ON THE GROWTH OF A RANDOM WALK

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1. Introduction and summary

Let X_i, $i=1, 2, 3, \cdots$, be a sequence of independent and identically distributed random variables and write $S_n=\sum_{i=1}^{n} X_i$, $n\geqq 1$, $F(x)=P(X_i<x)$. The object of this paper is to establish the following theorem.

THEOREM. *Let $EX_i=0$, $EX_i^2=1$, $EX_i^2 \log^+|X_i|<\infty$, and $\{\phi(n), n\geqq 1\}$ be a monotone non-decreasing sequence of positive numbers. Then, the following three conditions are equivalent:*

(A) $P(|S_n|>\phi(n)\sqrt{n} \text{ i.o.})=0$,

(B) $\sum_{n=1}^{\infty} n^{-1}\phi^2(n)\, P(|S_n|>\phi(n)\sqrt{n})<\infty$,

(C) $\sum_{n=1}^{\infty} n^{-1}\phi(n)e^{-\{\phi^2(n)\}/2}<\infty$.

(Here, $\log^+ x=\max(0, \log x)$. The "i.o." in (A) stands for "infinitely often".)

The equivalence of (A) and (C) under rather weaker conditions than the above was obtained by Feller [4]. In fact, the conditions $EX_i=0$, $EX_i^2=1$, and $EX_i^2 \log^+ \log^+ |X_i|<\infty$ suffice. Various authors have subsequently worked on the subject of the equivalence of (B) and (C). Baum and Katz [1] showed that (B) and (C) are equivalent under the additional assumption that $EX_i^2(\log^+|X_i|)^{1+\delta}<\infty$ for some $\delta>0$. Their work was later improved upon by Davis [2] who obtained the equivalence when $EX_i^2 \log^+|X_i| \log^+ \log^+|X_i|<\infty$. In this present paper we sharpen the methods used in [1] and [2] and are able to further improve on the moment condition required.

2. Preliminary lemmas

We require two lemmas before proceeding to the proof of the theorem. Both of these deal with the convergence

$$F_n(x)=P(S_n<x\sqrt{n})\to\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du\ ,$$

as $n\to\infty$.

LEMMA 1. *Suppose* $EX_i=0$, $EX_i^2=1$, $EX_i^2\log^+\log^+|X_i|<\infty$ *and let* $B_n^2=\int_{|x|<\sqrt{n}} x^2dF(x)$. *Then,*

$$\sum_{n=3}^{\infty} n^{-1}\log\log n \sup_x |F_n(x)-\Phi(xB_n^{-1})|<\infty\ . \tag{1}$$

PROOF. Making use of Theorem 1 of Davis [2], we have, under the above mentioned conditions,

$$\sum_{n=3}^{\infty} n^{-1}\log\log n \sup_x |P(S_n<x\sqrt{n})-\Phi(x\beta_n^{-1})|<\infty\ ,$$

where

$$\beta_n^2=\int_{|x|<\sqrt{n}} x^2dF(x)-\Big(\int_{|x|<\sqrt{n}} xdF(x)\Big)^2\ .$$

Consequently, the result (1) holds provided that

$$\sum_{n=3}^{\infty} n^{-1}\log\log n \sup_x |\Phi(xB_n^{-1})-\Phi(x\beta_n^{-1})|<\infty\ . \tag{2}$$

Now,

$$B_n^2=\beta_n^2+\Big(\int_{|x|<\sqrt{n}} xdF(x)\Big)^2\ ,$$

and it is easily seen by expansion in Taylor series that there is a positive constant C such that

$$\sup_x |\Phi(xB_n^{-1})-\Phi(x\beta_n^{-1})|\leqq C\Big(\int_{|x|<\sqrt{n}} xdF(x)\Big)^2\ .$$

Furthermore, since $EX_i=0$ and $EX_i^2\log^+\log^+|X_i|<\infty$,

$$\Big|\int_{|x|<\sqrt{n}} xdF(x)\Big|=\Big|\int_{|x|\geqq\sqrt{n}} xdF(x)\Big|\leqq\int_{|x|\geqq\sqrt{n}} |x|dF(x)$$
$$=0\ (\sqrt{n}\log\log n)^{-1}\ ,$$

so that

$$\sum_{n=3}^{\infty} n^{-1}\log\log n\Big(\int_{|x|<\sqrt{n}} xdF(x)\Big)^2<\infty\ ,$$

and therefore (2) holds. This completes the proof of the lemma.

LEMMA 2. *Suppose* $EX_i=0$ *and* $EX_i^2=1$. *Let* $\{B_n\}$ *be a sequence of positive constants with* $B_n\to 1$ *as* $n\to\infty$ *and write*

$$\Delta(n)=\sup_x |F_n(x)-\Phi(xB_n^{-1})|.$$

Then, for any number $a\geqq 1$ *such that* $F_n(x)$ *is continuous at* $x=\pm a$,

$$(1+x^2)|F_n(x)-\Phi(xB_n^{-1})|\leqq 1-B_n^2+2B_n^2\int_{|y|\geqq aB_n^{-1}} y^2d\Phi(y)+5a^2\Delta(n).$$

This result is an extension of Theorem 1 ([3] p. 70) which deals with the case $B_n=1$.

PROOF. We have,

$$\begin{aligned}\int_{-a}^{a} x^2dF_n(x)&=\int_{-a}^{a} x^2d(F_n(x)-\Phi(xB_n^{-1}))+\int_{-a}^{a} x^2d\Phi(xB_n^{-1})\\&=a^2(F_n(a)-\Phi(aB_n^{-1}))-a^2(F_n(-a)-\Phi(-aB_n^{-1}))\\&\quad-2\int_{-a}^{a} x(F_n(x)-\Phi(xB_n^{-1}))dx+\int_{-a}^{a} x^2d\Phi(xB_n^{-1})\\&\geqq -4a^2\Delta(n)+\int_{-a}^{a} x^2d\Phi(xB_n^{-1}),\end{aligned}$$

and consequently,

$$\int_{|x|\geqq a} x^2dF_n(x)=1-\int_{-a}^{a} x^2dF_n(x)\leqq 1-\int_{-a}^{a} x^2d\Phi(xB_n^{-1})+4a^2\Delta(n). \tag{3}$$

Furthermore,

$$\int_{|x|\geqq a} x^2dF_n(x)\geqq\begin{cases} y^2(1-F_n(y)) & \text{for } y\geqq a\\ y^2F_n(y) & \text{for } y\leqq -a,\end{cases} \tag{4}$$

and

$$\int_{|x|\geqq a} x^2d\Phi(xB_n^{-1})\geqq\begin{cases} y^2(1-\Phi(yB_n^{-1})) & \text{for } y\geqq a\\ y^2\Phi(yB_n^{-1}) & \text{for } y\leqq -a,\end{cases} \tag{5}$$

so that from (3), (4) and (5), we have for $|y|\geqq a$,

$$\begin{aligned}y^2|F_n(y)-\Phi(yB_n^{-1})|&\leqq \int_{|x|\geqq a} x^2dF_n(x)+\int_{|x|\geqq a} x^2d\Phi(xB_n^{-1})\\&\leqq 1-\int_{-a}^{a} x^2d\Phi(xB_n^{-1})+\int_{|x|\geqq a} x^2d\Phi(xB_n^{-1})+4a^2\Delta(n)\end{aligned}$$

$$=1-B_n^2+B_n^2\left\{1-\int_{|x|\leqq aB_n^{-1}} x^2d\Phi(x)\right.$$

$$\left.+\int_{|x|\geqq aB_n^{-1}} x^2d\Phi(x)\right\}+4a^2\Delta(n)$$

$$=1-B_n^2+2B_n^2\int_{|x|\geqq aB_n^{-1}} x^2d\Phi(x)+4a^2\Delta(n)\,.$$

Then, since $a\geqq 1$, we have for all y,

$$(1+y^2)\,|\,F_n(y)-\Phi(yB_n^{-1})\,|\leqq 1-B_n^2+2B_n^2\int_{|x|\geqq aB_n^{-1}} x^2d\Phi(x)+5a^2\Delta(n)\,,$$

which is the required result.

3. Proof of the theorem

The equivalence of (A) and (C) under the stated conditions has been obtained by Feller [4]; we shall obtain the equivalence of (B) and (C).

Let $B_n^2=\int_{|x|<\sqrt{n}} x^2dF(x)$. Then, using Lemma 2, we have

$$n^{-1}\phi^2(n)\,|\,P(|\,S_n\,|>\phi(n)\sqrt{n}\,)-2\{1-\Phi(\phi(n)B_n^{-1})\}\,|$$
$$\leqq n^{-1}\sup_{x\geqq 0}\,(1+x^2)\,|\,P(|\,S_n\,|>x\sqrt{n}\,)-2\{1-\Phi(xB_n^{-1})\}\,|$$
$$\leqq 2n^{-1}\left\{1-B_n^2+2\int_{|x|\geqq a} x^2d\Phi(xB_n^{-1})+5a^2\Delta(n)\right\}\,,$$

where $\pm a$ are continuity points of $P(S_n<x\sqrt{n}\,)$ and $\Delta(n)=\sup_x|\,P(S_n< x\sqrt{n}\,)-\Phi(xB_n^{-1})\,|$. Now ε can be chosen arbitrarily small and positive such that $\pm a_n=\pm\sqrt{(2+\varepsilon)\log\log n}$ are continuity points for each n, and then

$$(6)\qquad \sum_{n=1}^{\infty} n^{-1}\phi^2(n)\,|\,P(|\,S_n\,|>\phi(n)\sqrt{n}\,)-2\{1-\Phi(\phi(n)B_n^{-1})\}\,|<\infty$$

provided that

(i) $\sum n^{-1}(1-B_n^2)<\infty$,

(ii) $\sum n^{-1}\int_{|x|\geqq a_n} x^2d\Phi(xB_n^{-1})<\infty$,

(iii) $\sum n^{-1}\log\log n\Delta(n)<\infty$.

We have shown in Lemma 1 that (iii) is satisfied while for (ii) we have

$$B_n^2 \int_{|x|\geqq a_n B_n^{-1}} x^2 d\Phi(x) \leqq B_n^2 \int_{|x|\geqq a_n} x^2 d\Phi(x) \sim \int_{|x|\geqq a_n} x^2 d\Phi(x)\,,$$

as $n\to\infty$, and integrating by parts,

$$\int_{|x|\geqq a_n} x^2 d\Phi(x) = \sqrt{\frac{2}{\pi}}\int_{a_n}^{\infty} x^2 e^{-x^2/2}dx = \sqrt{\frac{2}{\pi}}\left\{a_n e^{-a_n^2/2} + \int_{a_n}^{\infty} e^{-x^2/2}dx\right\}$$
$$\sim \sqrt{\frac{2}{\pi}}\, a_n e^{-a_n^2/2}\,.$$

Consequently,

$$n^{-1}\int_{|x|\geqq a_n} x^2 d\Phi(x) = 0\left(\frac{(\log\log n)^{1/2}}{n(\log n)^{1+\epsilon/2}}\right)$$

as $n\to\infty$ and (ii) holds. Finally, for (i) we have, again using integration by parts,

$$\sum_{n=1}^{\infty} n^{-1}(1-B_n^2) = \sum_{n=1}^{\infty} n^{-1}\int_{|x|\geqq\sqrt{n}} x^2 dF(x)$$
$$= \sum_{n=1}^{\infty} P(|X|>\sqrt{n}) + 2\sum_{n=1}^{\infty} n^{-1}\int_{\sqrt{n}}^{\infty} xP(|X|>x)dx$$
$$= \sum_{n=1}^{\infty} P(|X|>\sqrt{n}) + 2\sum_{n=1}^{\infty} n^{-1}\sum_{r=n}^{\infty}\int_{\sqrt{r}}^{\sqrt{r+1}} xP(|X|>x)dx$$
$$\leqq \sum_{n=1}^{\infty} P(|X|>\sqrt{n}) + \sum_{n=1}^{\infty} n^{-1}\sum_{r=n}^{\infty} P(|X|>\sqrt{r})$$
$$< C\sum_{r=1}^{\infty} \log r P(|X|>\sqrt{r}) < \infty\,,$$

C being a positive constant and the convergence of the last series being implied by the moment condition $EX_i^2 \log^+|X_i|<\infty$. It follows, then, that (6) always holds under the conditions of the theorem and therefore the condition (B) is equivalent to the condition

$$\sum_{n=1}^{\infty} n^{-1}\phi^2(n)\{1-\Phi(\phi(n)B_n^{-1})\} < \infty\,. \tag{7}$$

Now as $n\to\infty$,

$$1-\Phi(\phi(n)B_n^{-1}) \sim \frac{1}{\sqrt{2\pi}}\frac{B_n}{\phi(n)}\exp\left\{-\frac{1}{2}\phi^2(n)B_n^{-2}\right\}$$
$$\sim \frac{1}{\sqrt{2\pi}}[\phi(n)]^{-1}\exp\left\{-\frac{1}{2}\phi^2(n)B_n^{-2}\right\}\,,$$

so that (7) is equivalent to

$$\sum_{n=1}^{\infty} n^{-1}\phi(n) \exp\left\{-\frac{1}{2}\phi^2(n)B_n^{-2}\right\} < \infty . \tag{8}$$

Furthermore, since $B_n^2 \leqq 1$,

$$\sum n^{-1}\phi(n) \exp\left\{-\frac{1}{2}\phi^2(n)B_n^{-2}\right\} \leqq \sum n^{-1}\phi(n) \exp\left\{-\frac{1}{2}\phi^2(n)\right\} ,$$

so in order to complete the proof it just remains to show that convergence in (8) implies that in (C). Suppose the contrary, namely that for some $\{\phi(n)\}$,

$$\sum n^{-1}\phi(n) \exp\left\{-\frac{1}{2}\phi^2(n)B_n^{-2}\right\} < \infty ,$$

$$\sum n^{-1}\phi(n) \exp\left\{-\frac{1}{2}\phi^2(n)\right\} = \infty .$$

It is easily seen that this is only possible if there is a subsequence of integers n_i with $\phi^2(n_i)(B_{n_i}^{-2}-1) \to \infty$ as $n_i \to \infty$. That is, writing

$$A_n = 1 - B_n^2 = \int_{|x| \geqq \sqrt{n}} x^2 dF(x) ,$$

$\phi^2(n_i)A_{n_i} \to \infty$ as $n_i \to \infty$. With this in mind we define sets

$$N_1 = \{n: \phi^2(n)A_n \leqq 8\} , \qquad N_2 = \{n: \phi^2(n)A_n > 8\} ,$$

and clearly for $n \in N_1$,

$$n^{-1}\phi(n) \exp\left\{-\frac{1}{2}\phi^2(n)\right\} = 0\left(n^{-1}\phi(n) \exp\left\{-\frac{1}{2}\phi^2(n)B_n^{-2}\right\}\right)$$

so that

$$\sum_{N_1} n^{-1}\phi(n)e^{-\{\phi^2(n)\}/2} < \infty . \tag{9}$$

Furthermore,

$$e^{\{\phi^2(n)\}/2} > 1 + \frac{1}{2}\phi^2(n) + \frac{1}{8}\phi^4(n) > \frac{1}{8}\phi^4(n) ,$$

so that for $n \in N_2$ and sufficiently large,

$$\phi(n)e^{-\{\phi^2(n)\}/2} < 8(\phi(n))^{-3} < 8(\phi(n))^{-2} < A_n .$$

However, we have shown above that $\sum n^{-1}A_n < \infty$, and consequently,

$$\sum_{N_2} n^{-1}\phi(n)e^{-\{\phi^2(n)\}/2} < \infty . \tag{10}$$

(9) and (10) give the required contradiction and the result of the theorem follows.

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