ON THE USE OF A LINEAR MODEL FOR THE IDENTIFICATION OF FEEDBACK SYSTEMS

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Summary

A basic linear model of stationary stochastic processes is proposed for the analysis of linear feedback systems. The model suggests a simple computational procedure which gives estimates of the response characteristics of the system and the spectra of the noise source. These estimates are obtained through the estimate of the linear predictor of the process, which is obtained by the ordinary least squares method.

The necessary assumption for the validity of the estimation procedure is so general that the procedure can be applied to the analysis of wide variety of practical systems with feedback.

The content of the present paper forms an answer to the problem discussed by the author in a former paper [1].

1. Introduction

The cross-spectral method has been extensively applied to the estimation of frequency response functions [2], [3], [8]. The discrete time parameter model which forms the basis of the cross-spectral method of estimation is of the form

$$x_0(n)=\sum_{j=1}^{K}\sum_{m}a_{jm}\,x_j(n-m)+u_0(n)\,,$$

where the unobservable noise $u_0(n)$ is assumed to be uncorrelated with, or orthogonal to, the input series $x_j(n)$ $(j=1, 2, \cdots, K)$. This last assumption is essential for the validity of ordinary cross-spectral approach [1].

In many important practical situations, the existence of feedback loops which connect the output $x_0(n)$ to the inputs $x_j(n)$'s is quite common and the present assumption of orthogonality of $u_0(n)$ to $x_j(n)$'s seriously limits the practical applicability of the method. As was briefly touched in the former paper [1] this difficulty may be due to our inability of including the condition of physical realizability, which requires the output of a system to be determined without using the future values of the input, into the cross-spectral method.

In the present paper we treat the problem in the time domain and directly get estimates of the impulse response functions $\{a_{jm}: m=0, 1, 2, \cdots\}$ $(j=1, 2, \cdots, K)$. The estimation procedure can be explained as follows: we first conceptually whiten the spectrum of the additive disturbance $u_0(n)$, or actually a driving input to the system, by a physically realizable linear transformation and then apply the least squares method to get an estimate of the regression coefficients of the linear predictor of the process and finally back-transform the estimated predictor into the original form of the system structure. The only practical condition to assure the validity of the estimation procedure is that, besides the existence of necessary driving input for each $x_j(n)$ such as $u_0(n)$ for $x_0(n)$, there should be some delays in the feedback loops so that we can effectively assume that an instantaneous return from the output to itself through the feedback loops is prohibited. When we are observing a physical process, this condition will be satisfied at least approximately if we limit our attention to some frequency band.

In the next section we shall give a precise description of the basic model and in the following section we propose an estimation procedure. The procedure is directly applicable to practical data and the consistency of the estimates is discussed. Some numerical examples are given to show the practical applicability of the procedure. Possible bias due to the incorrect specification of the model is discussed in the last section.

2. Basic model

Here we consider a set of observation points $\{i; i=0, 1, 2, \cdots, K\}$. The model of the system we are going to treat in this paper is given by the relation

$$x_i(n)=\sum_{j=0}^{K}\sum_{m=0}^{M} a_{ijm}\, x_j(n-m)+u_i(n)$$

$$i=0, 1, 2, \cdots, K, \qquad n=0, 1, 2, \cdots,$$

where $\{u_i(n); n=0, 1, 2, \cdots\}$ is the driving input at i and $\{x_i(n); i=0, 1, \cdots, K, n=0, 1, \cdots\}$ is the response of the system and the initial condition of the system is given by $\{x_i(n); i=0, 1, \cdots, K, n=-1, -2, \cdots, -M\}$. $\{a_{ijm}; m=0, 1, \cdots, M\}$ is the impulse response of the output $x_i(n)$ to the input $x_j(n)$ and we shall assume

$$a_{iim}=0 \qquad i=0, 1, \cdots, K, \quad m=0, 1, 2, \cdots, M.$$

This assumption is a salient feature of our model and if there is a feedback loop from i to i without going through other j's its effect is included into all of the responses $\{a_{ijm}\}$ and also the noise within the loop is included into $u_i(n)$. This fact should be taken into account when we analyze the result of application of our model to a practical problem.

In the following, we shall denote by $|G|$ the determinant of a matrix G and by $(G)_{ij}$ the (i, j) element of G and by I_n the n-dimensional identity matrix. We define

$$A(z)=\sum_{m=0}^{M} A_m z^m ,$$

where z is a complex scalar variable and A_m is a $(K+1)$-dimensional matrix with $(A_m)_{ij}=a_{ijm}$. We shall assume that the absolute values of the roots of characteristic equation $|I_{K+1}-A(z)|=0$ are all greater than 1. Under this assumption, $B(z)=(I_{K+1}-A(z))^{-1}$ has a Taylor expansion $B(z)=\sum_{m=0}^{\infty} B_m z^m$ with radius of convergence greater than 1. Thus if we put $b_{ijm}=(B_m)_{ij}$ we have $\sum_{m=0}^{\infty}|b_{ijm}|<\infty$ for every (i, j). From the relation $(I_{K+1}-A(z))\,B(z)=I_{K+1}$, we have

$$(I_{K+1}-A_0)\,B(z)-A'(z)\,B(z)=I_{K+1} ,$$

where $A'(z)=\sum_{m=1}^{M} A_m z^m$. Thus we get

$$(I_{K+1}-A_0)\,B_0=I_{K+1} ,$$

$$(I_{K+1}-A_0)\,B_m=\sum_{k=1}^{m} A_k B_{m-k} ,$$

where it is assumed that $A_k=0$ (null matrix) for $k>M$. This result shows that b_{ijm} is the response of the system at i at time m when a unit impulse $\{u_j(n)\}$ with $u_j(0)=1$ and $u_j(n)=0$ $(n\neq 0)$ is applied at j and other $u_k(n)$'s are all kept equal to zero. We have noticed that under our assumption $\sum_{m=0}^{\infty}|b_{ijm}|<\infty$ holds for every pair (i, j). Thus we can see that our system is absolutely stable in the sense that starting from arbitrary initial condition the response of the system eventually damps out if it is without driving input. It can be shown that our assumption on the absolute values of the characteristic equation $|I_{K+1}-A(z)|=0$ is just equivalent to this absolute stability of the system. It also should be noticed that the assumption implies $|I_{K+1}-A_0|\neq 0$.

Now we turn our attention to the stochastic situation where $\{u_i(n); i=0, 1, \cdots, K;\ n=0, \pm 1, \pm 2, \cdots\}$ is a $(K+1)$-dimensional stationary process with zero mean vector and finite second order moments. We assume $\{u_i(n)\}$ $(i=0, 1, \cdots, K)$ to be mutually uncorrelated, or orthogonal, and that each $u_i(n)$ is a regular process in the sense that it admits a one-sided moving average representation $u_i(n)=\sum_{l=0}^{\infty} b_{il}\,\varepsilon_i(n-l)$ with a white noise $\varepsilon_i(n)$ satisfying $E\varepsilon_i(n)=0$, $E\varepsilon_i^2(n)=\sigma_i^2$ (>0) and $E\varepsilon_i(n)\,\varepsilon_i(m)=0$ $(n\neq m)$. Hereafter the convergence and equality of random quantities are to be understood in the sense of mean square. We shall assume $b_{i0}=1$ $(i=0, 1, \cdots, K)$. To assume the regularity of $u_i(n)$ is just equivalent to assuming the influence of its infinitely remote past history to be vanishing in $u_i(n)$ in the sense of mean square. Thus the regularity assumption is a natural one for various practical situations. Now $\sum_{l=1}^{\infty} b_{il}\,\varepsilon_i(n-l)$ is the projection of $u_i(n)$ into the space spanned by its own past history and $\varepsilon_i(n)$ is the innovation. Thus $\sum_{l=1}^{\infty} b_{il}\,\varepsilon_i(n-l)$ can be approximated arbitrarily closely in the sense of mean square by a finite linear combination of $u_i(n-m)$ $(m=1, 2, \cdots)$. Thus, as a simplification, we assume that for some finite L $u_i(n)$ satisfies the relation

$$\text{(A)} \qquad u_i(n)=\sum_{l=1}^{L} c_{il}\,u_i(n-l)+\varepsilon_i(n)\,.$$

As we have assumed the orthogonality between $\{u_i(n)\}$'s, it holds that $E\varepsilon_i(n)\,\varepsilon_j(m)=0$ $(i\neq j)$. Here we define $x_i(n)$ by

$$x_i(n)=\sum_{j=0}^{K}\sum_{m=0}^{\infty} b_{ijm}\,u_j(n-m)\,.$$

Then it can be seen that $\{x_i(n)\,;\ i=0, 1, \cdots, K\}$ satisfies the relation

$$x_i(n)=\sum_{j=0}^{K}\sum_{m=0}^{M} a_{ijm}\,x_j(n-m)+u_i(n)\,.$$

Now let us assume that $\{z_i(n)\,;\ i=0, 1, \cdots, K\}$ satisfies the same relation as $\{x_i(n)\,;\ i=0, 1, \cdots, K\}$ and is stationarily correlated with $\{u_i(n)\}$ or with $\{x_i(n)\}$. Then $y_i(n)=x_i(n)-z_i(n)$ satisfies the relation

$$y_i(n)=\sum_{j=0}^{K}\sum_{m=0}^{M} a_{ijm}\,y_j(n-m) \quad i=0, 1, \cdots, K.$$

From this it follows that

$$\int_{-1/2}^{1/2} \exp(\sqrt{-1}\,2\pi f m)(I_{K+1}-A(\exp(-\sqrt{-1}\,2\pi f)))\,dF_y(f)=O \qquad m=0, \pm1, \pm2, \cdots ,$$

where $F_y(f)$ is the matrix spectral function of $\{y_i(n)\,;\ i=0, 1, \cdots, K\}$ and is defined by the relation $Ey_i(n+m)\,y_j(n)=\int_{-1/2}^{1/2}\exp(\sqrt{-1}\,2\pi f m)\,d(F_y(f))_{ij}$

and O is the $(K+1)\times(K+1)$ null matrix. Thus we get, formally,

$$(I_{K+1}-A(\exp(-\sqrt{-1}2\pi f)))\,dF_y(f)=O\,,$$

and by multiplying $B(\exp(-\sqrt{-1}2\pi f))$ from left we get

$$dF_y(f)=O\,.$$

This shows that under our assumption of the absolute stability of the system, above relation between $\{x_i(n)\,;\,i=0,1,\cdots,K\}$ and $\{u_i(n)\,;\,i=0,1,\cdots,K\}$ uniquely determines $\{x_i(n)\}$ in the sense of mean square. Thus, as our basic model we adopt

$$\text{(B)}\qquad x_i(n)=\sum_{j=0}^{K}\sum_{m=0}^{M}a_{ijm}\,x_j(n-m)+u_i(n)$$
$$i=0,1,\cdots,K,\quad n=0,\pm1,\pm2,\cdots.$$

It should be remembered that we are assuming $a_{iim}=0$ and not assuming $u_i(n)$ to be a white noise.

Now by using the relation (A), we can transform the original representation (B) to whiten $u_i(n)$ and get

$$\text{(C)}\qquad x_i(n)=\sum_{j=0}^{K}\sum_{m=0}^{M+L}A_{ijm}\,x_j(n-m)+\varepsilon_i(n)$$

where

$$\begin{aligned}
&A_{ii0}=0\\
&A_{iim}=c_{im} && \text{for}\quad m=1,2,\cdots,L\,,\\
&A_{ij0}=a_{ij0}\\
&A_{ijm}=a_{ijm}-\sum_{l=1}^{m}c_{il}\,a_{ijm-l} && \text{for}\quad m=1,2,\cdots,M+L\,,
\end{aligned}$$

and it is assumed that $c_{il}=0$ for $l>L$ and $a_{ijm}=0$ for $m>M$. Conversely $\{a_{ijm}\}$ and $\{c_{im}\}$ can be obtained from $\{A_{ijm}\}$ by the relation

$$\begin{aligned}
&c_{im}=A_{iim} && \text{for}\quad m=1,2,\cdots,L\,,\\
&a_{ij0}=A_{ij0}\\
&a_{ijm}=A_{ijm}+\sum_{l=1}^{m}c_{il}\,a_{ijm-l} && \text{for}\quad m=1,2,\cdots,M\,,
\end{aligned}$$

where we are assuming $c_{il}=0$ for $l>L$.

Our present representation (C) corresponds to the so called reduced form of mutually related multiple time series when $A_{ij0}=a_{ij0}=0$ for all (i,j) and in this case we can readily apply the method of least squares to get an estimate of $\{A_{ijm}\}$ [5]. In physical processes with continuous time parameter, there are usually lags in the responses and if we limit our attention to some frequency band and select the length of sampling interval between consecutive observations short enough, we shall generally be able to expect this condition to hold for the equi-spaced time-sampled data. But using a too short time interval usually introduces inefficiency of estimation procedure and sometimes it is not quite practical to ask all the A_{ij0}'s to be vanishing. What we need here for the validity of the least squares method is that the values of $x_j(n)$ with $A_{ij0}\neq 0$ should be (practically at least approximately) uncorrelated with $\varepsilon_i(n)$.

From this, we can see that for our requirement to be filled it is necessary and sufficient that

$$(A_0)_{ij}((I_{K+1}-A_0)^{-1})_{ji}=0 \qquad \text{for} \quad i, j=0, 1, \cdots, K.$$

If we are going to approximate a physical process with continuous time parameter by our present model, A_0 will represent the effect of quick responses of the system and A_0^n will have to approximate the effect of quick responses traveling through n observation points during a unit of sampling interval. In this case, we shall have to select the sampling interval short enough so that at least we can expect that every possible quick response of i to i is effectively negligible during the unit of sampling interval. Otherwise, we shall generally never be able to expect the orthogonality of $\varepsilon_i(n)$ to all of $x_j(n)$'s with $a_{ij0}\neq 0$. Thus we assume

$$(A_0^n)_{ii}=0 \qquad \text{for} \quad i=0, 1, \cdots, K \quad \text{and} \quad n=1, 2, \cdots.$$

This condition is satisfied if and only if, after proper rearrangement of the vector components of $\{x_0(n), \cdots, x_K(n)\}$ or relabelling of the observation points, A_0 has zeros on and below the diagonal, i.e., it takes the form

$$A_0=\begin{bmatrix} 0 & * & * \cdots * & * \\ 0 & 0 & * \cdots * & * \\ 0 & 0 & 0 \cdots * & * \\ \vdots & \vdots & \vdots \quad\; \vdots & \vdots \\ 0 & 0 & 0 \cdots 0 & * \\ 0 & 0 & 0 \cdots 0 & 0 \end{bmatrix}.$$

It can be seen that $(A_0)_{ij}((I_{K+1}-A_0)^{-1})_{ji}=0$ holds for this type of A_0. We shall hereafter assume this shape of A_0. Taking into account the relation $A_{ij0}=a_{ij0}$, we can see that in this case (C) takes the form

$$\text{(D)} \qquad x_i(n)=\sum_{j=i+1}^{K} A_{ij0}\, x_j(n)+\sum_{j=0}^{K}\sum_{m=1}^{M+L} A_{ijm}\, x_j(n-m)+\varepsilon_i(n)$$
$$i=0, 1, \cdots, K.$$

This corresponds to the so-called primary form of a linear causal chain system which has been extensively discussed by Wold [11], [12] in the field of econometric model building. It is fairly clear from our observation that our model with present assumption will be useful for the analysis of various physical processes. The assumed shape of A_0 means that, roughly speaking, we are arranging $x_j(n)$'s in the order of the speeds of their responses; starting from $x_0(n)$, the quickest one, to $x_K(n)$, the slowest one.

Obviously our present model is only a crude approximation for a physical process with continuous time parameter and if there exist any ambiguities of the values of estimates of a_{ij0} obtained by applying the model to equi-spaced time-sampled data we should try another analysis using a sample with twice as much frequency of sampling as of the original one and assuming $a_{ij0}=0$. Theoretical discussion of the approximation of continuous time parameter process by one with discrete time parameter would be important but is beyond the scope of the present paper.

For the proof of consistency of the least squares estimate of the coefficients of (D), we have to show the non-singularity of the variance covariance matrix of the regressor $\{x_j(n)\ (j=i+1, \cdots, K),\ x_j(n-m)\ (j=0, 1, \cdots, K;\ m=1, 2, \cdots, H)\}$ for any finite positive integer H. For this we have to notice that

$$(I_{K+1}-A_0)^{-1}=I_{K+1}+A_0+A_0^2+\cdots+A_0^K$$

and thus $(I_{K+1}-A_0)^{-1}$ has all zeros below the diagonal and 1's on the diagonal. The multi-dimensional prediction formula of $x_i(n)$ is then given by

$$x_i(n)=\sum_{j=0}^{K}\sum_{m=1}^{M+L}\sum_{k=0}^{K}((I-A_0)^{-1})_{ik}\,A_{kjm}\,x_j(n-m)+\delta_i(n)$$

where

$$\delta_i(n)=\sum_{k=0}^{K}((I-A_0)^{-1})_{ik}\,\varepsilon_k(n)=\varepsilon_i(n)+\sum_{k=i+1}^{K}((I-A_0)^{-1})_{ik}\,\varepsilon_k(n)\,.$$

As we have assumed $E\varepsilon_i^2(n)=\sigma_i^2>0$ and $\varepsilon_i(n)$'s are mutually orthogonal, the variance covariance matrix of $\delta_i(n)$ $(i=0, 1, \cdots, K)$ is non-singular. For any set of coefficients $\{B_{jm}\}$ $(m=0, 1, \cdots, H)$ we have

$$E\left|\sum_{j=0}^{K}\sum_{m=0}^{H}B_{jm}\,x_j(n-m)\right|^2=E\left|\sum_{j=0}^{K}B_{j0}\,\delta_j(n)\right|^2$$
$$+E\left|\text{linear combination of } x_j(n-m)\ (m=1,2,\cdots,H)\right|^2,$$

and for this quantity to be vanishing we have to ask $B_{j0}=0$ $(j=0, 1, \cdots, K)$ and accordingly $B_{jm}=0$ for all j and m. Thus, we have shown the non-singularity of the variance covariance matrix of $\{x_j(n)$ $(j=i+1, \cdots, K)$, $x_j(n-m)$ $(j=0, 1, \cdots, K;\ m=1, 2, \cdots, H)\}$. Taking into account the fact that $x_i(n)$ admits one-sided moving average representation by $\varepsilon_j(n)$'s, we can also get the present result as a direct consequence of a general theory of regular full rank process ([10], p. 147). We shall use the present result in the next section and see that the assumption $E\varepsilon_i^2(n)=\sigma_i^2>0$ $(i=0, 1, \cdots, K)$ is playing a definite role in our estimation procedure.

It is interesting to note that Parzen [7] has stressed the importance of the representation of time series in which the disturbances have the properties that least squares estimates are the efficient estimates. It is stated that "There is no guarantee that such representations exist. What to do in this case represents a development of the theory of statistical inference on stochastic processes which in my opinion would not be merely an extension of classical statistical inference." ([7], p. 45). Our present procedure uses such a representation as a pivot to get the final estimate, and it gives an example that the analysis of multiple time series is not completely reduced to the cross-spectral analysis if the latter is confined to the mere extension of classical regression analysis into the complex domain.

Obviously our transformed representation (D) is a kind of predictor formula for the process $x_i(n)$ and we are going to identify the necessary response characteristics and noise spectra through this representation. Thus it seems to be appropriate to call our present identification procedure predictive identification.

3. Estimation procedure

There is a fundamental paper on the estimation of the coefficients of a multiple autoregressive scheme by Mann and Wald [5]. Relation between the least squares estimation and the causal chain system was discussed by Wold [11], [12]. Also there are papers by Durbin [4] and by Whittle [9] which are closely related with our present subject.

Here we shall only describe a simple, though not necessarily efficient, estimation procedure for direct applications and very briefly discuss the consistency of the estimates. Practical applicability of the method will be illustrated by some numerical examples. An example of application to the analysis of a physical process will be discussed elsewhere [6]. It is quite desirable to have a practical formula for the evaluation of sampling variabilities of our estimates. We shall discuss this in a subsequent paper.

Our estimation procedure is as follows:

I First we arrange the records in the order of the speeds of their responses; the quickest one as $x_0(n)$ and the slowest one as $x_K(n)$.

We properly select the values of L and M. These are guessed values of L and M in the original model (B). For typographical simplicity we shall not distinguish these values from the true values but the difference will be clear from the context. We suggest to do the whole computation for several sets of values of L and M to get informations for the selection of L and M. The values of L and M may change for each set of the following normal equations.

II We solve the normal equation for each i;

$$\sum_{j=i+1}^{K} \hat{A}_{ij0} C(x_j, x_k)(l) + \sum_{j=0}^{K} \sum_{m=1}^{M+L} \hat{A}_{ijm} C(x_j, x_k)(l-m) = C(x_i, x_k)(l)$$

$$\begin{aligned} &l=1, 2, \cdots, M+L \quad &\text{for } k=0, 1, 2, \cdots, i, \\ &l=0, 1, 2, \cdots, M+L \quad &\text{for } k=i+1, i+2, \cdots, K, \end{aligned}$$

where $C(\xi, \eta)(l)$ denotes the lagged sample covariance $C(\xi, \eta)(l) = \frac{1}{N}\sum_{n=1}^{N-l} \xi(n+l)\,\eta(n)$ of observed sequence $\{\xi(n), \eta(n);\ n=1, 2, \cdots, N\}$. We are assuming the mean values of $\xi(n)$ and $\eta(n)$ to be vanishing.

III We put

$$\begin{aligned} \hat{c}_{il} &= \hat{A}_{iil} \qquad (l=1, 2, \cdots, L)\,, \\ \hat{c}_{il} &= 0 \qquad (l>L) \end{aligned}$$

and get $\hat{a}_{ijm}$ by the relation

$$\begin{aligned} \hat{a}_{ij0} &= \hat{A}_{ij0}\,, \\ \hat{a}_{ijm} &= \hat{A}_{ijm} + \sum_{l=1}^{m} \hat{c}_{il}\, \hat{a}_{ijm-l} \quad (m=1, 2, \cdots, M)\,. \end{aligned}$$

These $\{\hat{a}_{ijm}\}$'s are the desired estimates of the impulse response functions $\{a_{ijm}\}$. We assume $\hat{a}_{ij0} = a_{ij0} = 0$ for $j=0, 1, \cdots, i$.

IV We get an estimate $\hat{a}_{ij}(f)$ of the frequency response function $a_{ij}(f) = \sum_{m=0}^{M} a_{ijm} \exp(-2\pi \sqrt{-1}\, fm)$ by

$$\hat{a}_{ij}(f) = \sum_{m=0}^{M} \hat{a}_{ijm} \exp(-2\pi \sqrt{-1}\, fm)\,.$$

V We get an estimate of the power spectral density function $p(u_i)(f)$ of $u_i(n)$ by

$$\hat{p}(u_i)(f) = \frac{S_i^2}{\left|1 - \sum_{l=1}^{L} \hat{c}_{il} \exp(-2\pi \sqrt{-1}\, fl)\right|^2}$$

where

$$S_i^2 = C(x_i, x_i)(0) - \sum_{k=0}^{i} \sum_{l=1}^{M+L} C(x_i, x_k)(l)\hat{A}_{ikl} - \sum_{k=i+1}^{K} \sum_{l=0}^{M+L} C(x_i, x_k)(l)\hat{A}_{ikl}$$

is an estimate of $E\varepsilon_i^2(n)$.

VI We compute an estimate $\hat{b}_{ij}(f)$ of the closed loop frequency response function from j to i by

$$[\hat{b}_{ij}(f)] = [\delta_{ij} - \hat{a}_{ij}(f)]^{-1}$$

where [ ] denotes a $(K+1)\times(K+1)$ matrix and

$$\delta_{ii}=1 \quad \text{and} \quad \delta_{ij}=0 \ (i \neq j) .$$

VII The power contribution of the noise source $u_j(n)$ to the output $x_i(n)$ can be estimated by the quantity

$$\hat{q}_{ij}(f) = |\hat{b}_{ij}(f)|^2 \hat{p}(u_j)(f) \quad j=0, 1, 2, \cdots, K .$$

It is expected that at least approximately

$$\hat{p}(x_i)(f) = \sum_{j=0}^{K} \hat{q}_{ij}(f)$$

holds, where $\hat{p}(x_i)(f)$ is an estimate of the power spectral density at f of $x_i(n)$, which is obtained from the record $\{x_i(n);\ n=1, 2, \cdots, N\}$. If this is not the case, some increase of L and/or M would be necessary. The equality is strict when the original model is strict and an infinitely long record and M and L greater than or equal to their true values, respectively, are used for the computation and $\hat{p}(x_i)(f)$ is replaced by its theoretical value.

The quantity $\gamma_{ij}(f)$ given by

$$\gamma_{ij}(f) = \frac{\hat{q}_{ij}(f)}{\sum_{k=0}^{K} \hat{q}_{ik}(f)}$$

will be used to evaluate the relative contribution of $u_j(n)$ to the power of $x_i(n)$ at frequency f, and the quantity $R_{ij}(f)$ defined by

$$R_{ij}(f) = \sum_{k=0}^{j} \gamma_{ik}(f) \quad j=0, 1, 2, \cdots, K-1$$

will conveniently be used for graphical representation.

We shall here very briefly discuss the consistency of our estimates. Under our assumption of A_0, $\{A_{ijm};\ j=0, 1, \cdots, K,\ m=0, 1, \cdots, M\}$ satisfies the normal equation of the step II when $C(x_j, x_k)(l)$'s are replaced by the corresponding $E(x_j(n+l)\, x_k(n))$'s. We have already shown in the preceding section the non-singularity of the variance covariance matrix of $\{x_j(n)\,(j=i+1, \cdots, K),\ x_j(n-m)\,(j=0, 1, \cdots, K,\ m=1, 2, \cdots, H)\}$ (H: any finite positive integer). Thus if we can assume the convergence of $C(x_j, x_k)(l)$ to $E(x_j(n+l)\,x_k(n))$ in probability or with probability one, we can show the convergence of $\hat{A}_{ijm}$, properly defined when the solution of the normal equation does not exist, to A_{ijm} in probability or with probability one, respectively. $\hat{c}_{il}$, $\hat{a}_{ijm}$, S_i^2 and other related quantities converge to their theoretical values correspondingly. The simplest and sometimes most natural assumption in practical situations would be the assumption of ergodicity of the noise $\{u_i(n);\ i=0, 1, \cdots, K\colon n=0, \pm 1, \pm 2, \cdots\}$. Under this assumption the convergence is with probability one.

To show the practical applicability of our estimation procedure we give here some numerical examples. We have used as a realization of

Table 1

m	a_{01m}	$\hat{a}_{01m}$ ($L=6$, $M=6$)	$\hat{a}_{01m}$ ($L=6$, $M=2$)
0	0	−0.053	−0.052
1	0.12	0.120	0.122
2	0.20	0.178	0.175
3	0.05	−0.033	
4	0	−0.048	
5	0	0.012	
6	0	−0.052	

m	a_{10m}	$\hat{a}_{10m}$ ($L=6$, $M=6$)	$\hat{a}_{10m}$ ($L=6$, $M=2$)
0	0	0	0
1	−0.10	−0.113	−0.111
2	−0.10	−0.069	−0.068
3	−0.10	−0.146	
4	0	−0.067	
5	0	0.010	
6	0	0.050	

Table 2

m	a_{01m}	$\hat{a}_{01m}$ ($L=6$, $M=6$)	a_{10m}	$\hat{a}_{10m}$ ($L=6$, $M=6$)
0	0.25	0.197	0	0
1	0.15	0.148	−0.1	−0.114
2	0.08	0.057	−0.2	−0.169
3	0.03	−0.056	−0.1	−0.151
4	0	−0.050	0	−0.077
5	0	0.005	0	0.007
6	0	−0.055	0	0.064

our noise $\{u_i(n)\ (i=0, 1;\ n=1, 2, \cdots, 500)\}$ the results of two independent observations of a physical process and generated $x_i(n)$ $(i=0, 1)$ by the formula (B). We assumed $x_i(n)=0$ for $n\leqq 0$. The results of computation are illustrated in Tables 1 and 2. From the result of Table 1 we can see that our estimate is rather insensitive to the change of M. This tendency has been observed in numerous other practical applications and suggests a kind of robustness of our procedure. The result illustrated in Table 2 is concerned with the numerical example of artificial series reported in [1]. The result shows that our present procedure is quite promising even in the identification of this kind of model where $a_{010}\neq 0$, if only the specification of the model or the ordering of the variables is correct. We shall discuss this last point in the next section. It also should be mentioned that in an example of application to a physical process [6], the present procedure gave a quite reasonable result, where the conventional cross-spectral method of estimation of the frequency response function completely failed.

4. Bias due to incorrect specification

Here we shall analyse the effect of incorrectly assuming $a_{ij0}\neq 0$ for some (i, j). We treat the case where $K=1$, i.e., the 2-dimensional case. Thus as our original form we have

$$x_0(n)=\sum_{m=0}^{M} a_{01m}\, x_1(n-m)+u_0(n)$$

$$x_1(n)=\sum_{m=1}^{M} a_{10m}\, x_0(n-m)+u_1(n)\,.$$

After whitening of $u_0(n)$ and $u_1(n)$ we get

$$x_0(n)=\sum_{m=1}^{L} A_{00m}\, x_0(n-m)+\sum_{m=0}^{M+L} A_{01m}\, x_1(n-m)+\varepsilon_0(n)$$

$$x_1(n)=\sum_{m=1}^{M+L} A_{10m}\, x_0(n-m)+\sum_{m=1}^{L} A_{11m}\, x_1(n-m)+\varepsilon_1(n)\,.$$

We are assuming $A_{100}=0$. If we incorrectly specify the model as

$$x_1(n)=\sum_{m=0}^{M+L} c_{10m}\, x_0(n-m)+\sum_{m=1}^{L} c_{11m}\, x_1(n-m)+\eta(n)$$

and apply the method of least squares we shall have to solve the following normal equation:

$$\sum_{m=0}^{M+L} \hat{c}_{10m}\, C(x_0, x_0)(l-m)+\sum_{m=1}^{L} \hat{c}_{11m}\, C(x_1, x_0)(l-m)=C(x_1, x_0)(l)$$

$$l=0, 1, 2, \cdots, M+L\,,$$

$$\sum_{m=0}^{M+L} \hat{c}_{10m} C(x_0, x_1)(l-m) + \sum_{m=1}^{L} \hat{c}_{11m} C(x_1, x_1)(l-m) = C(x_1, x_1)(l) \qquad l=1, 2, \cdots, L.$$

Here we assume that our computation is based on an infinitely long record and $C(x_i, x_j)(l) = Ex_i(n+l)\,x_j(n)$. Then, from the definition of the model, we have

$$\sum_{m=1}^{M+L} A_{10m} C(x_0, x_0)(l-m) + \sum_{m=1}^{L} A_{11m} C(x_1, x_0)(l-m) = C(x_1, x_0)(l) \qquad l=1, 2, \cdots, M+L,$$

$$\sum_{m=1}^{M+L} A_{10m} C(x_0, x_1)(l-m) + \sum_{m=1}^{L} A_{11m} C(x_1, x_1)(l-m) = C(x_1, x_1)(l) \qquad l=1, 2, \cdots, L.$$

We also have

$$C(x_1, x_0)(0) = \sum_{m=1}^{L+M} A_{10m} C(x_0, x_0)(-m) + \sum_{m=1}^{L} A_{11m} C(x_1, x_0)(-m) + C(\varepsilon_1, x_0)(0),$$

where

$$C(\varepsilon_1, x_0)(0) = A_{010} C(\varepsilon_1, x_1)(0) = A_{010} C(\varepsilon_1, \varepsilon_1)(0).$$

Thus the bias $B_{1jm} = c_{1jm} - A_{1jm}$ is given as

$$\begin{bmatrix} B_{100} \\ \vdots \\ B_{10M+L} \\ B_{111} \\ \vdots \\ B_{11L} \end{bmatrix} = \begin{bmatrix} C(x_0, x_0)(0) & \cdots C(x_0, x_0)(M+L) & C(x_1, x_0)(-1) & \cdots C(x_1, x_0)(-L) \\ \vdots & \vdots & \vdots & \vdots \\ C(x_0, x_0)(M+L) & \cdots C(x_0, x_0)(0) & C(x_1, x_0)(M+L-1) & \cdots C(x_1, x_0)(M) \\ C(x_0, x_1)(1) & \cdots C(x_0, x_1)(1-M-L) & C(x_1, x_1)(0) & \cdots C(x_1, x_1)(L-1) \\ \vdots & \vdots & \vdots & \vdots \\ C(x_0, x_1)(L) & \cdots C(x_0, x_1)(-M) & C(x_1, x_1)(L-1) & \cdots C(x_1, x_1)(0) \end{bmatrix}^{-1} \times \begin{bmatrix} A_{010} C(\varepsilon_1, \varepsilon_1)(0) \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the elements of the last vector are all zeros except the first one. As the inverse matrix is positive definite we can see that the following relation holds:

$$c_{100}=B_{100}=A_{010}\,C(\varepsilon_1,\ \varepsilon_1)(0)\times(\text{positive number})\,.$$

Taking into account the relation $a_{ij0}=A_{ij0}$, this result tells us that by missing to specify $a_{100}=0$ we shall introduce into the estimate of a_{100}, the bias B_{100} which is of the same sign as A_{010} or a_{010}. This effect is clearly seen in our numerical example shown in Table 3. The result was obtained by using the same data as the result of Table 2 but with incorrect specification of the order of variables. The present observation will be of help to understand why it is possible that we sometimes get an estimate of a_{010} with unexpected sign.

Table 3

m	a_{01m}	$\hat{a}_{01m}$ ($L=6$, $M=6$)	a_{10m}	$\hat{a}_{10m}$ ($L=6$, $M=6$)
0	0	0.232	0.25	0
1	−0.1	−0.048	0.15	0.210
2	−0.2	−0.118	0.08	0.072
3	−0.1	−0.119	0.03	−0.067
4	0	−0.010	0	−0.016
5	0	0.045	0	0.011
6	0	0.072	0	−0.051

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