GROUP DIVISIBLE ROTATABLE DESIGNS —SOME FURTHER CONSIDERATIONS

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1. Introduction

In an earlier paper Das & Dey [2] introduced a series of response surface designs called "Group divisible rotatable designs". They presented a series of response surface designs in which the v factors of the design can be split into two groups of factors, one containing p factors ($p<v$) and the other rest ($v-p$) factors. The designs are rotatable within each of the groups when the levels of the factors in other group are held constant.

The construction of these designs as well as the analysis of the designs has been discussed in the above paper. The designs presented in that paper have each of the factors at 5-levels. The present paper aims at getting some designs of the type with factors each at three levels and also getting some further 5-level designs with smaller number of points.

2. Designs with 5-levels for each of the factors

Let the design points of a second order rotatable design, having 5-levels for each factor, be identified as the rows of a matrix $A_i^{N_i \times p_i}$, ($i=1, 2$). Then, we have the following.

THEOREM 2.1. *A group divisible rotatable design with each factor having 5-levels will be obtained by considering the rows of the following matrix D_1 as design points:*

$$D_1=\left[\begin{array}{c|c} A_1^{N_1\times p_1} & O^{N_1\times p_2} \\ \hline O^{N_2\times p_1} & A_2^{N_2\times p_2} \end{array}\right]$$

where A_1 and A_2 are as defined earlier and $O^{N_i\times p_j}$ ($i\neq j$); ($i, j=1, 2$) are null matrices of order $N_i\times p_j$, such that $N_1+N_2=N$, the number of design points and $p_1+p_2=v$, the number of factors in the design.

PROOF. The conditions to be satisfied by the levels of the factors in a "Group divisible rotatable design", as shown by Das & Dey [2], are as follows:

(A) $$\sum x_i=\sum x_i x_j=\sum x_i x_j^2=\sum x_i^3=\sum x_i x_j^3=\sum x_i x_j x_k$$
$$=\sum x_i x_j x_k^2=\sum x_i x_j x_k x_l=0 \qquad \text{for all } i\neq j\neq k\neq l\,,$$

(B) $$\sum x_j^2=\text{constant},$$
$$\sum x_j^4=\text{constant} \qquad \text{for all } j=1,\cdots,p_1\,,$$

(C) $$3\sum x_j^2 x_{j'}^2=\sum x_j^4 \qquad \text{for all } j\neq j'=1,\cdots,p_1\,,$$

(D) $$\sum x_i^2=\text{constant},$$
$$\sum x_i^4=\text{constant} \qquad \text{for all } i=p_i+1,\cdots,v\ (=p_1+p_2)\,,$$

(E) $$3\sum x_i^2 x_{i'}^2=\sum x_i^4 \qquad \text{for all } i\neq i'=p_1+1,\cdots,v\,,$$

(F) $$\sum x_i^2 x_j^2=\text{constant} \qquad i\neq j;\ i=1,\cdots,p_1,\quad j=p_1+1,\cdots,v\,,$$

where x_i denotes the levels of the ith factor ($i=1,\cdots,v$) and the summation is over the design points. Now, since the rows of A_i are the design points of a second order rotatable design, all the conditions (A) to (E) are satisfied by the levels of the factors. The condition (F) is also satisfied, as in this case

$$\sum x_i^2 x_j^2=0;\quad i\neq j,\quad i=1,\cdots,p_1$$
$$j=p_1+1,\cdots,p_1+p_2\,.$$

Hence we get the theorem.

This method of getting the present series of designs not only helps in reducing the number of points but also simplifies the solutions of the normal equations for estimating the "parameters" of the surface. This is due to the fact that the term $\sum x_i^2 x_j^2=\theta$ (as given in the paper by Das and Dey) now turns out to be zero.

Below we give an example with $v\ (=p_1+p_2)=8$, $p_1=4=p_2$. Each factor has 5-levels. The A_1 and A_2 matrices are as follows:

$$A_1=\begin{bmatrix} 1 & 2 & 3 & 4 \\ \hline a & a & a & a \\ a & a & a & -a \\ a & a & -a & a \\ a & a & -a & -a \\ a & -a & a & a \\ a & -a & a & -a \\ a & -a & -a & a \\ a & -a & -a & -a \\ -a & a & a & a \\ -a & a & a & -a \\ -a & a & -a & a \\ -a & a & -a & -a \\ -a & -a & a & a \\ -a & -a & a & -a \\ -a & -a & -a & a \\ -a & -a & -a & -a \\ b & 0 & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2=\begin{bmatrix} 5 & 6 & 7 & 8 \\ \hline a & a & a & 0 \\ a & a & -a & 0 \\ a & -a & a & 0 \\ a & -a & -a & 0 \\ -a & a & a & 0 \\ -a & a & -a & 0 \\ -a & -a & a & 0 \\ -a & -a & -a & 0 \\ a & a & 0 & a \\ a & a & 0 & -a \\ a & -a & 0 & a \\ a & -a & 0 & -a \\ -a & a & 0 & a \\ -a & a & 0 & -a \\ -a & -a & 0 & a \\ -a & -a & 0 & -a \\ a & 0 & a & a \\ a & 0 & a & -a \\ a & 0 & -a & a \\ a & 0 & -a & -a \\ -a & 0 & a & a \\ -a & 0 & a & -a \\ -a & 0 & -a & a \\ -a & 0 & -a & -a \\ 0 & a & a & a \\ 0 & a & a & -a \\ 0 & a & -a & a \\ 0 & a & -a & -a \\ 0 & -a & a & a \\ 0 & -a & a & -a \\ 0 & -a & -a & a \\ 0 & -a & -a & -a \\ c & 0 & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & -c \end{bmatrix}.$$

The total number of points in this case is $25+40=65$ as against 160 points required by the method given by Das & Dey [2].

3. Designs with 3-levels

It is well known that a second order rotatable design in 3 levels can always be constructed if there exists a balanced incomplete block design (BIBD) with $r=3\lambda$ where r and λ have their usual meaning. Then we have the following.

THEOREM 3.1. *A group divisible rotatable design in 3 levels will be obtained by considering the rows of the matrix D_2 as design points, where D_2 is defined as follows:*

$$D_2=\left[\begin{array}{c|c} A_1^{*N_1\times p_1} & O^{N_1\times p_2} \\ \hline O^{N_2\times p_1} & A_2^{*N_2\times p_2} \end{array}\right].$$

$A_i^{*N_i\times p_i}$ $(i=1, 2)$ *are matrices of order* $N_i\times p_i$ $(i=1, 2)$ *whose rows are the design points of a second order rotatable design obtained through BIBD with* $r=3\lambda$.

The proof of this theorem follows from Theorem 2.1.

4. A result of some interest

Let there be a group divisible rotatable design in v factors obtained through any of the above methods. Then a group divisible rotatable design in $(v-x)$ factors $(x<v)$ can be obtained by omitting any x columns of the design in v factors.

Summary

The paper discusses a method of construction of "Group divisible rotatable designs". Through this method 5-level designs are obtained with smaller number of points. A series of 3-level designs has also been put forward.

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