ASYMPTOTIC EQUIVALENCE OF REAL PROBABILITY DISTRIBUTIONS*

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Introduction

Asymptotic distribution theory based on the concept of convergence has been widely discussed in the literature, for which the requirement of the existence of limiting distribution and that of fixed basic spaces are restrictive in some cases of applications. In fact, the asymptotic independence problem of a system of random variables can not be handled by means of the usual theory of asymptotic distribution when the number of the variables increases indefinitely under a limiting process being considered. It would assure us wider applicability of the theory to remove the restrictions stated above, and such an attempt has already been made in the previous paper [1], in which some of the concepts of asymptotic equivalence between two sequences of probability distributions were introduced with applications in a fairly general situation. These notions, however, appear too strong for some practical applications.

In the present paper, we shall confine ourselves to the case where the basic spaces are Euclidean, and introduce several types of asymptotic equivalence of real probability distributions, some of which are weaker than those given in the paper [1].

In the first section, we shall give some preliminaries, and in section 2, definitions of stronger notions of asymptotic equivalence are introduced in the general case of basic spaces, together with their mutual inclusion relations. Section 3 is devoted to discussions on conditions under which different types of the notions are mutually equivalent in the case of equal basic spaces.

In section 4, we shall discuss some of the fundamental properties of a certain type of the notions which is practically important, and in section 5, discussions will be made on measurable transformations which transfer a type of asymptotic equivalence to the same or to another. Such transformations are of use in some practical situations.

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The final section is devoted to discuss the asymptotic equivalence of marginal probability distributions, when the other marginals are replaced in a manner by another sequence of probability distributions, or by a constant to which the other marginals converge in probability.

The present article is based on a series of works [2], [3], [4], [5], and an application of the present theory will be seen in the papers [6], [7].

1. Preliminaries

In this section, necessary notations and results on real probability distributions are stated.

For any given positive integer n, let $R_{(n)}$ be the n-dimensional Euclidean space, and $\boldsymbol{B}_{(n)}$ the usual Borel field of subsets of $R_{(n)}$. Let us denote the family of all probability distributions, or of all random variables, defined over the measurable space $(R_{(n)}, \boldsymbol{B}_{(n)})$ by $\mathcal{F}(R_{(n)}, \boldsymbol{B}_{(n)})$, the members of which will be designated by random variables, $X_{(n)}, Y_{(n)}, \cdots$, say, with corresponding probability measures $P^{X_{(n)}}, P^{Y_{(n)}}, \cdots$, respectively.

Let $\nu_{(n)}$ be any given σ-finite measure over $(R_{(n)}, \boldsymbol{B}_{(n)})$, and $\mathcal{P}(R_{(n)}, \boldsymbol{B}_{(n)}, \nu_{(n)})$ be the family of all probability distributions over $(R_{(n)}, \boldsymbol{B}_{(n)})$ which are absolutely continuous with respect to $\nu_{(n)}$. Throughout the present paper, $\mu_{(n)}$ designates the usual Lebesgue measure over $R_{(n)}$.

Let $\boldsymbol{C}_{(n)}$ be any given non-empty subclass of $\boldsymbol{B}_{(n)}$, and let us define for any given pair of members of $\mathcal{F}(R_{(n)}, \boldsymbol{B}_{(n)})$, $X_{(n)}$ and $Y_{(n)}$, the following quantities:

$$(1.1) \qquad \delta_d(X_{(n)}, Y_{(n)}: \boldsymbol{C}_{(n)}) = \sup_{E \in \boldsymbol{C}_{(n)}} |P^{X_{(n)}}(E) - P^{Y_{(n)}}(E)|\,,$$

and

$$(1.2) \qquad \delta_r(X_{(n)}, Y_{(n)}: \boldsymbol{C}_{(n)}) = \sup_{E \in \boldsymbol{C}_{(n)}} \left| \frac{P^{X_{(n)}}(E)}{P^{Y_{(n)}}(E)} - 1 \right|$$

with the convention that $0/0=1$.

Note that the first quantity defines a distance over the family $\mathcal{F}(R_{(n)}, \boldsymbol{B}_{(n)})$ if we identify those random variables which have the same probability measure over $\boldsymbol{C}_{(n)}$, $X_{(n)}$ and $Y_{(n)}$ being said to have the same probability measure over $\boldsymbol{C}_{(n)}$ if it holds that $P^{X_{(n)}}(E) = P^{Y_{(n)}}(E)$ for every E belonging to the class.

It should also be remarked that, in the case when both $X_{(n)}$ and $Y_{(n)}$ belong to the family $\mathcal{P}(R_{(n)}, \boldsymbol{B}_{(n)}, \nu_{(n)})$ for some σ-finite measure $\nu_{(n)}$, it holds that

$$(1.3) \qquad 2\delta_d(X_{(n)}, Y_{(n)}: \boldsymbol{B}_{(n)}) = \int_{R_{(n)}} |f-g|\, d\nu_{(n)}\,,$$

where f and g are the generalized probability density functions with respect to $\nu_{(n)}$ ($gpdf(\nu_{(n)})$, in short) of $X_{(n)}$ and $Y_{(n)}$, respectively.

It is immediate from the definition that, if $m \leqq n$ and $C_{(n)}$ contains all the subsets of the form $E_{(m)} \times R_{(n-m)}$, $E_{(m)} \in C_{(m)}$, then

$$\delta_d(X_{(m)}, Y_{(m)} : C_{(m)}) \leqq \delta_d(X_{(n)}, Y_{(n)} : C_{(n)}) , \tag{1.4}$$

and

$$\delta_r(X_{(m)}, Y_{(m)} : C_{(m)}) \leqq \delta_r(X_{(n)}, Y_{(n)} : C_{(n)}) \tag{1.5}$$

where $X_{(n)} = (X_{(m)}, X_{(n-m)})$ and $Y_{(n)} = (Y_{(m)}, Y_{(n-m)})$ are the decompositions corresponding to the decomposition of the space, $R_{(n)} = R_{(m)} \times R_{(n-m)}$.

The following inequalities are also easy to prove:

$$\delta_d(X_{(n)}, Y_{(n)} : C_{(n)}) \leqq \min \{\delta_r(X_{(n)}, Y_{(n)} : C_{(n)}), \delta_r(Y_{(n)}, X_{(n)} : C_{(n)})\} , \tag{1.6}$$

and

$$\frac{\delta_r(Y_{(n)}, X_{(n)} : C_{(n)})}{1 + \delta_r(Y_{(n)}, X_{(n)} : C_{(n)})} \leqq \delta_r(X_{(n)}, Y_{(n)} : C_{(n)}) \tag{1.7}$$

where the rôle of $X_{(n)}$ and $Y_{(n)}$ can, of course, be exchanged.

Now, we shall consider some of the familiar subclasses of $\boldsymbol{B}_{(n)}$.

Let $\boldsymbol{M}_{(n)}$ be the class of all subsets of $R_{(n)}$ which are of the form

$$E(a_{(n)}) = \{z_{(n)} = (z_1, \cdots, z_n) \mid -\infty \leqq z_i < a_i,\ i = 1, \cdots, n\} , \tag{1.8}$$

for all extended real vectors, $a_{(n)} = (a_1, \cdots, a_n)$, a_i being admitted to take the values $\pm\infty$. This is a multiplicative class and contains the empty set and the whole space as its elements.

The class $\boldsymbol{S}_{(n)}$ is defined to be the class of all subsets of the form

$$E(a_{(n)} : b_{(n)}) = \{z_{(n)} = (z_1, \cdots, z_n) \mid b_i \leqq z_i < a_i,\ i = 1, \cdots, n\}$$

where $a_{(n)} = (a_i, \cdots, a_n)$ and $b_{(n)} = (b_1, \cdots, b_n)$ are any extended real vectors.

Let $\boldsymbol{A}_{(n)}$ be the finitely additive class over $\boldsymbol{M}_{(n)}$, and finally, $\boldsymbol{G}_{(n)}$ the class of all open subsets of $R_{(n)}$ with respect to the usual Euclidean distance.

For these classes, it is well-known that (a) the class $\boldsymbol{A}_{(n)}$ consists of all finite unions (may, or may not disjoint) of the members of $\boldsymbol{S}_{(n)}$, and (b) for any subset E belonging to $\boldsymbol{S}_{(n)}$, there exist a set of members of $\boldsymbol{M}_{(n)}$, $\{F_1, \cdots, F_N\}$ say, and a set of constants, c_i's, taking the values ± 1, such that $N \leqq 2^n$ and

$$\nu_{(n)}(E) = \sum_{i=1}^{N} c_i \nu_{(n)}(F_i) , \tag{1.9}$$

for any σ-finite measure $\nu_{(n)}$ over $(R_{(n)}, \boldsymbol{B}_{(n)})$.

From (1.9) it immediately follows that

$$(1.10)\qquad \delta_d(X_{(n)}, Y_{(n)}: \boldsymbol{M}_{(n)}) \leqq \delta_d(X_{(n)}, Y_{(n)}: \boldsymbol{S}_{(n)}) \leqq 2^n \delta_d(X_{(n)}, Y_{(n)}: \boldsymbol{M}_{(n)}) .$$

Since, for any given σ-finite measure $\nu_{(n)}$ over $(R_{(n)}, \boldsymbol{B}_{(n)})$ and for any given subset E belonging to $\boldsymbol{B}_{(n)}$,

$$\nu_{(n)}(E) = \inf \{ \sum_i \nu_{(n)}(F_i) \mid E \subseteqq \bigcup_i F_i,\ F_i \in \boldsymbol{A}_{(n)} \} ,$$

it holds that

$$(1.11)\qquad \delta_d(X_{(n)}, Y_{(n)}: \boldsymbol{A}_{(n)}) = \delta_d(X_{(n)}, Y_{(n)}: \boldsymbol{B}_{(n)}) .$$

In the final place, let

$$(1.12)\qquad X_{(n)} = (X_{(n_1)}, \cdots, X_{(n_k)})$$

be a decomposition of a random variable $X_{(n)}$, corresponding to some decomposition of $R_{(n)}$, $R_{(n)} = R_{(n_1)} \times \cdots \times R_{(n_k)}$. For this decomposition, the set of k marginals,

$$(1.13)\qquad \{X_{(n_1)}, \cdots, X_{(n_k)}\} ,$$

is said to be an *independent system of random variables* if it holds that

$$P^{X_{(n)}}(E) = P^{X_{(n_1)}}(E_1) \times \cdots \times P^{X_{(n_k)}}(E_k) ,$$

for every subset E of the form $E = E_1 \times \cdots \times E_k$ with $E_i \in \boldsymbol{B}_{(n_i)}$, $i = 1, \cdots, k$.

It is also well-known that for any given $X_{(n)}$ decomposed in the form (1.12) there exists a random variable, $Y_{(n)} = (Y_{(n_1)}, \cdots, Y_{(n_k)})$, such that the k-marginals constitute an independent system and $Y_{(n_i)}$ is distributed identically with $X_{(n_i)}$, $i = 1, \cdots, k$.

2. Definition of some types of asymptotic equivalence and of asymptotic independence

In this section, we shall introduce two types of stronger notions of asymptotic equivalence in the general case where the dimensions of the basic spaces may or may not vary with underlying parameters, and two types of weaker notions in the case of equal basic spaces.

Corresponding notions of asymptotic independence of system of random variables will also be defined.

Let $\{X^s_{(n_s)}\}$ $(s = 1, 2, \cdots)$ and $\{Y^s_{(n_s)}\}$ $(s = 1, 2, \cdots)$ be two sequences of random variables, for which $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ are assumed to belong to the family $\mathscr{F}(R_{(n_s)}, \boldsymbol{B}_{(n_s)})$ for each s. The sequence of the underlying spaces, $\{R_{(n_s)}\}$ $(s = 1, 2, \cdots)$, will be called *the sequence of basic spaces*. The case where the dimensions of the basic spaces, n_s, are identical with some positive integer n is called *the case of equal basic spaces*, otherwise *the case of unequal basic spaces*. An important case of unequal basic spaces is that n_s tends to infinity with increasing s.

Together with the above sequences of random variables, we shall consider a *sequence of basic classes*, $\{C_{(n_s)}\}$ $(s=1, 2, \cdots)$, with a non-empty subclass $C_{(n_s)}$ of $B_{(n_s)}$ for each s.

Under the situation mentioned above, we give the following.

DEFINITION 2.1. Two sequences of random variables, $\{X^s_{(n_s)}\}$ $(s=1, 2, \cdots)$ and $\{Y^s_{(n_s)}\}$ $(s=1, 2, \cdots)$ are said to be *asymptotically equivalent in the sense of type* $(C)_d$ or *of type* $(C)_r$ as $s \to \infty$ and are denoted by

$$X^s_{(n_s)} \sim Y^s_{(n_s)}\, (C)_d \qquad (s \to \infty)\,, \tag{2.1}$$

or

$$X^s_{(n_s)} \sim Y^s_{(n_s)}\, (C)_r \qquad (s \to \infty)\,, \tag{2.2}$$

if it holds that

$$\delta_d(X^s_{(n_s)}, Y^s_{(n_s)} : C_{(n_s)}) \to 0 \qquad (s \to \infty)\,, \tag{2.3}$$

or

$$\delta_r(X^s_{(n_s)}, Y^s_{(n_s)} : C_{(n_s)}) \to 0 \qquad (s \to \infty)\,, \tag{2.4}$$

respectively.

This definition gives two notions of *uniform asymptotic equivalence* basing upon the absolute difference and the relative difference of two probability measures. In this sense, we shall use the brief notations, $AEUD(C)$ and $AEUR(C)$ for type $(C)_d$ and $(C)_r$ asymptotic equivalence defined above respectively.

In the case of equal basic spaces where $n_s=n$, if we fix $Y^s_{(n)}$ independently of s, i.e., $Y^s_{(n)}=Y_{(n)}$, then the above definition gives us notions of uniform convergence. We shall say that the sequence of random variables, $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ *converges in the sense of type* $(C)_d$, or *of type* $(C)_r$, to $Y_{(n)}$ as $s \to \infty$, according as

$$\delta_d(X^s_{(n)}, Y_{(n)} : C_{(n)}) \to 0 \qquad (s \to \infty)\,, \tag{2.5}$$

or

$$\delta_r(X^s_{(n)}, Y_{(n)} : C_{(n)}) \to 0 \qquad (s \to \infty)\,, \tag{2.6}$$

respectively. We say, in short, $X^s_{(n)}$ *converges* $(C)_d$ or *converges* $(C)_r$ to $Y_{(n)}$ as $s \to \infty$ in the respective cases stated above.

In the case of equal basic spaces, weaker types of asymptotic equivalence can be defined as in the following.

DEFINITION 2.2. $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ and $\{Y^s_{(n)}\}$ $(s=1, 2, \cdots)$ are said to be *asymptotically equivalent in the sense of type* $((C))_d$ or *of type* $((C))_r$, ($AED((C))$ or $AER((C))$ for short) as $s\to\infty$, and are denoted by

$$X^s_{(n)} \sim Y^s_{(n)}\ ((C))_d \qquad (s\to\infty)\,, \tag{2.7}$$

or

$$X^s_{(n)} \sim Y^s_{(n)}\ ((C))_r \qquad (s\to\infty)\,, \tag{2.8}$$

if it holds respectively that

$$|P^{X^s_{(n)}}(E)-P^{Y^s_{(n)}}(E)|\to 0 \qquad (s\to\infty)\,, \tag{2.9}$$

or

$$\left|\frac{P^{X^s_{(n)}}(E)}{P^{Y^s_{(n)}}(E)}-1\right|\to 0 \qquad (s\to\infty)\,, \tag{2.10}$$

for every subset E belonging to $C_{(n)}$, where we use the convention $0/0 =1$.

As before, when $Y^s_{(n)}$ is fixed independently of s, the above definition gives us notions of convergence, i.e., type $((C))_d$ and type $((C))_r$. By using the notions of asymptotic equivalence given above, we can define those of asymptotic independence of system of random variables.

Suppose we are given for each s a decomposition of $X^s_{(n_s)}$ in the form

$$X^s_{(n_s)}=(X^s_{(m^s_1)}, \cdots, X^s_{(m^s_k)}) \tag{2.11}$$

where $X^s_{(m^s_i)}$ belongs to $\mathscr{F}(R_{(m^s_i)}, \boldsymbol{B}_{(m^s_i)})$, $i=1, \cdots, k$, and k and m^s_i may be dependent on s. Let us consider for each s the system of marginal random variables of $X^s_{(n_s)}$, i.e.,

$$\{X^s_{(m^s_1)}, \cdots, X^s_{(m^s_k)}\}\,. \tag{2.12}$$

As was noted in the preceding section, there exists a sequence of random variables,

$$\{Y^s_{(n_s)}=(Y^s_{(m^s_1)}, \cdots, Y^s_{(m^s_k)})\} \qquad (s=1, 2, \cdots)$$

such that $Y^s_{(m^s_i)}$ and $X^s_{(m^s_i)}$ are identically distributed for all i, $i=1, \cdots, k$, and the set of marginals, $\{Y^s_{(m^s_1)}, \cdots, Y^s_{(m^s_k)}\}$ is an independent system of random variables for each s.

Under this situation, we shall give the following.

DEFINITION 2.3. A system of random variables given by (2.12) is *asymptotically independent in the sense of type* $(C)_d$ ($AIUD(C)$, for short) as $s\to\infty$ if it holds that

$$X^s_{(n_s)} \sim Y^s_{(n_s)}\ (C)_d\,, \qquad (s\to\infty)\,. \tag{2.13}$$

In the same manner, one can define the other types of asymptotic independence, namely, type $(C)_r$ in the general case and type $((C))_d$ and $((C))_r$ in the case of equal basic spaces.

Now, we shall proceed to the properties of notions of asymptotic equivalence. First, it is often useful to give the following.

LEMMA 2.1. *In order that two sequences of random variables,* $\{X^s_{(n_s)}\}$ $(s=1, 2, \cdots)$ *and* $\{Y^s_{(n_s)}\}$ $(s=1, 2, \cdots)$ *are* $AEUD(C)$ *or* $AEUR(C)$, *it is necessary and sufficient respectively, that*

$$|P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)})| \to 0 \qquad (s\to\infty)\,, \tag{2.14}$$

or

$$\left|\frac{P^{X^s_{(n_s)}}(E^s_{(n_s)})}{P^{Y^s_{(n_s)}}(E^s_{(n_s)})} - 1\right| \to 0 \qquad (s\to\infty) \tag{2.15}$$

for any given sequence of subsets, $\{E^s_{(n_s)}\}$ $(s=1, 2, \cdots)$ *with* $E^s_{(n_s)}$ *belonging to* $C_{(n_s)}$ *for each* s.

The proof of this lemma is easy and is omitted.

By using the results in the preceding section, it is not so difficult to establish the inclusion relations for several types of asymptotic equivalence which are obtained by taking the familiar classes given in the preceding section as the basic classes (Comments by R. M. Meyer, Matsunawa [8]): In the general case we have

$$\begin{array}{ccccccccc} (G)_d & \rightleftarrows & (B)_d & \rightleftarrows & (A)_d & \longrightarrow & (S)_d & \longrightarrow & (M)_d \\ & & \uparrow & & & & & & \uparrow \\ (G)_r & \rightleftarrows & (B)_r & \rightleftarrows & (A)_r & \rightleftarrows & (S)_r & \longrightarrow & (M)_r \end{array}\,, \tag{2.16}$$

and in the case of equal basic spaces,

$$\begin{array}{ccccccccc} (G)_d & \rightleftarrows & (B)_d & \rightleftarrows & (A)_d & \longrightarrow & (S)_d & \rightleftarrows & (M)_d \\ & & \uparrow & & & & & & \uparrow \\ (G)_r & \rightleftarrows & (B)_r & \rightleftarrows & (A)_r & \rightleftarrows & (S)_r & \longrightarrow & (M)_r \end{array} \tag{2.17}$$

for stronger notions, and

$$
\begin{array}{ccccccc}
 & \nearrow & & ((\boldsymbol{G}))_d & & \searrow & \\
 & & & \uparrow & & & \\
((\boldsymbol{B}))_d & \longrightarrow & ((\boldsymbol{A}))_d & \rightleftarrows & ((\boldsymbol{S}))_d & \rightleftarrows & ((\boldsymbol{M}))_d \\
\uparrow & & \uparrow & | & \uparrow & & \uparrow \\
((\boldsymbol{B}))_r & \longrightarrow & ((\boldsymbol{A}))_r & \rightleftarrows & ((\boldsymbol{S}))_r & \longrightarrow & ((\boldsymbol{M}))_r \\
 & \searrow & & | & & \nearrow & \\
 & & & ((\boldsymbol{G}))_r & & &
\end{array}
\tag{2.18}
$$

for weaker notions.

Thus, so far as we are interested in these types of asymptotic equivalence, we may only consider the following notions:

$$(\boldsymbol{B})_d,\ (\boldsymbol{S})_d,\ (\boldsymbol{M})_d;\ (\boldsymbol{B})_r,\ (\boldsymbol{M})_r\ , \tag{2.19.1}$$

in the case of unequal basic spaces,

$$(\boldsymbol{B})_d,\ (\boldsymbol{M})_d;\ (\boldsymbol{B})_r,\ (\boldsymbol{M})_r\ , \tag{2.19.2}$$

and

$$((\boldsymbol{B}))_d,\ ((\boldsymbol{G}))_d,\ ((\boldsymbol{M}))_d;\ ((\boldsymbol{B}))_r,\ ((\boldsymbol{G}))_r,\ ((\boldsymbol{S}))_r,\ ((\boldsymbol{M}))_r\ , \tag{2.19.3}$$

in the case of equal basic spaces.

In the subsequent section, we shall give conditions under which some of the notions given above are mutually equivalent in the case of equal basic spaces.

3. Inclusion relations in the case of equal basic spaces

In the present section, we shall discuss the conditions under which some of the notions of asymptotic equivalence given in (2.19.2) and (2.19.3) are mutually equivalent.

In the first place, we give two kinds of properties of a sequence of random variables.

Let $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ be a sequence of random variables belonging to $\mathcal{F}(R_{(n)}, \boldsymbol{B}_{(n)})$, and let $\boldsymbol{C}_{(n)}$ be a subclass of $\boldsymbol{B}_{(n)}$, for which the following definitions are meaningful.

DEFINITION 3.1. We say that $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ has property $B(\boldsymbol{C})$ if for any given $\varepsilon>0$, there exist a bounded subset, B says, belonging to $\boldsymbol{C}_{(n)}$ and a positive integer s_0 such that

$$P^{X^s_{(n)}}(B)>1-\varepsilon \qquad \text{for all } s\geqq s_0\,.$$

Note that $B(\boldsymbol{B})$, $B(\boldsymbol{G})$, $B(\boldsymbol{A})$ and $B(\boldsymbol{S})$ are mutually equivalent properties. We therefore use property $B(\boldsymbol{S})$ for them.

DEFINITION 3.2. $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ is said to have property $C(\boldsymbol{C})$ if, for any given $\varepsilon>0$, these exist a positive number δ and a positive integer s_0 such that the conditions, $E\in \boldsymbol{C}_{(n)}$ and $\mu_{(n)}(E)<\delta$, imply that

$$P^{X^s_{(n)}}(E)<\varepsilon$$

for all $s\geqq s_0$, where $\mu_{(n)}$ is, as before, the usual Lebesgue measure over $R_{(n)}$.

It should be remarked that property $C(\boldsymbol{B})$ is equivalent to the *uniform* $\mu_{(n)}$*-continuity* defined by Loéve [9].

We shall list some of the results on the above two properties in the following lemmas, whose proofs are straightforward and will be omitted.

LEMMA 3.1. (i) *Type* $((\boldsymbol{M}))_d$ *asymptotic equivalence brings over property* $B(\boldsymbol{S})$, *i.e., if* $X^s_{(n)}\sim Y^s_{(n)}\ ((\boldsymbol{M}))_d$ *and either one of the sequences has property* $B(\boldsymbol{S})$, *then the other has the same property.*

(ii) *If* $X^s_{(n)}\to Y_{(n)}\ ((\boldsymbol{M}))_d$ *as* $s\to\infty$ *for some* $Y_{(n)}$ *belonging to* $\mathcal{F}(R_{(n)}, \boldsymbol{B}_{(n)})$ *then the sequence has property* $B(\boldsymbol{S})$.

(iii) *A necessary and sufficient condition for* $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ *to have property* $B(\boldsymbol{S})$ *is that for some decomposition,* $X^s_{(n)}=(X^s_{1(m_1)}, \cdots, X^s_{k(m_k)})$, k *and* $m_1, \cdots, m_k$ *being fixed independently of* s, *every sequence of marginals,* $\{X^s_{i(m_i)}\}$ $(s=1, 2, \cdots)$ *has property* $B(\boldsymbol{S})$, $i=1, \cdots, k$.

(iv) *If* $\{X^s_{(n)}=(X^s_1, \cdots, X^s_n)\}$ $(s=1, 2, \cdots)$ *has property* $B(\boldsymbol{S})$, *then* $\{\bar{X}^s_{(m)}=(X^s_{i_1}, \cdots, X^s_{i_m})\}$ $(s=1, 2, \cdots)$ *has the same property where* m *is fixed independently of* s, *while the choice of* $\{i_1, \cdots, i_m\}$ *may depend on* s.

LEMMA 3.2. (i) *Property* $C(\boldsymbol{C})$ *implies property* $C(\boldsymbol{C}')$ *if* $\boldsymbol{C}'_{(n)}\subseteqq \boldsymbol{C}_{(n)}$.

(ii) *Property* $C(\boldsymbol{C})$ *is brought over by type* $((\boldsymbol{C}))_d$ *asymptotic equivalence.*

(iii) *If* $\{X^s_{(n)}=(X^s_1, \cdots, X^s_n)\}$ $(s=1, 2, \cdots)$ *has both properties* $C(\boldsymbol{S})$ *and* $B(\boldsymbol{S})$, *then the sequence of any* m *marginals,* $\{\bar{X}^s_{(m)}=(X^s_{i_1}, \cdots, X^s_{i_m})\}$ $(s=1, 2, \cdots)$, *has the same properties when* m *is fixed independently of* s, *while the choice of* $\{i_1, \cdots, i_m\}$ *out of* $\{1, \cdots, n\}$ *may depend on* s.

(iv) *If* $\boldsymbol{C}^*_{(n)}$ *is the class of all finite unions of the members of* $\boldsymbol{C}_{(n)}$, *then properties* $C(\boldsymbol{C})$ *and* $C(\boldsymbol{C}^*)$ *are mutually equivalent. Thus,* $C(\boldsymbol{S})$ *and* $C(\boldsymbol{A})$ *are equivalent to each other.*

(v) *If* $X^s_{(n)}\to Y_{(n)}\ ((\boldsymbol{C}))_d$ *as* $s\to\infty$ *for some* $Y_{(n)}$ *belonging to the family* $\mathcal{P}(R_{(n)}, \boldsymbol{B}_{(n)}, \mu_{(n)})$, *then the sequence* $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ *has property* $C(\boldsymbol{C})$.

As was shown in the diagram (2.18), there hold the inclusion relations

$$((\boldsymbol{B}))_d \longrightarrow ((\boldsymbol{G}))_d \longrightarrow ((\boldsymbol{M}))_d$$

among the first three types of asymptotic equivalence given in (2.19.3).

The following theorem gives sufficient conditions under which these notions are mutually equivalent.

THEOREM 3.1. (i) *If* $X^s_{(n)} \sim Y^s_{(n)}\,((\boldsymbol{M}))_d$, $(s \to \infty)$, *and the sequences have properties* $C(S)$ *and* $B(S)$, *then it holds that* $X^s_{(n)} \sim Y^s_{(n)}\,((\boldsymbol{G}))_d$, $(s \to \infty)$.

(ii) *If both of the sequences have properties* $C(\boldsymbol{B})$ *and* $B(S)$, *then* $X^s_{(n)} \sim Y^s_{(n)}\,((\boldsymbol{G}))_d$ *implies* $X^s_{(n)} \sim Y^s_{(n)}\,((\boldsymbol{B}))_d$ *as* $s \to \infty$.

The proof of this theorem is not so difficult and is omitted.

It should be noted that, by the Lemmas 3.1 and 3.2, the conditions for the sequences to have properties $C(S)$ and $B(S)$, stated in (i) of the above theorem, can be replaced by the condition that at least one of the sequences has the properties simultaneously or one for each. The same is not necessarily true for the assertion (ii) of the above theorem. It is also remarked that the first three notions in (2.19.3) are mutually equivalent under the conditions stated in (ii) of the above theorem.

The following theorem provides a sufficient condition under which $(\boldsymbol{M})_d$ and $((\boldsymbol{M}))_d$ are mutually equivalent.

THEOREM 3.2. *If* $X^s_{(n)} \sim Y^s_{(n)}\,((\boldsymbol{M}))_d$ $(s \to \infty)$ *and the sequences have properties* $C(S)$ *and* $B(S)$, *then it holds that* $X^s_{(n)} \sim Y^s_{(n)}\,(\boldsymbol{M})_d$ $(s \to \infty)$.

PROOF. If the assertion were not true, then one could assume without any harm in the proof below that there exist a positive constant η and a sequence of members of $\boldsymbol{S}_{(n)}$, $\{E_s\}$ $(s=1, 2, \cdots)$ say, such that

$$|P^{X^s_{(n)}}(E_s) - P^{Y^s_{(n)}}(E_s)| < 2\eta \tag{3.1}$$

for all s.

By property $B(S)$ of both the sequences, there exists a member, B, of $\boldsymbol{S}_{(n)}$ whose closure being compact such that

$$P^{X^s_{(n)}}(B) > 1 - \eta/2 \quad \text{and} \quad P^{Y^s_{(n)}}(B) > 1 - \eta/2 \tag{3.2}$$

for all s greater than some positive integer s_0. Thus, putting $A_s = B \cap E_s$, we have, by (3.1) and (3.2)

$$|P^{X^s_{(n)}}(A_s) - P^{Y^s_{(n)}}(A_s)| > \eta \tag{3.3}$$

for all $s > s_0$. Note that A_s belongs to the class $\boldsymbol{S}_{(n)}$ for each s.

Now, since A_s are bounded uniformly for all s, there can be found a subsequence, $A_{s'}$ $(s' \to \infty)$ say, of $\{A_s\}$ $(s=1, 2, \cdots)$ and a subset of $\boldsymbol{R}_{(n)}$, A say, such that

$$\lim_{s' \to \infty} A_{s'} (= \underline{\lim}_{s' \to \infty} A_{s'} = \overline{\lim}_{s' \to \infty} A_{s'}) = A\,, \tag{3.4}$$

and for A there exists a member E of $S_{(n)}$ such that the symmetric difference $E\Delta A$ is contained in the boundary set of E and hence the closures of E and A are identical and compact.

Hence, it follows from the assumption and the Lemma 3.2, (iv), that for any given ε positive there exists a positive integer s_0' such that

$$(3.5)\qquad |P^{X_{(n)}^{s'}}(E)-P^{X_{(n)}^{s'}}(A)|<\varepsilon \quad \text{and} \quad |P^{Y_{(n)}^{s'}}(E)-P^{Y_{(n)}^{s'}}(A)|<\varepsilon$$

for all $s'>s_0'$. It is also easy to see that

$$(3.6)\qquad |P^{X_{(n)}^{s'}}(A)-P^{X_{(n)}^{s'}}(A_{s'})|<\varepsilon \quad \text{and} \quad |P^{Y_{(n)}^{s'}}(A)-P^{Y_{(n)}^{s'}}(A_{s'})|<\varepsilon$$

for all $s>s_0''$, where s_0'' is some positive integer.

From (3.5) and (3.6) it now follows that

$$(3.7)\qquad |P^{X_{(n)}^{s'}}(A_{s'})-P^{Y_{(n)}^{s'}}(A_{s'})|<|P^{X_{(n)}^{s'}}(E)-P^{Y_{(n)}^{s'}}(E)|+4\varepsilon\,,$$

for all $s'>\max(s_0', s_0'')$. But, since the first member of the right-hand side tends to zero with increasing s' and ε can be taken arbitrarily small, the above result contradicts (3.3), which completes the proof of the theorem.

It should be noted that the same remark just after the preceding theorem is true for the condition of the above theorem. Immediately from the above theorem, one has the following corollary, which is well-known as Polya's theorem.

COROLLARY 3.1. *If* $X_{(n)}^s \to Y_{(n)}^s\ ((\boldsymbol{M}))_d$ *as* $s\to\infty$ *for some* $Y_{(n)}$ *belonging to* $\mathcal{P}(R_{(n)}, \boldsymbol{B}_{(n)}, \mu_{(n)})$, *then the convergence is of type* $(\boldsymbol{M})_d$.

Note that type $((\boldsymbol{M}))_d$ convergence in the above case is equivalent to the convergence in law.

A similar result to that of the above theorem would be obtained in the case of more abstract basic spaces [11], [12].

It is an open question to find out further conditions under which the notions of asymptotic equivalence given in (2.19.1) through (2.19.3) are mutually equivalent.

4. Some properties of type $(M)_d$ asymptotic equivalence in the case of equal basic spaces

In many applications, type $(\boldsymbol{M})_d$ asymptotic equivalence seems to play an important rôle. In this section, we shall discuss some of the fundamental properties of the concept, mainly in relation to the so-called in probability convergence.

In the first place, the following lemma is straightforward from the definition.

LEMMA 4.1. *If* $X^s_{(n)} \sim Y^s_{(n)} ((M))_d$ *and* $Y^s_{(n)} \to c_{(n)}$ *(in prob.) as* $s \to \infty$, *then* $X^s_{(n)} \to c_{(n)}$ *(in prob.) as* $s \to \infty$, *where* $c_{(n)}$ *is a point of* $R_{(n)}$.

It is easy to establish the following theorem, whose proof will be omitted.

THEOREM 4.1. *Suppose that the following conditions are satisfied.*

(i) $\{X^s_{(n)} = (X^s_1, \cdots, X^s_n)\}$ $(s=1, 2, \cdots)$ *has properties* $C(S)$ *and* $B(S)$.

(ii) $\{c^s_{(n)} = (c^s_1, \cdots, c^s_n)\}$ $(s=1, 2, \cdots)$ *is any given sequences of points of* $R_{(n)}$ *which converges to the point* $1_{(n)} = (1, \cdots, 1)$.

Then, it holds that

$$X^s_{(n)} \sim Y^s_{(n)} (M)_d , \qquad (s \to \infty) , \tag{4.1}$$

where $Y^s_{(n)} = (c^s_1 X^s_1, \cdots, c^s_n X^s_n)$ *for each* s.

It should be remarked that the condition (i) of the above theorem can not be removed, which will be seen by simple counter examples. It is also easy to see the following.

THEOREM 4.2. *Let* $\{(X^s_{(n)}, Y^s_{(n)})\}$ $(s=1, 2, \cdots)$ *be a sequence of members of* $(R_{(2n)}, \boldsymbol{B}_{(2n)})$, *satisfying the following conditions*:

(i) *The first marginals* $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ *has property* $C(S)$.

(ii) *The second marginals* $\{Y^s_{(n)}\}$ $(s=1, 2, \cdots)$ *converges in probability to a point* $c_{(n)}$ *of* $R_{(n)}$.

Then, it holds that

$$X^s_{(n)} + Y^s_{(n)} \sim X^s_{(n)} + c_{(n)} (M)_d , \qquad (s \to \infty) . \tag{4.2}$$

As an immediate consequence of this theorem, we have the following.

COROLLARY 4.1. *Under the same situation as in the above theorem, suppose that the following conditions are satisfied.*

(i) $X^s_{(n)} \to Z_{(n)}$ *(in law) as* $s \to \infty$, *where* $Z_{(n)}$ *is any given member of* $(R_{(n)}, \boldsymbol{B}_{(n)}, \mu_{(n)})$.

(ii) $Y^s_{(n)} \to c_{(n)}$ *(in prob.) as* $s \to \infty$.

Then, it holds that

$$X^s_{(n)} + Y^s_{(n)} \to Z_{(n)} + c_{(n)} (M)_d , \qquad (s \to \infty) . \tag{4.3}$$

Now, in the final place, we shall show the following.

THEOREM 4.3. *Under the same situation as in the theorem 4.2, suppose that the following conditions are satisfied.*

(i) $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ *has properties* $C(S)$ *and* $B(S)$.

(ii) $Y^s_{(n)} \to c_{(n)}$ *(in prob.) as* $s \to \infty$, *where* $c_{(n)} = (c_1, \cdots, c_n)$ *is a point of* $R_{(n)}$ *such that* $c_i \neq 0$, $i=1, \cdots, n$.

Then, it holds that

(4.4) $$X^s_{(n)}/Y^s_{(n)} \sim X^s_{(n)}/c_{(n)}\ (\boldsymbol{M})_d \qquad (s\to\infty)\,,$$

and

(4.5) $$X^s_{(n)}\cdot Y^s_{(n)} \sim X^s_{(n)}\cdot c_{(n)}\ (\boldsymbol{M})_d \qquad (s\to\infty)$$

where we have used the notations, $a_{(n)}/b_{(n)}=(a_1/b_1,\cdots,a_n/b_n)$ *and* $a_{(n)}\cdot b_{(n)}=(a_1b_1,\cdots,a_nb_n)$ *for* $a_{(n)}=(a_1,\cdots,a_n)$ *and* $b_{(n)}=(b_1,\cdots,b_n)$ *in general.*

PROOF. Since (4.5) follows from (4.4), we shall prove (4.4).

Assume first that $c_i>0$ for all i, and hence, without any loss of generality, that $c_{(n)}=1_{(n)}$. Using the notation $b_{(n)}<d_{(n)}$ for $b_i<d_i$, $i=1,\cdots,n$, and $b_{(n)}\leqq d_{(n)}$ for $b_i\leqq d_i$, $i=1,\cdots,n$, one easily has

(4.6) $$\begin{aligned}&|P(X^s_{(n)}/Y^s_{(n)}<a_{(n)})-P(X^s_{(n)}<a_{(n)})|\\ &\leqq \sup_{\substack{|u_i-v_i|<2|a_i|\delta_i\\ i=1,\cdots,n}} |P(X^s_{(n)}<u_{(n)})-P(X^s_{(n)}<v_{(n)})|\\ &\quad +2\{1-P(1_{(n)}-\delta_{(n)}\leqq Y^s_{(n)}<1_{(n)}+\delta_{(n)})\}\,,\end{aligned}$$

for any given $a_{(n)}=(a_1,\cdots,a_n)$ and $\delta_{(n)}=(\delta_1,\cdots,\delta_n)$ with $0<\delta_i<1$, $i=1,\cdots,n$.

The condition (i) of the theorem assures us that the first member of the right-hand side of (4.6) is sufficiently small if δ_i are small, and the condition (ii) does the second member is sufficiently small if s is large. In other words, for any given ε positive integer s_0 such that

$$\sup_{a_{(n)}} |P(X^s_{(n)}/Y^s_{(n)}<a_{(n)})-P(X^s_{(n)}<a_{(n)})<\varepsilon$$

for all $s>s_0$, which proves (4.4) in the case when all the c_i are positive. It is not so difficult to see that (4.4) holds true for any given $c_{(n)}$. This completes the proof of the theorem.

It should be noted that the results obtained above are extensions, in a sense of Cramér's convergence theorems [10].

5. Measurable transformations preserving asymptotic equivalence in the general case

Lemma 1.3.4 of [1] states that every measurable transformation preserves type $(\boldsymbol{B})_d$ asymptotic equivalence in the general case of basic spaces, i.e., if for each s,

(5.1) $$f_s(z_{(n_s)})=(f^s_1(z_{(n_s)},\cdots,f^s_{m_s}(z_{(n_s)}))$$

is a measurable transformation from $R_{(n_s)}$ to $R_{(m_s)}$, and if $\{X^s_{(n_s)}\}$ $(s=1,2,\cdots)$ and $\{Y^s_{(n_s)}\}$ $(s=1,2,\cdots)$ are $AEUD(\boldsymbol{B})$, then the sequences, $\{U^s_{(m_s)}\}$ $(s=1,2,\cdots)$ and $\{V^s_{(m_s)}\}$ $(s=1,2,\cdots)$ defined by

$$(5.2)\qquad U^s_{(m_s)}=f_s(X^s_{(n_s)}) \quad\text{and}\quad V^s_{(m_s)}=f_s(Y^s_{(n_s)})\,, \qquad s=1, 2, \cdots,$$

are $AEUD(\boldsymbol{B})$ as $s\rightarrow\infty$.

The purpose of the present section is to discuss the same kind of problem for other types of asymptotic equivalence. First, it is easy to show the following.

THEOREM 5.1. *Let, for each s,*

$$(5.3)\qquad f_s(z_{(n_s)})=(f_1^s(z_1), \cdots, f_{n_s}^s(z_{n_s}))$$

be a measurable transformation from $R_{(n_s)}$ *into itself, and put*

$$(5.4)\qquad U^s_{(n_s)}=f_s(X^s_{(n_s)}) \quad\textit{and}\quad V^s_{(n_s)}=f_s(Y^s_{(n_s)})\,.$$

Under this situation, suppose that $f_i^s(z)$ *is a continuous and monotone non-decreasing function in z on the real line for each i and s,* $i=1, \cdots, n_s$: $s=1, 2, \cdots$. *Then, it holds that* (a) $X^s_{(n_s)}\sim Y^s_{(n_s)}\,(\boldsymbol{M})_d$ *implies that* $U^s_{(n_s)}\sim V^s_{(n_s)}\,(\boldsymbol{M})_d$, *and* (b) $X^s_{(n_s)}\sim Y^s_{(n_s)}\,(\boldsymbol{S})_d$ *implies that* $U^s_{(n_s)}\sim V^s_{(n_s)}\,(\boldsymbol{S})_d$, *as* $s\rightarrow\infty$.

Furthermore, if $X^s_{(n_s)}$ *and* $Y^s_{(n_s)}$ *belong to* $\mathcal{P}(R_{(n_s)}, \boldsymbol{B}_{(n_s)}, \mu_{(n_s)})$ *for each s, the assertions* (a) *and* (b) *hold true under the conditions that each* $f_i^s(z)$ *is continuous and monotone* (*non-decreasing or non-increasing*).

PROOF. The first part follows from the fact that the inverse images of $\boldsymbol{M}_{(n_s)}$ and $\boldsymbol{S}_{(n_s)}$ with respect to the transformation (5.3) are contained respectively in $\boldsymbol{M}_{(n_s)}$ and $\boldsymbol{S}_{(n_s)}$ for each s.

To prove the second part, let us consider the transformation

$$h_s(z_{(n_s)})=(c_1^s z_1, \cdots, c_{n_s}^s z_{n_s})\,,$$

for each s where $c_i^s=+1$, or $=-1$, according as $f_i^s(z)$ is monotone non-decreasing or non-increasing, and let us put $\boldsymbol{M}^*_{(n_s)}=h_s^{-1}(\boldsymbol{M}_{(n_s)})$ and $\boldsymbol{S}^*_{(n_s)}=h_s^{-1}(\boldsymbol{S}_{(n_s)})$. Then, under the condition of the theorem it holds that

$$(5.5)\qquad \delta_d(X^s_{(n_s)}, Y^s_{(n_s)}: \boldsymbol{M}_{(n_s)})=\delta_d(X^s_{(n_s)}, Y^s_{(n_s)}: \boldsymbol{M}^*_{(n_s)})\,,$$

and

$$(5.6)\qquad \delta_d(X^s_{(n_s)}, Y^s_{(n_s)}: \boldsymbol{S}_{(n_s)})=\delta_d(X^s_{(n_s)}, Y^s_{(n_s)}: \boldsymbol{S}^*_{(n_s)})\,,$$

for each s. It also holds that

$$(5.7)\qquad f_s^{-1}(\boldsymbol{M}_{(n_s)})\subseteqq\boldsymbol{M}^*_{(n_s)} \quad\text{and}\quad f_s^{-1}(\boldsymbol{S}_{(n_s)})\subseteqq\boldsymbol{S}^*_{(n_s)}\,,$$

for each s.

From (5.5) through (5.7) it follows that

$$\delta_d(X^s_{(n_s)}, Y^s_{(n_s)}: \boldsymbol{M}_{(n_s)})\geqq\delta_d(U^s_{(n_s)}, V^s_{(n_s)}: \boldsymbol{M}_{(n_s)})\,,$$

and

$$\delta_d(X^s_{(n_s)}, Y^s_{(n_s)}: S_{(n_s)}) \geqq \delta_d(U^s_{(n_s)}, V^s_{(n_s)}: S_{(n_s)}),$$

for each s which imply the assertions (a) and (b) in turn. This completes the proof of the theorem.

In the case of equal basic spaces, it is easy to see the following theorem, whose proof will be omitted.

THEOREM 5.2. *Let*

$$f(z_{(n)}) = (f_1(z_{(n)}), \cdots, f_m(z_{(n)})) \tag{5.8}$$

be a measurable transformation from $R_{(n)}$ *to* $R_{(m)}$, *and put*

$$U^s_{(m)} = f(X^s_{(n)}) \quad \textit{and} \quad V^s_{(m)} = f(Y^s_{(n)}), \tag{5.9}$$

for each s.

Under this situation, suppose that $f(z_{(n)})$ *is continuous, and at least one of the sequences,* $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ *and* $\{Y^s_{(n)}\}$ $(s=1, 2, \cdots)$, *has properties* $C(S)$ *and* $B(S)$ *simultaneously or one for each. Then,* $X^s_{(n)} \sim Y^s_{(n)}\ ((M))_d$ *implies that* $U^s_{(m)} \sim V^s_{(m)}\ ((M))_d$ *as* $s \to \infty$.

As an immediate consequence of this theorem, we have the following well-known result.

COROLLARY 5.1. *If the transformation* (5.8) *is continuous, and if* $X^s_{(n)} \to Y_{(n)}$ *(in law) as* $s \to \infty$, $Y_{(n)}$ *being some member of* $\mathcal{P}(R_{(n)}, B_{(n)}, \mu_{(n)})$, *then* $U^s_{(m)} \to V_{(m)}$ *(in law) as* $s \to \infty$, *where* $V_{(m)} = f(Y_{(n)})$.

Even if the basic spaces are fixed, when the transformation under consideration depends on s, the problem is quite complicated. The following theorem gives an answer to the problem in a special case of such situations.

THEOREM 5.3. *Let, for each* s,

$$f_s(z_{(n)}) = (f^s_1(z_1), \cdots, f^s_n(z_n)) \tag{5.10}$$

be a measurable transformation from $R_{(n)}$ *into itself, and put, as before,*

$$U^s_{(n)} = f_s(X^s_{(n)}) \quad \textit{and} \quad V^s_{(n)} = f_s(Y^s_{(n)}). \tag{5.11}$$

Under this situation, suppose that $f^s_i(z)$ *is a continuous and monotone function of* z *on the real line, and at least one of the sequences,* $\{X^s_{(n)}\}$ $(s=1, 2, \cdots)$ *and* $\{Y^s_{(n)}\}$ $(s=1, 2, \cdots)$ *has properties* $C(S)$ *and* $B(S)$ *simultaneously or one for each. Then,* $X^s_{(n)} \sim Y^s_{(n)} (M)_d$ *implies that* $U^s_{(n)} \sim V^s_{(n)} (M)_d$ *as* $s \to \infty$.

The proof of this theorem is similar to that of Theorem 5.1, and is omitted.

Note that the last statement in the proof of the Theorem 4.3 follows from the above theorem.

6. Type $(M)_d$ asymptotic equivalence of marginal random variables

In the present section, we shall consider type $(\boldsymbol{M})_d$ asymptotic equivalence of marginal random variables when the rest are replaced in some way by another random variables which are asymptotically equivalent in a sense to the original ones.

Suppose that we are given a sequence of random variables, $\{(X^s_{(n_s)}, Z^s_{(m_s)})\}$ $(s=1, 2, \cdots)$, with $(X^s_{(n_s)}, Z^s_{(m_s)}) \in \mathcal{F}(R_{(n_s+m_s)}, \boldsymbol{B}_{(n_s+m_s)})$ for each s. The first problem to be discussed in this section is as follows: Let the cdf. of $X^s_{(n_s)}$ be $F_s(x_{(n_s)})$, and the conditional cdf. of $Z^s_{(m_s)}$ given $X^s_{(n_s)}=x_{(n_s)}$ be $P_s(z_{(m_s)}\,|\,x_{(n_s)})$. The cdf. of $Z^s_{(m_s)}$ is then given by

$$H_s(z_{(m_s)})=\int_{R_{(n_s)}} P_s(z_{(m_s)}\,|\,x_{(n_s)})dF_s(x_{(n_s)}) \,. \tag{6.1}$$

Let $\{\bar{X}^s_{(n_s)}\}$ $(s=1, 2, \cdots)$ be a sequence of random variables with cdf.'s $\bar{F}_s(x_{(n_s)})$, $s=1, 2, \cdots$, and $\{\bar{Z}^s_{(m_s)}\}$ $(s=1, 2, \cdots)$ be a sequence of random variables whose cdf.'s are given by

$$\bar{H}_s(z_{(m_s)})=\int_{R_{(n_s)}} P_s(z_{(m_s)}\,|\,x_{(n_s)})d\bar{F}_s(x_{(n_s)}) \,, \qquad s=1, 2, \cdots \,. \tag{6.2}$$

Under this situation, what type of asymptotic equivalence of $\{X^s_{(n_s)}\}$ $(s=1, 2, \cdots)$ and $\{\bar{X}^s_{(n_s)}\}$ $(s=1, 2, \cdots)$, probably with some additional conditions, implies type $(\boldsymbol{M})_d$ asymptotic equivalence of $\{Z^s_{(m_s)}\}$ $(s=1, 2, \cdots)$ and $\{\bar{Z}^s_{(m_s)}\}$ $(s=1, 2, \cdots)$?

The same question is asked in the case where $Z^s_{(m_s)}$ has the conditional pdf., $p_s(z_{(m_s)}\,|\,x_{(n_s)})$, given $X^s_{(n_s)}=x_{(n_s)}$ for each s, in which case $\bar{Z}^s_{(m_s)}$ stands for a random variable whose pdf. is given by

$$\bar{h}_s(z_{(m_s)})=\int_{R_{(n_s)}} p_s(z_{(m_s)}\,|\,x_{(n_s)})d\bar{F}_s(x_{(n_s)}) \tag{6.3}$$

for each s.

First, the following theorem gives an answer to the question stated above in the case of unequal basic spaces.

THEOREM 6.1. (i) *Under the situation stated above, let* $\bar{Z}^s_{(m_s)}$ *be a random variable with the corresponding cdf. given by* (6.2), *for each* s. *Then, the condition* $X^s_{(n_s)} \sim \bar{X}^s_{(n_s)}\ (\boldsymbol{B})_d$ *implies that* $Z^s_{(m_s)} \sim \bar{Z}^s_{(m_s)}\ (\boldsymbol{M})_d$ *as* $s \to \infty$.

(ii) *In the case when* $Z^s_{(m_s)}$ *has the conditional pdf. and* $\bar{Z}^s_{(m_s)}$ *is a random variable with the pdf. given by* (6.3), *the condition* $X^s_{(n_s)} \sim \bar{X}^s_{(n_s)}$ $(\boldsymbol{B})_d$ *implies that* $Z^s_{(m_s)} \sim \bar{Z}^s_{(m_s)}$ $(\boldsymbol{B})_d$ *as* $s \to \infty$.

PROOF. To prove the first part of the theorem, it is sufficient to show that

$$\delta_d(Z^s_{(m_s)}, \bar{Z}^s_{(m_s)} : \boldsymbol{M}_{(m_s)}) \leqq 2\delta_d(X^s_{(n_s)}, \bar{X}^s_{(n_s)} : \boldsymbol{B}_{(n_s)}) \tag{6.4}$$

for each s.

From (6.1) and (6.2) it follows that, for any given $\varepsilon > 0$ and any given point $z_{(m_s)}$ of $R_{(m_s)}$, there exists a $\boldsymbol{B}_{(n_s)}$-partition of $R_{(n_s)}$, $\boldsymbol{\Gamma}_s = \{E^s_i : E^s_i \in \boldsymbol{B}_{(n_s)},\ i = 1, 2, \cdots\}$, such that

$$|H_s(z_{(m_s)}) - \bar{H}_s(z_{(m_s)})| < |\sum_{\boldsymbol{\Gamma}_s} P(z_{(m_s)} | x^s_{(n_s)i}) \{P^{X^s_{(n_s)}}(E^s_i) - P^{\bar{X}^s_{(n_s)}}(E^s_i)\}| + \varepsilon\,, \tag{6.5}$$

where, for each i, $x^s_{(n_s)i}$ is any fixed point of E^s_i. Note that this inequality holds for any $\boldsymbol{B}_{(n_s)}$-partition of $R_{(n_s)}$ which is finer than $\boldsymbol{\Gamma}_s$.

By using the Hahn-Jordan decomposition theorem for the signed measure $P^{X^s_{(n_s)}} - P^{\bar{X}^s_{(n_s)}}$, we have by (6.5)

$$|H_s(z_{(m_s)}) - \bar{H}_s(z_{(m_s)})| \leqq 2\{P^{X^s_{(n_s)}}(A_s) - P^{\bar{X}^s_{(n_s)}}(A_s)\} + \varepsilon\,,$$

for some subset A_s belonging to $\boldsymbol{B}_{(n_s)}$, from which (6.4) follows. This proves the assertion (i) of the theorem.

To prove (ii), we shall show that

$$\delta_d(Z^s_{(m_s)}, \bar{Z}^s_{(m_s)} : \boldsymbol{B}_{(m_s)}) \leqq \delta_d(X^s_{(n_s)}, \bar{X}^s_{(n_s)} : \boldsymbol{B}_{(n_s)})\,, \tag{6.6}$$

for each s.

Let, for each s, $h_s(z_{(m_s)})$ be the pdf. of $Z^s_{(m_s)}$, i.e.,

$$h_s(z_{(m_s)}) = \int_{R_{(n_s)}} p_s(z_{(m_s)} | x_{(n_s)}) dF_s(x_{(n_s)})\,. \tag{6.7}$$

Then, it follows from (6.3) and (6.7) that

$$\int_{R_{(m_s)}} |h_s(z_{(m_s)}) - \bar{h}_s(z_{(m_s)})| d\mu_{(m_s)} \leqq \int_{R_{(n_s)}} |dF_s(x_{(n_s)}) - d\bar{F}_s(x_{(n_s)})|\,,$$

for each s.

Since for any given $\varepsilon > 0$ there can be found a $\boldsymbol{B}_{(n_s)}$-partition, $\boldsymbol{\Delta}_s$, of $R_{(n_s)}$, such that the right-hand member of the above inequality is less than

$$\sum_{E \in \boldsymbol{\Delta}_s} |P^{X^s_{(n_s)}}(E) - P^{\bar{X}^s_{(n_s)}}(E)| + \varepsilon\,,$$

for each s, we easily have (6.6) by a similar argument to the proof of (i) of the theorem and by (1.3), which completes the proof of the assertion (ii). Thus the proof of the theorem is completed.

In the case of equal basic spaces, the situation mentioned above can be restated as follows: Let $\{(X^s_{(n)}, Z^s_{(m)})\}$ $(s=1, 2, \cdots)$ be a sequence of random variables belonging to $\mathcal{F}(R_{(n+m)}, \boldsymbol{B}_{(n+m)})$ such that for each s, $X^s_{(n)}$ has the cdf. $F_s(x_{(n)})$ and $Z^s_{(m)}$ the conditional cdf. $P_s(z_{(m)}|x_{(n)})$ given $X^s_{(n)}=x_{(n)}$. Further, let $\{\bar{X}^s_{(n)}\}$ $(s=1, 2, \cdots)$ be a sequence of members of $\mathcal{F}(R_{(n)}, \boldsymbol{B}_{(n)})$ with the corresponding cdf.'s, $\bar{F}_s(x_{(n)})$, $s=1, 2, \cdots$. Corresponding to (6.1) and (6.2), cdf.'s of $Z^s_{(n)}$ and $\bar{Z}^s_{(n)}$ are given respectively by

$$H_s(z_{(m)})=\int_{R_{(n)}} P_s(z_{(m)}|x_{(n)})dF_s(x_{(n)})\,, \tag{6.8}$$

and

$$\bar{H}_s(z_{(m)})=\int_{R_{(n)}} P_s(z_{(m)}|x_{(n)})d\bar{F}_s(x_{(n)})\,, \tag{6.9}$$

for each s. When $Z^s_{(m)}$ has the conditional pdf. $p_s(z_{(m)}|x_{(n)})$ for each s, and hence the pdf. of $Z^s_{(m)}$ is given by

$$h_s(z_{(m)})=\int_{R_{(n)}} p_s(z_{(m)}|x_{(n)})dF_s(x_{(n)})\,, \tag{6.10}$$

the same notation $\bar{Z}^s_{(m)}$ is used to designate a random variable whose pdf. is given by

$$\bar{h}_s(z_{(m)})=\int_{R_{(n)}} p_s(z_{(m)}|x_{(n)})d\bar{F}_s(x_{(n)})\,, \tag{6.11}$$

for each s.

Under the situation stated above, we can state the following.

THEOREM 6.2. *Let $\bar{Z}^s_{(m)}$ be a random variable with the cdf. given by (6.9), and assume that the following conditions are satisfied:*

(i) *For some real $c^s_{(n)}=(c^s_1, \cdots, c^s_n)$ and $d^s_{(n)}=(d^s_1, \cdots, d^s_n)$, $s=1, 2, \cdots$, the sequence of random variables, $\{\bar{Y}^s_{(n)}\}$ $(s=1, 2, \cdots)$, has property $B(S)$ where*

$$\bar{Y}^s_{(n)}=((\bar{X}^s_1-d^s_1)/c^s_1, \cdots, (\bar{X}^s_n-d^s_n)/c^s_n)$$

for each s.

(ii) *For the constants given in the above condition* (i), *put $x^s_{(n)}=(c^s_1x_1+d^s_1, \cdots, c^s_nx_n+d^s_n)$ for $x_{(n)}=(x_1, \cdots, x_n)$ and $Q_s(z_{(m)}|x_{(n)})=P_s(z_{(m)}|x^s_{(n)})$*

for each s. *Then, for any given* $\varepsilon>0$ *and any given bounded subset* B *belonging to* $S_{(n)}$, *there exist a positive number* $\delta=\delta(\varepsilon, B)$ *and a positive integer* $s_0=s_0(\varepsilon, B)$ *such that* $|x_{(n)}-y_{(n)}|<\delta$ *with* $x_{(n)}$, $y_{(n)}$ *belonging to* B *implies that*

$$\sup_{z_{(m)}\in R_{(m)}} |Q_s(z_{(m)}|x_{(n)})-Q_s(z_{(m)}|y_{(n)})|<\varepsilon$$

for all $s\geqq s_0$.

Then, the condition $X^s_{(n)}\sim\bar{X}^s_{(n)}\,(M)_d$ *implies that* $Z^s_{(m)}\sim\bar{Z}^s_{(m)}\,(M)_d$ *as* $s\to\infty$.

PROOF. Let $\bar{G}_s(y_{(n)})$ and $G_s(y_{(n)})$ be the cdf.'s of $\bar{Y}^s_{(n)}$ and $Y^s_{(n)}$ respectively where $Y^s_{(n)}$ is defined by

$$Y^s_{(n)}=((X^s_1-d^s_1)/c^s_1,\cdots,(X^s_n-d^s_n)/c^s_n)$$

for each s. Then, the cdf.'s of $Z^s_{(m)}$ and $\bar{Z}^s_{(m)}$ can be rewritten as

$$H_s(z_{(m)})=\int_{R_{(n)}} Q_s(z_{(m)}|x_{(n)})dG_s(x_{(n)})$$

and

$$\bar{H}_s(z_{(m)})=\int_{R_{(n)}} Q_s(z_{(m)}|x_{(n)})d\bar{G}_s(x_{(n)}),$$

respectively.

It follows from the condition of the theorem that $Y^s_{(n)}\sim\bar{Y}^s_{(n)}\,(M)_d$ as $s\to\infty$, and therefore, the sequence $\{Y^s_{(n)}\}$ $(s=1, 2, \cdots)$ has property $B(S)$. Hence, for any given $\varepsilon>0$, there exist a member of $S_{(n)}$, B say whose closure being compact and a positive integer $s_0=s_0(\varepsilon)$ such that

$$(6.12)\qquad |H_s(z_{(m)})-\bar{H}_s(z_{(m)})| \leqq\left|\int_B Q_s(z_{(m)}|x_{(n)})dG_s(x_{(n)})-\int_B Q_s(z_{(m)}|x_{(n)})d\bar{G}_s(x_{(n)})\right|+\varepsilon,$$

for all $z_{(m)}$ in $R_{(m)}$ and all $s\geqq s_0$.

Let $\Delta=\{E_i,\ i=1,\cdots,N\}$ be a $S_{(n)}$-partition of the subset B such that

$$\sup_{z_{(m)}\in R_{(m)}}\{\sup_{x_{(n)}\in E_i} Q_s(z_{(m)}|x_{(n)})-\inf_{x_{(n)}\in E_i} Q_s(z_{(m)}|x_{(n)})\}<\varepsilon$$

for each i; $i=1,\cdots,N$, and for all $s\geqq s_0'$, s_0' being some positive integer.

Evaluating the first member of the right-hand side of (6.12), we then easily obtain

$$|H_s(z_{(m)})-\bar{H}_s(z_{(m)})|\leqq N\delta_d(Y^s_{(n)},\bar{Y}^s_{(n)}:S_{(n)})+3\varepsilon,$$

from which follows the assertion of the theorem, because N is independent of s. This completes the proof of the theorem.

In the case when $X^s_{(n)} \to \bar{X}_{(n)} (\boldsymbol{M})_d$ as $s \to \infty$, $\bar{X}_{(n)}$ being some member of $\mathscr{F}(R_{(n)}, \boldsymbol{B}_{(n)})$, the first condition of the above theorem is automatically fulfilled, and we immediately obtain the following

COROLLARY 6.1. *Suppose that, for any given* $\varepsilon > 0$ *and any given bounded subset* B *belonging to* $S_{(n)}$, *there exist a positive number* $\delta = \delta(\varepsilon, B)$ *and a positive integer* $s_0 = s_0(\varepsilon, B)$ *such that* $|x_{(n)} - y_{(n)}| < \delta$ *with* $x_{(n)}$, $y_{(n)}$ *belonging to* B *implies that*

$$\sup_{z_{(m)} \in R_{(m)}} |P_s(z_{(m)} | x_{(n)}) - P_s(z_{(m)} | y_{(n)})| < \varepsilon$$

for all $s \geqq s_0$.

Then, the condition $X^s_{(n)} \to \bar{X}_{(n)} (\boldsymbol{M})_d$ *implies that* $Z^s_{(m)} \sim \bar{Z}^s_{(m)} (\boldsymbol{M})_d$ *as* $s \to \infty$, *where* $\bar{Z}^s_{(m)}$ *stands for a random variable whose cdf. is given by*

$$\bar{H}_s(z_{(m)}) = \int_{R_{(n)}} P_s(z_{(m)} | x_{(n)}) d\bar{F}(x_{(n)}) ,$$

$\bar{F}(x_{(n)})$ *being the cdf. of* $\bar{X}_{(n)}$.

Sometimes it is known previously that one of the sequences, $\{Z^s_{(m)}\}$ $(s=1, 2, \cdots)$ and $\{\bar{Z}^s_{(m)}\}$ $(s=1, 2, \cdots)$ given in the preceding theorem has properties $C(S)$ and $B(S)$. In such cases, the second condition (ii) of the Theorem 6.2 can be weakened in the following form:

(ii)* For any given $\varepsilon > 0$ and any given bounded subset B belonging to $S_{(n)}$, there exist a positive number $\delta = \delta(\varepsilon, B, z_{(m)})$ and a positive integer $s_0 = s_0(\varepsilon, B, z_{(m)})$ such that the condition $|x_{(n)} - y_{(n)}| < \delta$ with $x_{(n)}$, $y_{(n)}$ belonging to B implies that

$$|Q_s(z_{(m)} | x_{(n)}) - Q_s(z_{(m)} | y_{(n)})| < \varepsilon$$

for every fixed $z_{(m)}$ in $R_{(m)}$, and for $s \geqq s_0$.

In the next place, we shall consider the case where $Z^s_{(m)}$ has the conditional pdf. for each s. It should first be noted that in such a case a random variable $\bar{Z}^s_{(m)}$ with the cdf. defined by (6.9) has the pdf. given by (6.11).

The following lemma is easy to prove and the proof will be omitted.

LEMMA 6.1. *For the condition* (ii) *of the Theorem* 6.2 *to hold it is sufficient that the following two conditions are satisfied*:

(i) *For any given* $z_{(m)}$ *in* $R_{(m)}$ *and for each* s, *the function* $q_s(z_{(m)} | x_{(n)}) = p_s(z_{(m)} | x^s_{(n)})$ *is totally differentiable with respect to* $x_{(n)}$ *over* $R_{(n)}$.

(ii) *For any given bounded subset* B *belonging to* $S_{(n)}$, *there exist a* *positive number* $\eta=\eta(B)$ *and a positive integer* $s_0=s_0(B)$ *such that*

$$\sup_{x_{(n)}\in B}\int_{R_{(m)}}|\phi_{si}(z_{(m)}\,|\,x_{(n)})|d\mu_{(m)}\leqq\eta \qquad i=1,\cdots,n\,, \tag{6.13}$$

for all $s\geqq s_0$, *where*

$$\phi_{si}(z_{(m)}\,|\,x_{(n)})=\frac{\partial}{\partial x_i}q_s(z_{(m)}\,|\,x_{(n)}) \qquad i=1,\cdots,n\,. \tag{6.14}$$

It follows, then, from the Theorem 6.2 and the above lemma that the following corollary holds true.

COROLLARY 6.2. *Suppose that the conditions,* (i) *of Theorem* 6.2 *and* (i) *and* (ii) *of Lemma* 6.1, *are satisfied. Then, the condition* $X^s_{(n)}\sim\bar{X}^s_{(n)}(\boldsymbol{M})_d$ *implies that* $Z^s_{(m)}\sim\bar{Z}^s_{(m)}(\boldsymbol{M})_d$ *as* $s\to\infty$, *where* $\bar{Z}^s_{(m)}$ *stands for a random variables whose pdf. is given by* (6.11) *for each* s.

It should be remarked that if we replace the condition (6.13) of (ii) of Lemma 6.1 by

$$\sup_{x_{(n)}\in B}\left|\int_{R_{(m)}}\phi_{si}(z_{(m)}\,|\,x_{(n)})d\mu_{(m)}\right|\leqq\eta\,, \qquad i=1,\cdots,n\,, \tag{6.15}$$

then the condition (ii)** thus modified and the condition (iii) of Lemma 6.1 imply the condition (ii)* stated below the Corollary 6.1.

The last half of this section is devoted to the discussion of the second problem, which is set forth in the case of equal basic spaces as follows: Let, as before, $\{(X^s_{(n)}, Z^s_{(m)})\}$ $(s=1,2,\cdots)$ be a sequence of random variables belonging to $\mathcal{F}(R_{(n+m)}, \boldsymbol{B}_{(n+m)})$ and let $F_s(x_{(n)})$ and $P_s(z_{(m)}\,|\,x_{(n)})$ be the cdf. of $X^s_{(n)}$ and the conditional cdf. of $Z^s_{(m)}$ given $X^s_{(n)}=x_{(n)}$, respectively. When $Z^s_{(m)}$ has the conditional pdf., then let it be $p_s(z_{(m)}\,|\,x_{(n)})$. Suppose, for some real c^s_i and d^s_i, $i=1,\cdots,n$ and $s=1,2,\cdots$, the random variables defined by

$$Y^s_{(n)}=((X^s_1-d^s_1)/c^s_1,\cdots,(X^s_n-d^s_n)/c^s_n)\,, \qquad s=1,2,\cdots\,, \tag{6.16}$$

converge in probability to the point $1_{(n)}=(1,\cdots,1)$ as $s\to\infty$. Under this situation, a question is asked whether $\{Z^s_{(m)}\}$ $(s=1,2,\cdots)$ and $\{\bar{Z}^s_{(m)}\}$ $(s=1,2,\cdots)$ are $AEUD(\boldsymbol{M})$ or not, as $s\to\infty$, $\bar{Z}^s_{(m)}$ stands for a random variable whose cdf. is given by $P_s(z_{(m)}\,|\,c^s_{(n)}+d^s_{(n)})$ for each s, provided that it is eligible for a cdf. of a member of $\mathcal{F}(R_{(m)}, \boldsymbol{B}_{(m)})$. The same question is asked also when $Z^s_{(m)}$ has the conditional pdf. for each s.

Firstly, we shall prove the following theorem, which is an answer to the above question.

THEOREM 6.3. *Under the situation stated above, suppose that there exist sequences of real vectors,* $\{c^s_{(n)}=(c^s_1,\cdots,c^s_n)\}$ $(s=1,2,\cdots)$ *and* $\{d^s_{(n)}=(d^s_1,\cdots,d^s_n)\}$ $(s=1,2,\cdots)$ *such that the sequence of random variables,* $\{Y^s_{(n)}\}$ $(s=1,2,\cdots)$ *defined by* (6.16) *converges in probability to the point* $1_{(n)}=(1,\cdots,1)$ *and* $P_s(z_{(m)}\mid c^s_{(n)}+d^s_{(n)})$ *is the cdf. of some member,* $\bar{Z}^s_{(m)}$ *say, of* $\mathfrak{F}(R_{(m)},\boldsymbol{B}_{(m)})$ *for each* s.

Then, in order that $Z^s_{(m)}\sim\bar{Z}^s_{(m)}(\boldsymbol{M})_d$ *as* $s\to\infty$, *it is sufficient that the following condition is satisfied:*

(i) *For any given* $\varepsilon>0$, *there exist a positive number* $\delta=\delta(\varepsilon)$ *and a positive integer* $s_0=s_0(\varepsilon)$ *such that* $|x_{(n)}-1_{(n)}|<\delta$ *implies that*

$$\sup_{z_{(m)}\in R_{(m)}}|Q_s(z_{(m)}\mid x_{(n)})-Q_s(z_{(m)}\mid 1_{(n)})|<\varepsilon$$

for all $s\geqq s_0$, *where* $Q_s(z_{(m)}\mid x_{(n)})$ *is the same as that defined in the condition* (ii) *of the Theorem* 6.2.

PROOF. Let, as before, $H_s(z_{(m)})$, $\bar{H}_s(z_{(m)})$ and $G_s(x_{(n)})$ be the cdf.'s of $Z^s_{(m)}$, $\bar{Z}^s_{(m)}$ and $Y^s_{(n)}$, respectively, for each s. It is then evident that

$$|H_s(z_{(m)})-\bar{H}_s(z_{(m)})|$$
$$\leqq\left(\int_{V_\delta}+\int_{V^c_\delta}\right)|Q_s(z_{(m)}\mid x_{(n)})-Q_s(z_{(m)}\mid 1_{(n)})|\,dG_s(x_{(n)})\,,$$

for each s where V_δ is the δ-neighbourhood of the point $1_{(n)}$.

By this inequality and the conditions of the theorem, it easily follows that

$$\sup_{z_{(m)}\in R_{(m)}}|H_s(z_{(m)})-\bar{H}_s(z_{(m)})|\to 0\,,\qquad(s\to\infty)\,,$$

which proves the theorem.

Note that if one of the sequences, $\{Z^s_{(m)}\}$ $(s=1,2,\cdots)$ and $\{\bar{Z}^s_{(m)}\}$ $(s=1,2,\cdots)$ has properties $C(S)$ and $B(S)$ simultaneously or one for each, then the condition (i) of the above theorem can be weakened in the following form:

(i)* For any given $\varepsilon>0$ and any fixed point $z_{(m)}$ of $R_{(m)}$, there exist a positive number $\delta=\delta(\varepsilon,z_{(m)})$ and a positive integer $s_0=s_0(\varepsilon,z_{(m)})$ such that the condition $|x_{(n)}-1_{(n)}|<\delta$ implies that

$$|Q_s(z_{(m)}\mid x_{(n)})-Q_s(z_{(m)}\mid 1_{(n)})|<\varepsilon$$

for all $s\geqq s_0$.

Now, in the final place, suppose that $Z^s_{(m)}$ has the conditional pdf., $p_s(z_{(m)}\mid x_{(n)})$ for each s. In this case, an answer to the problem is given by the following theorem, whose proof is easy and will be omitted.

THEOREM 6.4. *Suppose that there exist sequences of real vectors,* $\{c^s_{(n)}=(c^s_1, \cdots, c^s_n)\}$ $(s=1, 2, \cdots)$ *and* $\{d^s_{(n)}=(d^s_1, \cdots, d^s_n)\}$ $(s=1, 2, \cdots)$ *such that the sequence of random variables,* $\{Y^s_{(n)}\}$ $(s=1, 2, \cdots)$ *defined by* (6.16) *converges in probability to* $1_{(n)}=(1, \cdots, 1)$, *and* $p_s(z_{(m)}|c^s_{(n)}+d^s_{(n)})$ *is the pdf. of some member,* $\bar{Z}^s_{(n)}$ *say, of* $\mathcal{P}(R_{(m)}, \boldsymbol{B}_{(m)}, \mu_{(m)})$.

Then it is sufficient for $\{Z^s_{(m)}\}$ $(s=1, 2, \cdots)$ *and* $\{\bar{Z}^s_{(m)}\}$ $(s=1, 2, \cdots)$ *to be AEUD(M) that the following two conditions are satisfied:*

(i) *For any fixed* $z_{(m)}$ *in* $R_{(m)}$, *the function* $q_s(z_{(m)}|x_{(n)})=p_s(z_{(m)}|x^s_{(n)})$, $x^s_{(n)}$ *being the same as that given in the condition* (ii) *of Theorem* 6.2, *is totally differentiable with respect to* $x_{(n)}$ *in same neighbourhood of the point* $1_{(n)}$.

(ii) *There exists a neighbourhood of* $1_{(n)}$, $V_\delta=\{x_{(n)}: |x_{(n)}-1_{(n)}|<\delta\}$, *in which the above condition holds, a positive number* η *and a positive integer* s_0 *such that*

$$\sup_{x_{(n)}\in V_\delta} \int_{R_{(m)}} |\phi_{si}(z_{(m)}|x_{(n)})| d\mu_{(m)} \leqq \eta\,, \qquad i=1, \cdots, n\,,$$

for all $s\geqq s_0$, *where* $\phi_{si}(z_{(m)}|x_{(n)})$ *are the same as those defined by* (6.14).

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