ON THE RATE OF CONVERGENCE OF THE RANGE OF CUMULATIVE SUMS*

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1. Introduction and summary

Let $\{X_n, n\geqq 1\}$ be a sequence of independent random variables, and write $S_n=\sum_{k=1}^{n} X_k$. The range of cumulative sums, $\max(0, S_1, \cdots, S_n)-\min(0, S_1, \cdots, S_n)$ has been the subject of considerable research in the literature (see, for example, [3], [5] where further details and references may also be found). In what follows the random variables $X_1, X_2, \cdots$ are assumed to be independent and identically distributed with common law $\mathfrak{L}(X)$. We write $M_n=\max_{1\leq k\leq n} S_k$, $m_n=\min_{1\leq k\leq n} S_k$ and call $R_n=M_n-m_n$, the range of cumulative sums S_n. Take $0<r<2$ and write $F(x)=P(X\leqq x)$. Our purpose here is to prove an analogue of the Kolmogorov-Marcinkiewicz law of large numbers ([4], pp. 242–243) for the range R_n (Theorem 1) and obtain necessary and sufficient conditions, in terms of the order of magnitude of $P(|X|>n^{1/r})$, for the sequence $\{P(R_n>n^{1/r}\varepsilon), n\geqq 1\}$ to converge to zero for arbitrary $\varepsilon>0$ at specified rates (Theorem 2).

2. Results

THEOREM 1. *Let $E|X_i|^r<\infty$ with $0<r<2$ and write $EX_i=\mu$ whenever $E|X_i|<\infty$. Then,*

$$n^{-1/r}(R_n-nC_r) \xrightarrow{\text{a.s.}} 0$$

where $C_r=0$ if $0<r<1$, and $C_r=|\mu|$ if $1\leqq r<2$.

Remarks. Kolmogorov and Marcinkiewicz (see for example [4], pp. 242–243) obtained the corresponding almost sure convergence versions for the sums S_n and Heyde [1] obtained the corresponding version for the maxima M_n.

PROOF. For the proof we note that

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$$m_n = \min_{1 \leq k \leq n} S_k = -\{\max_{1 \leq k \leq n} (-S_k)\} .$$

The proof is now completed by applying Lemma 1 of Heyde [1] repeatedly to both M_n and m_n.

Let $L(\cdot)$ be a non-negative, non-decreasing and continuous function of slow variation. We then have the following theorem.

THEOREM 2. *If* $t \geqq 0$, *then*

a) $n^t L(n) P(R_n > n^{1/r}\varepsilon) \to 0$ *for all* $\varepsilon > 0$ *if and only if* $n^{t+1} L(n) P(|X| > n^{1/r}) \to 0$ *and* $n^{1-1/r} \int_{|x| < n^{1/r}} x dF(x) \to 0$,

b) $\sum_{n=1}^{\infty} n^{t-1} L(n) P(R_n > n^{1/r}\varepsilon) < \infty$ *for all* $\varepsilon > 0$ *if and only if* $\sum_{n=1}^{\infty} n^t L(n) P(|X| > n^{1/r}) < \infty$ *and* $n^{1-1/r} \int_{|x| < n^{1/r}} x dF(x) \to 0$.

Remarks. It is necessary to have the complementary forms of a) and b). For examples and other relevant remarks we refer to [2].

PROOF. The proof relies heavily on the results of [2] and [6]. For the sufficiency part of both a) and b) we note that

$$\begin{aligned} R_n &= M_n - \{-\max_{1 \leq k \leq n} (-S_k)\} \\ &\leqq 2 \max_{1 \leq k \leq n} |S_k| , \end{aligned}$$

and it follows that

$$P(R_n > n^{1/r}\varepsilon) \leqq P\left(\max_{1 \leq k \leq n} |S_k| > n^{1/r}\frac{\varepsilon}{2}\right).$$

A simple application of Theorems 3 and 4 of [6] and Theorems 1 and 2 of [2] now shows that the conditions in both a) and b) are sufficient.

As for the necessity part of both a) and b), we observe that

$$\begin{aligned} R_n &\geqq S_n - m_n \\ &= S_n + \max_{1 \leq k \leq n} (-S_k) \\ &\geqq \sum_{k=2}^{n} X_k , \end{aligned}$$

and

$$\begin{aligned} R_n &\geqq X_1 - m_n \\ &= X_1 + \max_{1 \leq k \leq n} (-S_k) \\ &\geqq -\sum_{k=2}^{n} X_k . \end{aligned}$$

Since $X_1, X_2, \cdots, X_n$ are identically distributed it follows that

$$2P(R_n > n^{1/r}\varepsilon) \leqq P(|S_{n-1}| > n^{1/r}\varepsilon) .$$

We once again appeal to Theorems 1 and 2 of [2] to complete the proof.

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