,R}(\omega,r)=\varphi(r)(\omega^{1/2})^{\nu/2+(I-1)/2-2}\exp\left\{-\frac{\sqrt{\omega}}{2\sigma_e^2}\left(\frac{1}{\sqrt{r}}+\frac{\sqrt{r}}{1+\rho J}\right)\right\}, \tag{1.11}$$

$0\leqq r,\ \omega\leqq\infty$. Finally, making the transformation $V=W^{1/2}$ (the positive root) we obtain

$$g_{V,R}(v,r)=\varphi(r)v^{(IJ-1)/2-1}\exp\left\{-\frac{\nu}{2\sigma_e^2}\left(\frac{1}{r^{1/2}}+\frac{r^{1/2}}{1+\rho J}\right)\right\}. \tag{1.12}$$

From (1.12) we arrive at (1.8).

Thus, from (1.7) and (1.8) one obtains

$$E\{S_a^2\mid R\}=\frac{\sigma_e^2R(1+\rho J)}{1+R+\rho J}(IJ-1), \tag{1.13}$$

$$E\{S_a^4\mid R\}=\frac{\sigma_e^4R^2(1+\rho J)^2}{(1+R+\rho J)^2}(I^2J^2-1), \tag{1.14}$$

and

$$E\{S_e^2S_a^2\mid R\}=\frac{\sigma_e^4R(1+\rho J)^2}{(1+R+\rho J)^2}(I^2J^2-1). \tag{1.15}$$

2. Bayes-equivariant estimators

2.1. *Estimators of* σ_e^2.

It is instructive to remember that if the variance ratio $\rho=\sigma_a^2/\sigma_e^2$ is known, there is an essentially unique best equivariant estimator of σ_e^2, namely

$$\hat{\sigma}_e^2(\rho)=\frac{1}{IJ+1}[S_e^2+S_a^2/(1+\rho J)]. \tag{2.1}$$

Indeed, if ρ is known there are only two unknown parameters: (μ,σ_e^2), $-\infty<\mu<\infty$, $0<\sigma_e^2<\infty$. The minimal sufficient statistic in this case is $(Y_{..}, S_e^2+S_a^2/(1+J\rho))$. Moreover, every equivariant estimator is of the form $\lambda[S_e^2+S_a^2/(1+J\rho)]$, where λ is a constant. Since $S_e^2+S_a^2/(1+J\rho)\sim\sigma_e^2\chi^2[IJ-1]$ it is simple to prove that the value of λ which minimizes uniformly the mean-square-error of the equivariant estimator is $\lambda=(IJ+1)^{-1}$. It is also easy to verify that the mean-square-error of $\hat{\sigma}_e^2(\rho)$ is equal to $2\sigma_e^4(IJ+1)^{-1}$.

When ρ is unknown there exists *no* uniformly minimum mean-square-error equivariant estimator. As discussed in the introduction any equivariant estimator of σ_e^2 can be written in the form (2), or equivalently, as

$$\hat{\sigma}_e^2=\frac{S_e^2}{IJ+1}(1+Rf(R)), \tag{2.2}$$

where $f(R)$ is a properly measurable function of the maximal invariant $R=S_a^2/S_e^2$. Formula (2.2) can be considered as a generalization of (2.1) to the case of unknown ρ, in which $f(R)$ is a properly chosen estimator of the unknown function $(1+J\rho)^{-1}$. We choose the function $f(R)$ so that the equivariant estimator (2.2) is Bayes against some prior distribution of (σ_e^2, ρ). In other words, given a prior distribution of (σ_e^2, ρ) we can associate with every equivariant estimator the corresponding prior risk (the expected mean-square-error risk with respect to the given prior distribution of (σ_e^2, ρ)). An equivariant estimator which minimizes the prior risk is called *Bayes equivariant*. It should be remarked that the Bayes equivariant estimators are not necessarily Bayes in the genral sense (one which minimizes the prior risk among all estimators). This is the case in estimating σ_e^2[1]. We shall further see that the prior distribution of the parameter σ_e^2 does not play any important role in the determination of the Bayes equivariant estimator. In order to obtain the Bayes equivariant estimator it is necessary to specify only the prior distribution of ρ, and then use the following formula

$$f_\xi(R)=\frac{E_{\rho|R,\xi}\{(1+\rho J)/(1+\rho J+R)^2\}}{E_{\rho|R,\xi}\{(1+\rho J)^2/(1+\rho J+R)^2\}}\,, \tag{2.3}$$

where $f_\xi(R)$ designates the function to be substituted in (2.2), and $E_{\rho|R,\xi}\{\cdot\}$ designates the posterior expectation of the function of ρ in the brackets, given R and the prior distribution ξ of ρ. We verify now that this indeed gives the Bayes equivariant estimator of σ_e^e.

Let $R_f(\sigma_e^2, \rho)$ designate the risk function (mean-square-error) of an equivariant estimator (2.2), as a function of (σ_e^2, ρ). We have,

$$\begin{aligned} R_f(\sigma_e^2, \rho) &= E\left\{\left[\frac{S_e^2}{IJ+1}(1+Rf(R))-\sigma_e^2\right]^2\right\} \\ &= E\left\{\left[\frac{S_e^2}{IJ+1}\left(1+R\cdot\frac{1}{1+\rho J}\right)\right.\right. \\ &\qquad \left.\left. -\sigma_e^2+\frac{S_a^2}{IJ+1}\left(f(R)-\frac{1}{1+\rho J}\right)\right]^2\right\} \\ &= \frac{2\sigma_e^4}{IJ+1}+2E\left\{\left[\frac{S_e^2}{IJ+1}\left(1+\frac{R}{1+\rho J}\right)-\sigma_e^2\right]\right. \\ &\qquad \left. \cdot\frac{S_a^2}{IJ+1}\left[f(R)-\frac{1}{1+\rho J}\right]\right\} \\ &\quad +E\left\{\frac{S_a^4}{(IJ+1)^2}\left(f(R)-\frac{1}{1+\rho J}\right)^2\right\}. \end{aligned} \tag{2.4}$$

By the iterated expectation law, we can compute the R.H.S. of (2.4) by determining first its conditional expectation given R, and then the total expectation. Substituting in (2.4) the conditional expectations given by (1.13)–(1.15) we obtain, after some algebraic manipulations,

[1] This statement is substantiated in Section 4.

$$(2.5)\qquad R_f(\sigma_e^2, \rho)=\frac{2\sigma_e^4}{IJ+1}+\sigma_e^4\frac{IJ-1}{IJ+1} \cdot E_{R|\rho}\left\{R^2\frac{(1+\rho J)^2}{(1+R+\rho J)^2}\left[f(R)-\frac{1}{1+\rho J}\right]^2\right\}.$$

Formula (2.5) shows that the decision problem does not involve the parameter σ_e^2. If ρ is known we choose $f(R)\equiv(1+\rho J)^{-1}$ and minimize $R_f(\sigma_e^2, \rho)$ uniformly. Furthermore, for values of (σ_e^2, ρ), the risk function is not smaller than $2\sigma_e^4/(IJ+1)$ for all choices of $f(R)$ when ρ is unknown. Moreover, since the distribution of R depends only on the parameter ρ we have to assume, without loss of generality, only a prior distribution for ρ. Let ξ designate the prior distribution law of ρ[1]. Let $R_f(\sigma_e^2, \xi)$ designate the prior risk under f, given σ_e^2 and ξ. $R_f(\sigma_e^2, \xi)$ is the expectation of $R_f(\sigma_e^2, \rho)$ with respect to ξ. Let

$$(2.6)\qquad Q_f(\xi)=E_{\rho|\xi}\left\{E_{R|\rho}\left\{R^2\frac{(1+\rho J)^2}{(1+R+\rho J)^2}\left[f(R)-\frac{1}{1+\rho J}\right]^2\right\}\right\}.$$

By virtue of the Fubini theorem[2], we can write

$$(2.7)\qquad Q_f(\xi)=E_{R|\xi}\left\{R^2E_{\rho|R,\xi}\left\{\frac{(1+\rho J)^2}{(1+r+\rho J)^2}\left[f(R)-\frac{1}{1+\rho J}\right]^2\right\}\right\}.$$

The function $f_\xi(R)$ given by (2.3) minimizes the posterior expectation $E_{\rho|R,\xi}\{\cdot\}$ in the R.H.S. of (2.7) for (almost) all R, and hence it minimizes $Q_f(\xi)$, and $R_f(\sigma_e^2, \xi)$. We remark that if the prior distribution law ξ concentrates the whole probability mass on a point ρ_0 one attains from (2.2) and (2.3) that the corresponding Bayes equivariant estimator of σ_e^2 is the locally best equivariant estimator

$$(2.8)\qquad \hat{\sigma}_e^2(\rho_0)=\frac{S_e^2}{IJ+1}\left[1+\frac{R}{1+\rho_0 J}\right].$$

2.2. *Estimators of* σ_a^2.

When ρ is known the best equivariant estimator is obviously $\hat{\sigma}_a^2(\rho)=\rho\hat{\sigma}_e^2(\rho)=\rho S_e^2(1+R/(1+\rho J))/(IJ+1)$. When ρ is unknown we can present all Bayes equivariant estimators of σ_a^2 in the analogous form

$$(2.9)\qquad \hat{\sigma}_a^2=\frac{S_e^2}{IJ+1}\,\frac{E_{\rho|R,\xi}\{\rho(1+\rho J)/(1+R+\rho J)\}}{E_{\rho|R,\xi}\{(1+\rho J)^2/(1+R+\rho J)^2\}}.$$

In particular, when the prior distribution law assigns ρ a prior distribution concentrated on the point ρ_0 we obtain from (2.9) the locally best

[1] One can also say that ξ is the conditional prior distribution law of ρ, given σ_e^2.

[2] One can prove that the Fubini theorem holds, when $f(R)\equiv f_\xi(R)$ for all proper prior distribution laws ξ of ρ,

equivariant estimator

$$\hat{\sigma}_a^2(\rho_0)=\rho_0\frac{S_e^2}{IJ+1}\left(1+\frac{R}{1+\rho_0 J}\right). \tag{2.10}$$

To verify (2.9) we write the mean-square-error of an equivariant estimator of the form (3) as

$$\begin{aligned} R_\varphi(\sigma_e^2,\rho)&=E\{(S_a^2\psi(R)-\sigma_a^2)^2\} \\ &=E\{E\{(S_a^2\psi(R)-\rho\sigma_e^2)^2 \mid R\}\}. \end{aligned} \tag{2.11}$$

Substituting the conditional expectations (1.13) and (1.15) we obtain

$$\begin{aligned} R_\varphi(\sigma_e^2,\rho)=\sigma_e^4 E_{R|\rho}\Big\{&\psi^2(R)\frac{R^2(1+\rho J)^2}{(1+R+\rho J)^2}(I^2J^2-1) \\ &-2\psi(R)\rho\frac{R(1+\rho J)}{1+R+\rho J}(IJ-1)+\rho^2\Big\}. \end{aligned} \tag{2.12}$$

Using the same method as in Section 2.1 we obtain that the function $\psi_\xi(R)$ which minimizes the posterior risk is

$$\psi_\xi(R)=\frac{1}{R(IJ+1)}\,\frac{E_{\rho|R,\xi}\{\rho(1+\rho J)/(1+R+\rho J)\}}{E_{\rho|R,\xi}\{(1+\rho J)^2/(1+R+\rho J)^2\}}. \tag{2.13}$$

Multiplying (2.13) by S_a^2 we obtain the Bayes equivariant estimator (2.9).

3. An example of non-trivial Bayes equivariant estimators

To derive an explicit expression for some non-trivial Bayes equivariant estimators we notice that, since $R\sim(1+\rho J)((I-1)/I(J-1))F[I-1, I(J-1)]$, where $F[\nu_1,\nu_2]$ designates a central F statistic with ν_1 and ν_2 d.f., the density function of R given ρ is

$$\begin{aligned} f_R^{(\rho)}(r\mid I-1, I(J-1))&=\frac{1}{(1+J\rho)^{(I-1)/2}B((I-1)/2, I(J-1)/2)} \\ &\quad\cdot\frac{r^{(I-3)/2}}{(1+r/(1+\rho J))^{(IJ-1)/2}}, \end{aligned} \tag{3.1}$$

$0\leq r<\infty$, where $B(a,b)$ is the beta function, $0<a$, $b<\infty$. Make the transformation $\varphi=(1+J\rho)^{-1}$, and let φ have a prior density function $h(\varphi)$ on $[0,1]$. Then, the posterior density of φ given R is

$$h(\varphi\mid R)=\frac{h(\varphi)\varphi^{(I-1)/2}(1+R\varphi)^{-(IJ-1)/2}}{\int_0^1 h(\varphi)\varphi^{(I-1)/2}(1+R\varphi)^{-(IJ-1)/2}d\varphi}. \tag{3.2}$$

We shall consider here the special case in which the prior distribution of φ is uniform on [0, 1]. Furthermore, we require that $I(J-1)>4$.

One can easily prove that for all $0<q,\ p<\infty$, such that $q-p>1$, and for all $0<\alpha<\infty$,

$$\int_0^1 \varphi^p(1+\alpha\varphi)^{-q}d\varphi=\frac{1}{\alpha^{p+1}}\int_{1/(1+\alpha)}^1 \mu^{q-p-2}(1-\mu)^p d\mu\,. \tag{3.3}$$

Hence, letting $\omega(R)=R/(1+R)$, we obtain

$$\begin{aligned}\int_0^1 &\varphi^{(I-1)/2}(1+R\varphi)^{-(IJ-1)/2}d\varphi \\ &=\frac{1}{R^{(I+1)/2}}B\left(\frac{I(J-1)}{2}-1,\ \frac{I+1}{2}\right)\left[1-I_{1-\omega(R)}\left(\frac{I(J-1)}{2}-1,\ \frac{I+1}{2}\right)\right],\end{aligned} \tag{3.4}$$

where $I_x(a, b)$ is the incomplete beta function ratio, $0\leqq x\leqq 1$, and $0<a$, $b<\infty$. The condition that $q-p>1$ implies the condition $I(J-1)>2$. Applying formula (3.3), and (3.4) and using the relationship $1-I_x(a, b)=I_{1-x}(b, a)$ we easily find that

$$\begin{aligned}E_{\rho|R}\left\{\frac{(1+\rho J)^2}{(1+R+\rho J)^2}\right\} &= E_{\varphi|R}\left\{\frac{1}{(1+R\varphi)^2}\right\} \\ &=\frac{I^2(J-1)^2-2I(J-1)}{I^2J^2-1} \\ &\quad\cdot\frac{I_{\omega(R)}((I+1)/2,\ I(J-1)/2+1)}{I_{\omega(R)}((I+1)/2,\ I(J-1)/2-1)}\,.\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}E_{\rho|R}\left\{\frac{(1+\rho J)}{(1+R+\rho J)^2}\right\} &= E_{\varphi|R}\left\{\frac{\varphi}{(1+\varphi R)^2}\right\} \\ &=\frac{1}{R}\,\frac{(I+1)[I(J-1)-2]}{I^2J^2-1} \\ &\quad\cdot\frac{I_{\omega(R)}((I+3)/2,\ I(J-1)/2)}{I_{\omega(R)}((I+1)/2,\ I(J-1)/2)}\,.\end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) in (2.3) we obtain that the Bayes equivariant estimator of σ_e^2, against the above prior distribution of ρ is

$$\hat\sigma_e^2=\frac{S_e^2}{IJ+1}\left[1+\frac{I+1}{I(J-1)}\cdot\frac{I_{\omega(R)}((I+3)/2,\ I(J-1)/2)}{I_{\omega(R)}((I+1)/2,\ I(J-1)/2+1)}\right]. \tag{3.7}$$

To obtain the Bayes equivariant estimator of σ_a^2 we have to supplement with the posterior expectation of $\rho(1+\rho J)/(1+R+\rho J)$. Since $\rho=((1-\varphi)/\varphi)/J$ we have

$$E_{\rho|R}\left\{\frac{\rho(1+J\rho)}{1+R+J\rho}\right\}=\frac{R}{J}\cdot\frac{IJ-3}{I-1}\cdot\frac{I_{\omega(R)}((I-1)/2,\ I(J-1)/2-1)}{I_{\omega(R)}((I+1)/2,\ I(J-1)/2-1)} \tag{3.8}$$

$$-\frac{1}{J}\cdot\frac{IJ-3}{I(J-1)-4}\cdot\frac{I_{\omega(R)}((I+1)/2,\ I(J-1)/2-2)}{I_{\omega(R)}((I+1)/2,\ I(J-1)/2-1)}\ .$$

Substituting (3.5) and (3.8) in (2.9) we obtain that the Bayes equivariant estimator of σ_a^2 is

$$(3.9)\qquad \hat{\sigma}_a^2=S_e^2\frac{(IJ-1)(IJ-3)}{IJ(J-1)[I(J-1)-2]}\left\{\frac{R}{I-1}\cdot\frac{I_{\omega(R)}((I-1)/2,\ I(J-1)/2-1)}{I_{\omega(R)}((I+1)/2,\ I(J-1)/2+1)}\right.$$
$$\left.-\frac{1}{I(J-1)-4}\cdot\frac{I_{\omega(R)}((I+1)/2,\ I(J-1)/2-2)}{I_{\omega(R)}((I+1)/2,\ I(J-1)/2+1)}\right\}.$$

We consider now some characteristics of the above Bayes estimators. First, since $\omega(R)\rightarrow 0$ as $R\rightarrow\infty$ we obtain that

$$(3.10)\qquad \lim_{R\rightarrow\infty}\hat{\sigma}_e^2=\frac{S_e^2}{I(J-1)}\ .$$

This is the uniformly minimum variance unbiased estimators of σ_e^2. Moreover, since $R\sim(1+\rho J)((I-1)/I(J-1))F(I-1,\ I(J-1))$ we can write $R=O_p(\rho)$ as $\rho\rightarrow\infty$[1]. Thus, the distribution function of the Bayes equivariant estimator (3.7) converges completely to that of $\sigma_e^2\chi^2[I(J-1)]/I(J-1)$. This shows that the estimator (3.7) has a mean-square-error which converges to $2\sigma_e^4/I(J-1)$ as $\rho\rightarrow\infty$. For large values of J and $I=2$ this is very close to the lower bound $2\sigma_e^4/(IJ+1)$ of the mean-square-error. The Bayes equivariant estimator of σ_a^2 (3.9) behaves like the estimator

$$S_a^2\frac{(IJ-1)I(J-3)}{I(I-1)J(J-1)[I(J-1)-2]}\left(1+O_p\left(\frac{1}{\rho}\right)\right),$$

as $\rho\rightarrow\infty$. The mean-square-error of $\hat{\sigma}_a^2$ is of order $O(\rho^2)$ as $\rho\rightarrow\infty$.

To conclude the present section we bring two equivariant estimators which are "formal" Bayes estimators, that is Bayes equivariant estimators in which improper prior distribution is assumed. The first one was derived by Tiao and Tan [9]. Let $\tau^2=\sigma_e^2(1+J\rho)$. Tiao and Tan imposed on $\theta=(\mu,\ \rho_e^2,\ \tau^2)$ the improper prior distribution of Jeffery, with a σ-finite measure

$$dm(\theta)=\begin{cases} d\mu\cdot\dfrac{d\sigma_e}{\sigma_e}\cdot\dfrac{d\tau}{\tau} & \tau>\sigma_e>0\\ 0 & \text{otherwise}\ ,\end{cases}$$

and arrived at the formal Bayes estimator

[1] For a definition of order of magnitude in probability, $O_p(\cdot)$ and $o_p(\cdot)$, see J. W. Pratt [6].

$$\tilde{\sigma}_e^2 = \frac{S_e^2}{I(J-1)-2} \cdot \frac{I_{\omega(R)}((I-1)/2,\, I(J-1)/2-1)}{I_{\omega(R)}((I-1)/2,\, I(J-1)/2)} . \tag{3.11}$$

For σ_a^2 they proposed the formal Bayes estimator

$$\tilde{\sigma}_a^2 = \left[\frac{S_e^2}{J} \frac{R}{I-3} \cdot \frac{I_{\omega(R)}((I-3)/2,\, I(J-1)/2)}{I_{\omega(R)}((I-1)/2,\, I(J-1)/2)} - \frac{1}{I(J-1)-2} \cdot \frac{I_{\omega(R)}((I-1)/2,\, I(J-1)/2-1)}{I_{\omega(R)}((I-1)/2,\, I(J-1)/2)}\right] . \tag{3.12}$$

There is some resemblance in the structure of estimators (3.9) and (3.12).

Stone and Springer [8] derived formal Bayes equivariant estimators by applying an improper prior distribution with a σ-finite measure defined by

$$dm^*(\theta) = \begin{cases} d\mu \cdot \dfrac{d\sigma_e}{\tau} \cdot \dfrac{d\tau}{\tau} & \tau > \sigma_e > 0 \\ 0 & \text{otherwise} . \end{cases} \tag{3.13}$$

Their estimator of σ_e^2 is then

$$\tilde{\sigma}_e^2 = \frac{S_e^2}{I(J-1)-3} \cdot \frac{I_{\omega(R)}(I/2,\, (I(J-1)-3)/2)}{I_{\omega(R)}(I/2,\, (I(J-1)-1)/2)} . \tag{3.14}$$

The estimator of σ_a^2 is

$$\tilde{\sigma}_a^2 = \frac{S_e^2}{J}\left[\frac{R}{I-2} \cdot \frac{I_{\omega(R)}((I-2)/2,\, (I(J-1)-1)/2)}{I_{\omega(R)}(I/2,\, (I(J-1)-1)/2)} - \frac{1}{I(J-1)-3} \cdot \frac{I_{\omega(R)}(I/2,\, (I(J-1)-3)/2)}{I_{\omega(R)}(I/2,\, (I(J-1)-1)/2)}\right] . \tag{3.15}$$

Since Tiao and Tans' estimators as well as Stone and Springer's are only formal Bayes equivariant, it is not necessarily true that they are admissible in the class of equivariant estimators. We do not have a proof of either admissibility or inadmissibility of these estimators. Numerical comparison of their mean-square-error functions to those of some commonly used equivariant estimators (see [4]) shows that Tiao and Tan's estimators and Stone and Springer's are quite inefficient. The proper Bayes equivariant estimators (3.7) and (3.9) are at least admissible in the class of equivariant estimators.

4. The inadmissibility of the Bayes equivariant estimators

In the present section we prove that, given any Bayes equivariant estimator of σ_e^2, one can construct a non-equivariant estimator which has a uniformly smaller mean-square-error. This proves that *all* equivariant estimators are *inadmissible* in the *general* class of all estimators of σ_e^2. The proof follows a method suggested by Stein [7], and the main theorem is therefore called Stein's theorem. This theorem has been proven previously in the paper of Klotz, Milton and Zacks [4] with only a slightly different notation. It is reproduced here for the sake of continuity and due to its elegance.

STEIN'S THEOREM. *Let* $W=S_e^2+S_a^2+IJY_{..}^2$, $U_1=S_e^2/W$ *and* $U_2=(S_e^2+S_a^2)/W$. *Let* $\theta=(\sigma_e^2, \rho, \mu)$ *designate a parameter point, and let*

$$\tilde{\sigma}_e^2=W\psi(U_1, U_2) \tag{4.1}$$

be an estimator of σ_e^2. *If, for some* θ_0

$$P_{\theta_0}\left\{W\psi(U_1, U_2)>\frac{W}{IJ+2}\right\}>0, \tag{4.2}$$

then $\tilde{\sigma}_e^2$ *is inadmissible for squared-error loss.*

PROOF. According to the conditional distribution laws (1.1), (1.3) and (1.5), and since when $\{(a_1,\cdots,a_I), M_1, M_2\}$ are given the statistics S_e^2, S_a^2 and $IJY_{..}^2$ are conditionally independent, we obtain, from a well known result concerning the gamma distributions that U_1 and U_2 are *conditionally* independent of W, given $\{(a_1,\cdots,a_I), M_1, M_2\}$.

The mean-square-error of $\tilde{\sigma}_e^2=W\psi(U_1, U_2)$ is

$$\begin{aligned} &E\{[W\psi(U_1, U_2)-\sigma_e^2]^2\} \\ &=E\{E\{[W\psi(U_1, U_2)-\sigma_e^2]^2 \mid (a_1,\cdots,a_I, M_1, M_2\}\} \\ &=\sigma_e^4E\{E\{[\chi^2[IJ+2M_1+2M_2]\psi(U_1, U_2)-1]^2 \mid \\ &\qquad (a_1,\cdots,a_I), M_1, M_2, U_1, U_2\}\} \\ &=\sigma_e^4E\{\psi^2(U_1, U_2)(IJ+2M_1+2M_2)(IJ+2M_1+2M_2+2) \\ &\qquad -2\psi(U_1, U_2)(IJ+2M_1+2M_2)+1\} \\ &=\sigma_e^4E\Big\{(IJ+2M_1+2M_2)(IJ+2M_1+2M_2+2) \\ &\qquad \cdot\left[\psi(U_1, U_2)-\frac{1}{IJ+2M_1+2M_2+2}\right]^2+\frac{2}{IJ+2M_1+2M_2+2}\Big\}. \end{aligned} \tag{4.3}$$

Let

$$\psi^*(U_1, U_2)=\min\left\{\psi(U_1, U_2), \frac{1}{IJ+2}\right\}. \tag{4.4}$$

We have

$$\begin{aligned} &\left[\psi^*(U_1, U_2)-\frac{1}{IJ+2M_1+2M_2+2}\right]^2 \\ &\qquad \leq\left[\psi(U_1, U_2)-\frac{1}{IJ+2M_1+2M_2+2}\right]^2 \end{aligned} \tag{4.5}$$

for *all* $M_1, M_2 = 0, 1, \cdots$. Hence, if we define the estimator $(\sigma_e^*)^2 = W\psi^*(U_1, U_2)$, then the mean-square-error of $(\sigma_e^*)^2$ does not exceed that of $\tilde{\sigma}_e^2$ for all $\theta = (\sigma_e^2, \rho, \mu)$. Finally, if there exists a parameter point θ_0 at which (4.2) holds, then at that point

$$E_{\theta_0}\{[W\psi^*(U_1, U_2) - \sigma_e^2]^2\} < E_{\theta_0}\{[W\psi(U_1, U_2) - \sigma_e^2]^2\} . \tag{4.6}$$

This proves that condition (4.2) is sufficient for the inadmissibility of $\tilde{\sigma}_e^2$.

COROLLARY. *Every Bayes equivariant estimator of σ_e^2 is inadmissible.*

PROOF. Let $\varphi_\xi(R) = (L + Rf_\xi(R))/(IJ+1)$, where $f_\xi(R)$ is given by (2.3). Then, the Bayes equivariant estimator of σ_e^2 corresponding to ξ is $S_e^2\varphi_\xi(R)$. We notice that if $\psi_\xi(U_1, U_2) = U_1\varphi_\xi((U_2 - U_1)/U_1)$ then the Bayes-equivariant estimator can be written in the form $W\psi_\xi(U_1, U_2)$. Moreover, $Rf_\xi(R) > 0$ a.s. [all θ]. Hence,

$$P_\theta\left\{W\psi_\xi(U_1, U_2) > \frac{W}{IJ+2}\right\} = P_\theta\left\{\frac{S_e^2}{IJ+1}(1 + Rf_\xi(R)) > \frac{W}{IJ+2}\right\} > 0 , \quad \text{for } all\ \theta . \tag{4.7}$$

Hence, according to the previous theorem, *all* Bayes equivariant estimators of σ_e^2 are inadmissible.

The method of constructing an estimator having a uniformly smaller mean-square-error given in the proof of Stein's Theorem cannot be applied to show that all Bayes equivariant estimators of σ_a^2 are inadmissible. We shall not give here a rigorous proof of the inadmissibility of these estimators, $\hat{\sigma}_a^2$, but only a heuristic indication. We argue that, even if ρ is known the best equivariant estimator of σ_a^2, namely $\hat{\sigma}_a^2(\rho) = \rho\hat{\sigma}_e^2(\rho)$ is inadmissible. Indeed, the best equivariant estimator $\hat{\sigma}_e^2(\rho) = S_e^2/(IJ+1)\cdot(1 + R/(1 + J\rho))$ satisfies condition (4.6) of Stein's theorem and is therefore inadmissible. We conjecture that when ρ is unknown the Bayes equivariant estimators of σ_a^2 are also inadmissible.

Acknowledgement

The author wishes to thank Professor R. A. Wijsman for reading an earlier version of the paper and for certain suggestions.

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