CHARACTERISTIC FUNCTIONS SATISFYING A FUNCTIONAL EQUATION (I)

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(Received Sept. 30, 1967)

1. Introduction and summary

Let p and n be given integers ($n\geqq 2$, and $0\leqq p\leqq n$) and $a_1, \cdots, a_n$ be given constants lying between 0 and 1 exclusive*. Then, there exists a unique positive number α such that

$$(1.1)\qquad a_1^\alpha+\cdots+a_n^\alpha=1\,.$$

We denote by $T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$ the set of all characteristic functions (abr. ch. f.) $\varphi(t)$ which satisfy the equation,

$$(1.2)\qquad \varphi(t)=\varphi(a_1t)\cdot\,\cdots\,\cdot\varphi(a_pt)\varphi(-a_{p+1}t)\cdot\,\cdots\,\cdot\varphi(-a_nt)\,.$$

When $p=n$ it contains every stable ch. f. with the characteristic exponent α, namely the one which satisfies the equation $\varphi(t)=\varphi(at)\varphi(bt)$ for any pair a and b of positive numbers such that $a^\alpha+b^\alpha=1$. Every stable ch. f. is infinitely divisible and represented by Levy's formula,

$$(1.3)\qquad \log\varphi(t)=i\gamma t-\frac{1}{2}\sigma^2t^2+\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x)+\int_{-\infty}^0\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x)$$

where (i) $M(x)\equiv 0,\ N(x)\equiv 0,\ \gamma=0$ if $\alpha\geqq 2$,

(ii) $\sigma=0,\ M(x)=-\lambda x^{-\alpha},\ N(x)=\mu|x|^{-\alpha},\ \lambda\geqq 0,\ \mu\geqq 0$ if $\alpha<2$,

(iii) $\lambda=\mu$ if $\alpha=1$,

(iv) γ is a suitably chosen constant if $\alpha\neq 1,\ \alpha<2$,

or more explicitly by**

* The requirement that $a_j<1$, $j=1, \cdots, n$ is imposed here to avoid the trivial case $|\varphi(t)|\equiv 1$.

** See [2], pp. 164-171.

$$(1.4)\qquad \varphi(t)=\begin{cases} \exp\left\{-c|t|^\alpha\left(1+\beta\frac{t}{|t|}\tan\frac{\pi}{2}\alpha\right)\right\} & \alpha\neq 1\,, \\ \exp\{i\gamma t-c|t|\}\ \ \text{(the Cauchy dist.)} & \alpha=1\,, \end{cases}$$

where

$$c\geqq 0\,, \qquad |\beta|\leqq 1\,.$$

When $p<n$, among stable ch. f.'s only symmetric ones (i.e. $\varphi(t)=\exp\{-c|t|^\alpha\}$) belong to $T_\alpha(a_1,\cdots,a_p,-a_{p+1},\cdots,-a_n)$. If $\varphi(t)$ is a symmetric stable ch.f. and if $a_1+\cdots+a_p-a_{p+1}-\cdots-a_n=1$, then for any real γ, $e^{i\gamma t}\varphi(t)$ also belongs to $T_\alpha(a_1,\cdots,a_p,-a_{p+1},\cdots,-a_n)$.

Does $T_\alpha(a_1,\cdots,a_p,-a_{p+1},\cdots,-a_n)$ contain any other ch.f.'s? The problems of this sort are treated by several authors; P. Levy* gave an example of non-stable ch.f. $\varphi(t)\in T_1\left(\frac{1}{2},\frac{1}{2}\right)$,

$$\varphi(t)=\prod_{-\infty}^{\infty}\exp\{2^{-k+1}(\cos 2^k t-1)\}\,.$$

This is an infinitely divisible ch.f. and is represented by (1.3) with $\gamma=0$, $\sigma^2=0$, and $M(x)=-N(-x)=-2^{-k}$ for $2^k\leqq x<2^{k+1}$, $k=0,\pm 1,\pm 2,\cdots$. This example was generalized to $T_\alpha(a,b)$, [9]; if $-(\log a)/\rho$ and $-(\log b)/\rho$ are relatively prime positive integers for some $\rho>0$, if $\lambda(t)\equiv M(e^{-t})e^{-\alpha t}$ is a periodic function with the period ρ, and if $M(x)$ is monotone non-decreasing, then

$$\varphi(t)=\exp\left\{\int_0^\infty(\cos tx-1)dM(x)\right\}$$

belongs to $T_\alpha(a,b)$.

Marcinkiewicz**, and Linnik [3] treated a more general equation,

$$(1.5)\qquad \varphi(a_1t)\cdot\cdots\cdot\varphi(a_nt)=\varphi(b_1t)\cdot\cdots\cdot\varphi(b_nt)\,,$$

where $(a_1,\cdots,a_n)$ is not a permutation of $(b_1,\cdots,b_n)$. Marcinkiewicz showed under the assumption of existence of moments of all order that (1.5) implies $\varphi(t)=\exp\left\{i\gamma t-\frac{1}{2}\sigma^2t^2\right\}$. Yu. V. Linnik obtained a necessary and sufficient condition on $a_1,\cdots,a_n$, $b_1,\cdots,b_n$ under which $\varphi(t)=\exp\left\{i\gamma t-\frac{1}{2}\sigma^2t^2\right\}$ is the unique solution of (1.5).

He also derived a necessary condition that a symmetric ch.f. $\varphi(t)$ satisfies (1.5) under the assumption $\max(|a_1|,\cdots,|a_n|)\neq\max(|b_1|,\cdots,$

* See [1] p. 538.

** See [5].

$|b_n|$). Laha and Lukacs [4] considered the equation $\varphi(t)=\prod_1^\infty \varphi(a_n t)$ and showed that if $\sum_1^\infty a_n^2 \geqq 1$, then $\varphi(t)=\exp\left\{i\gamma t-\frac{1}{2}\sigma^2 t^2\right\}$.

The purpose of the present paper is to give a necessary and sufficient condition that a complex valued function $\varphi(t)$ belongs to $T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$. The condition will be stated in the theorems of the next section. In section 4 we first prove that every $\varphi(t) \in T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$ is infinitely divisible and that the Poisson spectra $M(x)$ and $N(x)$ are related by (4.4) and (4.5). The problem is then essentially reduced to solving these equations. Section 3 contains a statement and a proof of the related theorem 5 which is of much interest. The results are used in section 4 to prove the theorems. Section 5 is devoted to examples.

2. Notations and statement of theorems

$\sigma_0(z)$, $\sigma_1(z)$ and $\sigma(z)$ are complex variable entire functions defined respectively by

$$\sigma_0(z)=1-a_1^z-\cdots-a_n^z\,,$$
$$\sigma_1(z)=1-a_1^z-\cdots-a_p^z+a_{p+1}^z+\cdots+a_n^z$$

and

$$\sigma(z)=\begin{cases}\sigma_0(z)\sigma_1(z) & \text{if } p<n\\ \sigma_0(z) & \text{if } p=n\,.\end{cases}$$

For any $\rho>0$, $A_n(0)$, $A_n(\rho)$, $B_n^p(\rho)$, and $C_n^p(\rho)$ are sets of all n-tuples $(b_1, \cdots, b_n)$ with $0<b_j<1$, $j=1, \cdots, n$ and

$A_n(0)$: for some j and k, $\log b_j/\log b_k$ is an irrational number,

$A_n(\rho)$: $l_k=-\log b_k/\rho$, $k=1, \cdots, n$ are mutually prime positive integers,

$B_n^p(\rho)$: a subset of $A_n(\rho)$ such that either at least one of $l_1, \cdots, l_p$ is odd or at least one of $l_{p+1}, \cdots, l_n$ is even, and,

$C_n^p(\rho)$: a subset of $A_n(\rho)$ such that $l_1, \cdots, l_p$ are all even and $l_{p+1}, \cdots, l_n$ are all odd.

$\boldsymbol{M}$ is the set of all monotone non-decreasing, right-continuous functions $Q(x)$ on $(0, \infty)$ such that $\lim_{x\to\infty} Q(x)=0$ and

$$\int_0^1 x^2 dQ(x)<\infty\,.$$

For any $\rho>0$, we denote by $\boldsymbol{P}^+(\rho)$ and by $\boldsymbol{P}^-(\rho)$ the set of all left-continuous function $\lambda(t)$ such that $\lambda(t+\rho)=\lambda(t)$ and $\lambda(t+\rho)=-\lambda(t)$, respectively.

We are now in the position to state theorems.

THEOREM 1. *Suppose* $(a_1, \cdots, a_n) \in A_n(0)$, *and* $\varphi(t) \in T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$. *If either* $p=n$, *or* $\alpha=1$, $\varphi(t)$ *is the stable ch.f. with the characteristic exponent* α. *If* $p<n$, *then there exists unique real number* γ *such that* $\varphi(t)e^{-i\gamma t}$ *is the symmetric stable ch.f.* γ *is taken to be* 0 *if* $\sigma_1(1)\neq 0$.

THEOREM 2. *If* $\alpha<2$, $p<n$ *and if* $(a_1, \cdots, a_n) \in B_n^p(\rho)$, *then a necessary and sufficient condition that a complex valued function* $\varphi(t)$ *belongs to* $T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$ *is that* $\varphi(t)$ *is represented as,*

$$\varphi(t)=\exp\left\{i\gamma t+2\int_0^\infty (\cos tx-1)dM(x)\right\}, \tag{2.1}$$

$$\gamma\sigma_1(1)=0, \tag{2.2}$$

where $M(x)$ *is a monotone non-decreasing function on* $(0, \infty)$ *such that* $\lambda(t)=M(e^{-t})e^{-\alpha t}$ *is an element of* $\boldsymbol{P}^+(\rho)$.

THEOREM 3. *If* $\alpha<2$, $p<n$, *and if* $(a_1, \cdots, a_n) \in C_n^p(\rho)$, *then a necessary and sufficient condition that the complex valued function* $\varphi(t)$ *belongs to* $T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$ *is that* $\varphi(t)$ *is represented as follows;*

(i) *case* $1<\alpha<2$

$$\log\varphi(t)=i\beta t+\int_0^\infty (e^{itx}-1-itx)dM(x)+\int_{-\infty}^0 (e^{itx}-1-itx)dN(x), \tag{2.3}$$

$$\beta\sigma_1(1)=0. \tag{2.4}$$

(ii) *case* $\alpha=1$

$$\log\varphi(t)=i\beta t+\int_0^\infty\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dM(x)+\int_{-\infty}^0\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dN(x), \tag{2.5}$$

$$\beta\sigma_1(1)+\sum_{j=1}^p \gamma(a_j)-\sum_{j=p+1}^n \gamma(a_j)=0. \tag{2.6}$$

(iii) *case* $0<\alpha<1$

$$\log\varphi(t)=\int_0^\infty (e^{itx}-1)dM(x)+\int_{-\infty}^0 (e^{itx}-1)dN(x) \tag{2.7}$$

where

$$\gamma(c)=c\left[\int_0^\infty\left(\frac{x}{1+(cx)^2}-\frac{x}{1+x^2}\right)dM(x)+\int_{-\infty}^0\left(\frac{x}{1+(cx)^2}-\frac{x}{1+x^2}\right)dN(x)\right]$$

and in each case $M(x)$ *and* $N(x)$ *are non-decreasing functions defined respectively on* $(0, \infty)$ *and* $(-\infty, 0)$ *and have the form*

$$M(x)=-[\lambda(-\log x)+\mu(-\log x)]/x^\alpha\,, \qquad x>0 \tag{2.8}$$

and

$$N(x)=[\lambda(-\log|x|)-\mu(-\log|x|)]/|x|^\alpha\,, \qquad x<0\,, \tag{2.9}$$

$\lambda(t)$ *and* $\mu(t)$ *being elements respectively of* $\boldsymbol{P}^+(\rho)$ *and* $\boldsymbol{P}^-(\rho)$.

THEOREM 4. *If* $\alpha<2$, $p=n$, *and if* $(a_1, \cdots, a_n)\in A_n(\rho)$, $\rho>0$, *then a necessary and sufficient condition that a complex valued function* $\varphi(t)$ *belongs to* $T_\alpha(a_1, \cdots, a_n)$ *is that* $\varphi(t)$ *is put in the form either* (2.3) *with* $\beta=0$ *(case* $1<\alpha<2$), *or* (2.5) *with* $\sum\gamma(a_j)$ *(case* $\alpha=1$), *or* (2.7) *(case* $0<\alpha<1$), *where* $M(x)$ *and* $N(x)$ *are monotone non-decreasing functions defined respectively on* $(0, \infty)$ *and* $(-\infty, 0)$ *such that*

$$M(x)=-\lambda(-\log x)/x^\alpha \qquad x>0 \tag{2.10}$$

and

$$N(x)=\mu(-\log|x|)/|x|^\alpha \qquad x<0\,, \tag{2.11}$$

$$\lambda(t)\in\boldsymbol{P}^+(\rho)\,, \qquad \mu(t)\in\boldsymbol{P}^+(\rho)\,.$$

3. Related theorem and its proof

We prove in this section the theorem 5 below. The relation to the main theorems will be clarified in the next section.

THEOREM 5. *Let* $A_j=\log a_j$, $j=1, \cdots, n$. *Let* $f(t)$ *and* $g(t)$ *be the monotone non-increasing left continuous functions defined on* $(-\infty, \infty)$ *such that* $\lim_{t\to-\infty} f(t)=\lim_{t\to-\infty} g(t)=0$.

Then a necessary and sufficient condition that the relations,

$$f(t)=f(t+A_1)+\cdots+f(t+A_p)+g(t+A_{p+1})+\cdots+g(t+A_n)\,, \tag{3.1}$$

and

$$g(t)=g(t+A_1)+\cdots+g(t+A_p)+f(t+A_{p+1})+\cdots+f(t+A_n) \tag{3.2}$$

hold is that $f(t)$ *and* $g(t)$ *are represented as follows:*

(i) *case* $(a_1, \cdots, a_n)\in A_n(0)$

$$f(t)=-\lambda e^{\alpha t}\,, \qquad g(t)=-\mu e^{\alpha t}\,,$$

where λ *and* μ *are constants, and* $\lambda=\mu$ *when* $p<n$.

(ii) *case* $(a_1, \cdots, a_n) \in B_n^p(\rho)$

$$f(t)=-\lambda(t)e^{\alpha t}\,, \qquad g(t)=-\mu(t)e^{\alpha t}\,,$$

where $\lambda(t) \in \boldsymbol{P}^+(\rho)$, $\mu(t) \in \boldsymbol{P}^+(\rho)$, *and* $\lambda(t)=\mu(t)$ *when* $p<n$.

(iii) *case* $p<n$, $(a_1, \cdots, a_n) \in C_n^p(\rho)$.

$$f(t)=-(\lambda(t)+\mu(t))e^{\alpha t}\,, \qquad g(t)=-(\lambda(t)-\mu(t))e^{\alpha t}\,,$$

where $\lambda(t) \in \boldsymbol{P}^+(\rho)$ *and* $\mu(t) \in \boldsymbol{P}^-(\rho)$.

PROOF. We can use Linnik's arguments [3]. See also Laha and Lukacs ([7], pp. 137–146).

For any $t \leqq 0$, we have

$$0 \geqq f(t) \geqq f(0) > -\infty \tag{3.3}$$

and

$$0 \geqq g(t) \geqq g(0) > -\infty\,. \tag{3.4}$$

We introduce

$$\chi_f(z)=\int_{-\infty}^0 e^{-zt}f(t)dt$$

and

$$\chi_g(z)=\int_{-\infty}^0 e^{-zt}g(t)dt\,,$$

which converge and are regular in the half plane $\mathrm{Re}\, z<0$. We can easily verify the relations,

$$\sigma_0(z)\chi_f(z)=E_f(z)+(a_{p+1}^z+\cdots+a_n^z)(\chi_g(z)-\chi_f(z))\,, \tag{3.5}$$

and

$$\sigma_0(z)\chi_g(z)=E_g(z)-(a_{p+1}^z+\cdots+a_n^z)(\chi_g(z)-\chi_f(z))\,, \tag{3.6}$$

where

$$E_f(z)=\sum_{j=1}^p a_j^z \int_0^{A_j} e^{-zt}f(t)dt+\sum_{j=p+1}^n a_j^z \int_0^{A_j} e^{-zt}g(t)dt$$

and

$$E_g(z)=\sum_{j=1}^p a_j^z \int_0^{A_j} e^{-zt}g(t)dt+\sum_{j=p+1}^n a_j^z \int_0^{A_j} e^{-zt}f(t)dt\,.$$

From (3.5) and (3.6), we obtain

$$\chi_f(z)\sigma(z)=E^f(z)\,, \qquad \text{for } \operatorname{Re} z<0\,, \tag{3.7}$$

where

$$E^f(z)=\begin{cases} (1-a_1^z-\cdots-a_p^z)E_f(z)+(a_{p+1}^z+\cdots+a_n^z)E_g(z) & \text{if } p<n \\ E_f(z) & \text{if } p=n\,. \end{cases}$$

For any real Λ,

$$\chi_f(z+\Lambda)=\int_{-\infty}^{0} e^{-zt}e^{-\Lambda t}f(t)dt$$

converges in the half plane $\operatorname{Re} z<-\Lambda$.

Complex inversion formula of the Laplace transform is then applied to obtain, for $t<0$,

$$\int_t^0 e^{-\Lambda\tau}f(\tau)d\tau=-\lim_{y\to\infty}\frac{1}{2\pi i}\int_{r-iy}^{r+iy} e^{tz}\frac{\chi_f(z+\Lambda)}{z}dz\,, \tag{3.8}$$

where

$$r<0\,, \qquad r<-\Lambda\,.$$

LEMMA 1. *All zeros of $\sigma(z)$ are located in some strip $x_0\leqq \operatorname{Re} z\leqq\alpha$. There exist two constants k_0 and k_1 such that the multiplicity of each zero does not exceed k_0, and the number of zeros in any horizontal strip of width 2 does not exceed k_1.*

PROOF. Let $a_0=\min(a_1, \cdots, a_n)$. Then for sufficiently large $x^*>0$, there exists a positive constant C such that $|\sigma(z)|\geqq Ca_0^{2x}>0$ holds if $x=\operatorname{Re} z<-x^*$. On the other hand, if $x>\alpha$, then $|a_1^z+\cdots+a_p^z\pm a_{p+1}^z\pm\cdots\pm a_n^z|\leqq a_1^x+\cdots+a_n^x<1$, or $|\sigma(z)|>0$. We have thus proved the first assertion. Since both $\sigma_0(z)$ and $\sigma_1(z)$ satisfy linear differential equations of degree at most n with constant coefficients, the multiplicity of any zero of $\sigma_0(z)$ and $\sigma_1(z)$ cannot exceed $n+1$. The last statement is proved by applying Jensen's theorem. (q.e.d.)

For $m=0, 1, 2, \cdots$, let $y=y_m$ be the line lying in the horizontal strip between $y=m$ and $y=m+1$ and the distance from each zero of $\sigma(z)$ is at least ε_1, say. We denote by L_m $(m=1, 2, \cdots)$ the contour bounded by the lines, $y=y_{m-1}$, $y=y_m$, $x=\alpha+\varepsilon_1$, and $x=x_0-\varepsilon_1$ and by L_{-m} the reflection of L_m by the real axis, while L_0 is the contour bounded by $y=y_0$, $y=-y_0$, $x=\alpha+\varepsilon$, and $x=x_0-\varepsilon_1$ $(\equiv x_1)$.

LEMMA 2. *$|E^f(z)|$ is bounded in every half plane* $\operatorname{Re} z\geqq C$.

PROOF. If $\operatorname{Re} z=x$, then

$$|E^f(z)|\leqq\sum_{j=1}^{n} a_j^x\int_{A_j}^{0} e^{-tx}|f(t)|dt\leqq -f(0)\sum_{j=1}^{n} a_j^x\int_{A_j}^{0} e^{-tx}dt\,.$$

Since the right-hand side is continuous in x, it is bounded in every finite interval $[-|C|, 0]$. If $x\geqq 0$, then the above inequalities imply,

$$|E^f(z)|\leqq -f(0)\sum_{j=1}^{n} a_j^x e^{-A_j x}\int_{A_j}^{0} dt=f(0)\sum_{1}^{n} A_j<\infty\,. \qquad \text{(q.e.d.)}$$

LEMMA 3 (Yu. V. Linnik*). *If* $|z_0-\zeta|\geqq\varepsilon$ *for all zeros* ζ *of* $\sigma(z)$, *then* $|\sigma(z_0)|\geqq C>0$ *where* C *is a constant depending only on* ε.

It follows from lemmas 2 and 3 that $|E^f(z)/\sigma(z)|$ is bounded on the contours $z=x+iy_m$, $x\geqq x^*$ (x^* is an arbitrary but fixed number), so that we have

$$\left|\int_{x_1\pm iy_m}^{\infty\pm iy_m} e^{t(z-\Lambda)}\frac{E^f(z)}{(z-\Lambda)\sigma(z)}dz\right|=O\left(\frac{1}{mt}e^{t(x_1-\Lambda)}\right)$$

and for fixed x_2 and t,

$$\left|\int_{x_2\pm iy_m}^{x_1\pm iy_m} e^{t(z-\Lambda)}\frac{E^f(z)}{(z-\Lambda)\sigma(z)}dz\right|=O\left(\frac{1}{m}\right)\,.$$

Hence if $(0>)\ x_1>\Lambda>x_2$, (3.8) becomes

$$\begin{aligned}(3.9)\qquad \int_t^0 e^{-\Lambda\tau}f(\tau)d\tau&=-\lim_{m\to\infty}\frac{1}{2\pi i}\int_{x_2-iy_m}^{x_2+iy_m} e^{t(z-\Lambda)}\frac{E^f(z)}{(z-\Lambda)\sigma(z)}dz\\ &=\frac{E^f(\Lambda)}{\sigma(\Lambda)}+\lim_{m\to\infty}\frac{1}{2\pi i}\int_{x_1-iy_m}^{x_1+iy_m} e^{t(z-\Lambda)}\frac{E^f(z)}{(z-\Lambda)\sigma(z)}dz\\ &=\frac{E^f(\Lambda)}{\sigma(\Lambda)}-\sum_{-m}^{m}\frac{1}{2\pi i}\int_{L_j} e^{t(z-\Lambda)}\frac{E^f(z)}{(z-\Lambda)\sigma(z)}dz+O\left(\frac{1}{mt}e^{t(x_1-\Lambda)}\right)\end{aligned}$$

where $E^f(\Lambda)/\sigma(\Lambda)$ is the residue of the integrand at Λ, and is equal to $\chi_f(\Lambda)=\int_{-\infty}^{0} e^{-\Lambda\tau}f(\tau)d\tau$. Thus we have

$$(3.10)\qquad \int_{-\infty}^{t} e^{-\Lambda t}f(\tau)d\tau=\lim_{m\to\infty}\sum_{j=-m}^{m}\frac{1}{2\pi i}\int_{L_j} e^{t(z-\Lambda)}\frac{E^f(z)}{(z-\Lambda)\sigma(z)}dz\,,$$

in which the series converges uniformly in t ($\leqq 0$).

A zero ζ of $\sigma(z)$ is said to be active if the residue of $E^f(z)e^{t(z-\Lambda)}/(z-\Lambda)\sigma(z)$ does not vanish at ζ. We shall show in lemmas 4–6 that every active zero is of the form $\alpha+iy$.

* Cf. [3], lemma IV.

LEMMA 4. *The infimum β of the real part of the active zeros of $\sigma(z)$ is an active zero.*

PROOF. Suppose β is not an active zero. Then there exists a real number Λ ($<x_0\leqq\beta$) and a complex active zero $\zeta_0=\mu_0+i\tau_0$ ($\tau_0>0$), such that every active zero except ζ_0 and $\bar{\zeta}_0$ is located outside the circle centred at Λ and passes through ζ_0 and $\bar{\zeta}_0$. Let $d_0=|\Lambda-\zeta_0|$ and $\delta_0=(\inf_{\zeta\neq\bar{\zeta}_0,\zeta_0}|\Lambda-\zeta|+d_0)/2$ where infimum is taken over all zeros ζ of $\sigma(z)$ other than ζ_0 and $\bar{\zeta}_0$. Clearly $\delta_0>d_0$. It follows from lemma 1 that for any $m\geqq0$, there exists a real number u_m between $(\beta+\Lambda)/2$ and β such that the line segment $x=u_m+iy$, $y_{m-1}\leqq y\leqq y_m$ ($-y_0\leqq y\leqq y_0$, if $m=0$) is at least at a distance of $\varepsilon_2=\min(\varepsilon_1, (\beta-\Lambda)/4k_2)$ from each zero of $\sigma(z)$. We can then replace in (3.10) the left wall $x=x_1$ of L_m by $x=u_m$.

For any function $h(t)$ integrable over $(-\infty, 0)$, let

$$L_r(h(t))=\underbrace{\int_{-\infty}^{t}dt\cdots\int_{-\infty}^{t}}_{r\ \text{times}}h(t)dt\,.$$

If $f_0(t)=\int_{-\infty}^{t}e^{-\Lambda\tau}f(\tau)d\tau$, then

$$L_r(f_0(t))=\sum_{-\infty}^{\infty}\frac{1}{2\pi i}\int_{L_m}e^{t(z-\Lambda)}\frac{E^f(z)}{(z-\Lambda)^{r+1}\sigma(z)}dz\,. \tag{3.11}$$

Here term by term integration is permissible because the series in (3.10) converges uniformly. The resulting series (3.11) converges absolutely and uniformly in $t<0$, for $r\geqq1$. Write

$$R_m=\frac{1}{2\pi i}\int_{L_m}e^{t(z-\Lambda)}\frac{E^f(z)}{(z-\Lambda)^{r+1}\sigma(z)}dz\,.$$

Then $|R_m|\leqq B_1e^{(u_m-\Lambda)t}M_m/\Delta_m^{r+1}$, where B_1 is a positive constant, M_m is supremum of $|E^f(z)/\sigma(z)|$ on $L_m\leqq C_0$ (by lemmas 2 and 3), Δ_m is infimum of $|z-\Lambda|$ on $L_m\geqq m$, and $u_m\geqq(\beta+\Lambda)/2$. Hence,

$$\sum_{|m|\geqq\delta_0}|R_m|\leqq B_2e^{(\beta-\Lambda)t/2}\sum_{m\geqq\delta_0}\left(\frac{1}{m}\right)^{r+1}=B_2e^{(\beta-\Lambda)t/2}\delta_0^{-(r+1)}\sum_{m\geqq\delta_0}\left(\frac{\delta_0}{m}\right)^{r+1}$$
$$\leqq B_2e^{(\beta-\Lambda)t/2}\delta_0^{-(r+1)}\sum_{m\geqq\delta_0}\left(\frac{\delta_0}{m}\right)^{2}=B_3e^{(\beta-\Lambda)t/2}/\delta_0^{r+1}\,,$$

where B_3 is a constant independent of r and t.

We next consider the active zeros of $\sigma(z)$ other than ζ_0 and $\bar{\zeta}_0$ lying in the horizontal strip bounded by the lines $y=\delta_0$ and $y=-\delta_0$. There exist only finite number of such zeros, $\zeta_1,\cdots,\zeta_s$, say. Let C_j, $j=1$,

$\cdots, s$, be mutually disjoint circles centred at ζ_j with radius at most $\delta_0 - d_0$. Since on each C_j, $|E^f(z)/\sigma(z)|$ is bounded (c.f. lemmas 2 and 3), we can find a positive constant B_4 such that

$$\sum_1^s \left| \frac{1}{2\pi i} \int_{C_j} e^{t(z-\Lambda)} \frac{E^f(z)}{(z-\Lambda)^{r+1}\sigma(z)} dz \right| \leqq B_4 e^{(\beta-\Lambda)t/2} / \delta_0^{r+1} .$$

Thus we have proved that the contribution of active zeros other than ζ_0 and $\bar{\zeta}_0$ to the sum (3.11) is bounded in absolute value, by

$$B_5 e^{(\beta-\Lambda)t/2} / \delta_0^{r+1} .$$

We finally consider the zeros ζ_0 and $\bar{\zeta}_0$. The sum of residues of $e^{t(z-\Lambda)} \cdot E^f(z)/(z-\Lambda)\sigma(z)$ at ζ_0 and $\bar{\zeta}_0$ is

$$Q(t) = e^{\Delta t} P(t) + e^{\bar{\Delta} t} \overline{P(t)} ,$$

where

$$\Delta = \zeta_0 - \Lambda = d_0 e^{i\alpha}$$

and

$$P(t) = b_0 t^\nu + \cdots + b_\nu , \qquad b_0 \neq 0 .$$

Since $\mathrm{Re}\, \Delta = \mu_0 - \Lambda > 0$, $I_m \Delta = \tau_0 > 0$, we have $\cos\alpha > 0$ and $\sin\alpha > 0$. Suppose first $\nu = 0$, then writing $b_0 = |b_0| e^{i\theta}$, we obtain

$$\begin{aligned} (3.12) \qquad L_r(Q(t)) &= 2\, \mathrm{Re}\, L_r(b_0 e^{\Delta t}) \\ &= 2|b_0| d_0^{-r}\, \mathrm{Re}\, \{\exp(\Delta t + i(\theta - r\alpha))\} \\ &= 2|b_0| d_0^{-r}\, \mathrm{Re}\, [\exp\{d_0 t \cos\alpha + i(d_0 t \sin\alpha + (\theta - rd))\}] . \end{aligned}$$

Let r be so large that

$$2|b_0| d_0^{-r} \exp\left\{ d_0 \cos\alpha \frac{-4\pi}{\sin\alpha} \right\} > 2B_5 \delta_0^{-(r+1)} \exp\left\{ \frac{-2\pi}{d_0 \sin\alpha} \frac{\beta - \Lambda}{2} \right\}$$

holds, and let it be fixed. Let k be an integer such that

$$-4\pi < 2k\pi - (\theta - r\alpha) \leqq -2\pi$$

and set

$$t_0 = (2k\pi - (\theta - r\alpha))/d_0 \sin\alpha \qquad (<0) .$$

Then,

$$\begin{aligned} (3.13) \qquad L_r(Q(t_0)) &= 2|b_0| d_0^{-r} \exp(d_0 t_0 \cos\alpha) \\ &\geqq 2|b_0| d_0^{-r} \exp\left(d_0 \frac{-4\pi}{\sin\alpha} \cos\alpha \right) \end{aligned}$$

$$\geqq 2B_5\delta_0^{-(r+1)} \exp\left(\frac{-2\pi}{d_0 \sin\alpha} \frac{\beta-\Lambda}{2}\right)$$

$$\geqq 2B_5\delta_0^{-(r+1)} \exp\left(\frac{\beta-\Lambda}{2} t_0\right).$$

Thus,

$$L_r(f_0(t_0)) \geqq B_5\delta_0^{-(r+1)} \exp\left(\frac{\beta-\Lambda}{2} t_0\right) > 0. \tag{3.14}$$

This is, however, impossible, because for any negative t, $f(t)$ is negative and $L_r(f_0(t_0))$ must also be negative.

When $\nu>0$, we can show by induction on r and k that

$$L_r(t^k e^{\Lambda t}) = e^{\Lambda t} \sum_{j=0}^{k} (-1)^j \frac{(j+r-1)!}{(r-1)!} \binom{k}{j} \frac{t^{k-j}}{\Lambda^{j+r}}. \tag{3.15}$$

Then,

$$\begin{aligned} L_r\{Q(t)\} &= 2\,\mathrm{Re}\, L_r\{P(t)e^{\Lambda t}\} \\ &= 2\,\mathrm{Re}\left\{(-1)^\nu b_0 \frac{(\nu+r-1)!}{(\nu-1)!} \frac{e^{\Lambda t}}{\Lambda^r}\left(1+B\frac{t^\nu}{r}\right)\right\} \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} B = &\sum_{j=0}^{\nu-1} (-1)^{j+\nu} \frac{(j+r-1)!}{(\nu+r-1)!} r \binom{\nu}{j} \frac{t^{-j}}{\Lambda^j} \\ &+ \sum_{k=0}^{\nu-1} \sum_{j=0}^{k} \frac{b_{\nu-k}}{b_0} (-1)^{j+\nu} \frac{(j+r-1)!}{(\nu+r-1)!} r \binom{k}{j} \frac{t^{-\nu+k-j}}{\Lambda^j} \end{aligned}$$

is bounded for $t \leqq -\varepsilon < 0$.

Again first select a sufficiently large r, keep it fixed and select a suitable $t=t_0$ in the interval $(-4\pi/d_0 \sin\alpha, -2\pi/d_0 \sin\alpha]$, from which we obtain $L_r(f_0(t_0))>0$.

LEMMA 5. *There exists a positive constant D_0 such that*

$$|L_r(f_0(t))| \leqq D_0 \frac{e^{(\alpha-\Lambda)t}}{|\alpha-\Lambda|^{r+1}}, \qquad \textit{for all } t<0. \tag{3.17}$$

PROOF. We first prove the lemma in the case $p=n$. Since in this case α is the only real zero, we conclude from lemmas 1 and 4, that every real zero must be of the form $\alpha+iy$, or more precisely there exists a monotone sequence $\{s_m\}$ of non-negative numbers such that active zeros are exhausted by,

$$\zeta_m = \alpha + is_m \quad \text{and} \quad \bar{\zeta}_m = \alpha - is_m.$$

They are all simple, and the residue of integrand $e^{t(z-\Lambda)}E_f(z)/(z-\Lambda)\sigma_0(z)$ at ζ_m is given by

$$e^{t(\zeta_m-\Lambda)}E_f(\zeta_m)/(\zeta_m-\Lambda)\sigma_0'(\zeta_m)\,.$$

Hence

$$L_r(f_0(t))=\sum_{-\infty}^{\infty}\frac{e^{t(\zeta_m-\Lambda)}E_f(\zeta_m)}{(\zeta_m-\Lambda)^{r+1}\sigma_0'(\zeta_m)}\,,\qquad \text{where } \zeta_{-m}=\bar{\zeta}_m\,.$$

Using lemma 1, we obtain

$$\begin{aligned}|\zeta_m-\Lambda|^{r+1}&=|\alpha-\Lambda|^{r+1}\left|1+i\frac{s_m}{\alpha-\Lambda}\right|^{r+1}\\&\geqq|\alpha-\Lambda|^{r+1}\left(1+\left(\frac{m}{(\alpha-\Lambda)k_1}\right)^2\right)\,.\end{aligned}$$

On the other hand, since $|\sigma_0'(\zeta_m)|\geqq|\sigma_0'(\alpha)|$, $|E_f(\zeta_m)/\sigma_0'(\zeta_m)|$ is bounded by a constant, we obtain (3.17).

In the general case we consider $F(t)=f(t)+g(t)$ instead of $f(t)$. Since then $F(t)$ satisfies the equation,

$$F(t)=F(t+A_1)+\cdots+F(t+A_n)\,,$$

the problem reduces to the case $p=n$, and we obtain

$$|L_r(F_0(t))|\leqq D_0\frac{e^{(\alpha-\Lambda)t}}{|\alpha-\Lambda|^{r+1}}$$

where

$$F_0(t)=\int_{\infty}^{t}e^{-\Lambda\tau}F(\tau)d\tau\,.$$

Considering the fact $|L_r(f_0(t))|\leqq|L_r(F_0(t))|$ we obtain (3.17). (q.e.d.)

LEMMA 6. $\beta=\alpha$.

PROOF. $\beta\leqq\alpha$ is clear. Suppose $\beta<\alpha$ holds, and set

$$d_1=|\Lambda-\beta|\,,\qquad \delta_1=(d_1+\inf_{\zeta\neq\beta}|\Lambda-\zeta|)/2\,,$$

where infimum is taken over all active zeros of $\sigma(z)$ other than β. We can use the argument of the proof of lemma 4 to show that the contribution of all zeros except β to the series (3.11) does not exceed in absolute value,

$$B_6e^{(\beta-\Lambda)t/2}/\delta_1^{r+1}\,.$$

The residue of $e^{t(z-\Lambda)}E^f(z)/(z-\Delta)\sigma(z)$ at β is given by,

$$e^{(\beta-\Lambda)t}P_1(t)\,,$$

where

$$P_1(t)=c_0t^{\mu}+\cdots+c_{\mu}\,,\qquad c_0\neq 0\,.$$

If $t_1>0$ is taken sufficiently large, either $P_1(t)\geqq P_1(-t_1)>0$ for all $t\leqq -t_1$ or $P_1(t)\leqq P(-t_1)<0$ for all $t\leqq -t_1$ holds.

Writing $s_1=|P_1(-t_1)|$, we obtain for $t_0<-t_1$,

$$\text{(3.18)}\qquad \begin{aligned}|L_r(e^{(\beta-\Lambda)t_0}P_1(t_0))|&=L_r(e^{(\beta-\Lambda)t_0}|P_1(t_0)|)\\ &\geqq s_1L_r(e^{(\beta-\Lambda)t_0})=s_1e^{(\beta-\Lambda)t_0}/(\beta-\Lambda)^r=s_1e^{d_1t_0}/d_1^r\,.\end{aligned}$$

Let r be so large that

$$\frac{1}{2}s_1e^{d_1t_0}/d_1^r>B_6e^{(d_1/2)t_0}/\delta_1^{r+1}>D_0e^{d_1t_0}/(\alpha-\Lambda)^{r+1}\,.$$

Then,

$$\begin{aligned}|L_rf_0(t_0)|&\geqq s_1e^{d_1t_0}/d_1^r-B_6e^{(d_1/2)t_0}/\delta_1^{r+1}\\ &\geqq\frac{1}{2}s_1e^{d_1t_0}/d_1^r>D_0e^{d_1t_0}/(\alpha-\Lambda)^{r+1}\,,\end{aligned}$$

contrary to lemma 5. (q.e.d.)

It follows from lemma 6 that every active zero of $\sigma(z)$ is of the form $\alpha+iy$. Note that zero of $\sigma_k(z)$ of the form $\alpha+iy$ is simple ($k=0,1$). In fact if $\sigma_k(\alpha+iy)=0$, then $\sigma_k'(\alpha+iy)=-(A_1a_1^{\alpha}+\cdots+A_na_n^{\alpha})>0$, $k=0,1$. We consider the three cases separately.

(i) case $(a_1,\cdots,a_n)\in A_n(0)$

In this case α is the only possible active zero. Since $\sigma_1(\alpha)\neq 0$ when $p<n$, α is simple zero of $\sigma(z)$, and (3.10) is reduced to

$$\text{(3.19)}\qquad \int_{-\infty}^{t}e^{-\Lambda\tau}f(\tau)d\tau=\lambda_1e^{(\alpha-\Lambda)t}\,,$$

where

$$\lambda_1=E^f(\alpha)/(\alpha-\Lambda)\sigma'(\alpha)\,,$$

or

$$\text{(3.20)}\qquad f(t)=-\lambda e^{\alpha t}\,,\qquad \text{for all } t<0$$

where

$$\lambda=-(\alpha-\Lambda)\lambda_1\,.$$

Similarly we obtain

(3.21) $$g(t)=-\mu e^{\alpha t}, \qquad \text{for all } t<0 .$$

When $p<n$, substituting (3.20) and (3.21) into (3.1) and (3.2) we obtain $\lambda=\mu$.

If $A=\min(|A_1|, \cdots, |A_n|)$, we can show by induction using (3.1) and (3.2) that (3.20) and (3.21) hold for all $t<mA$, $m=0, 1, 2, \cdots$. This means that they hold for all t.

(ii) case $(a_1, \cdots, a_n) \in B_n^p(\rho)$

In this case, possible active zeros are

$$\zeta_m=\alpha+i2m\pi/\rho , \qquad m=0, \pm 1, \cdots .$$

They are all simple. Writing $\xi_m=E^f(\zeta_m)/(\zeta_m-\Lambda)\sigma'(\zeta_m)$, the residue of $e^{t(z-\Lambda)}E^f(z)/(z-\Lambda)\sigma(z)$ at ζ_m is given by,

$$\xi_m e^{(\zeta_m-\Lambda)t} ,$$

and (3.10) is reduced to

(3.22) $$\int_{-\infty}^{t} f(\tau)e^{-\Lambda\tau}d\tau=\lim_{m\to\infty}\sum_{j=-m}^{m}\xi_j e^{i(2j\pi/\rho)t}e^{(\alpha-\Lambda)t} =\lambda_1(t)e^{(\alpha-\Lambda)t}$$

where

$$\lambda_1(t)=\lim_{m\to\infty}\sum_{j=-m}^{m}\xi_j e^{i(2j\pi/\rho)t} \in \boldsymbol{P}^+(\rho) .$$

Since $f(t)e^{-\Lambda t}$ is left continuous, both sides of (3.22) have derivatives on the left. Differentiating, we obtain

(3.23) $$f(t)=-\lambda(t)e^{\alpha t} , \qquad t<0$$

where

$$\lambda(t)=-\lambda_1^{-\prime}(t)-(\alpha-\Lambda)\lambda_1(t) \in \boldsymbol{P}^+(\rho) .$$

The desired result follows from the same argument as in the case (i).

(iii) case $(a_1, \cdots, a_n) \in C_n^p(\rho)$, $p<n$

$\sigma_0(\alpha+iy)=0$ implies $y=2m\pi/\rho$, and, $\sigma_1(\alpha+iy)=0$ implies $y=(2m+1)\pi/\rho$, $m=0, \pm 1, \cdots$. Possible active zeros of $\sigma(z)$ are,

$$\zeta_m=\alpha+i2m\pi/\rho , \qquad m=0, \pm 1, \pm 2, \cdots ,$$

and

$$\zeta'_m=\alpha+i(2m+1)\pi/\rho , \qquad m=0, \pm 1, \cdots ,$$

These are all simple. Writing

$$\eta_m = E^f(\zeta_m)/(\zeta_m - A)^2\sigma'(\zeta_m)$$

and

$$\eta'_m = E^f(\zeta'_m)/(\zeta'_m - A)^2\sigma'(\zeta'_m) ,$$

we obtain

(3.24)
$$L_1(f_0(t)) = \int_{-\infty}^{t} dt \int_{-\infty}^{t} f(\tau)e^{-A\tau}d\tau = (\lambda_1(t) + \mu_1(t))e^{(\alpha - A)t} , \qquad t<0$$

where

$$\lambda_1(t) = \sum_{-\infty}^{\infty} \eta_m e^{i(2m\pi/\rho)t} \in \boldsymbol{P}^+(\rho) ,$$

$$\mu_1(t) = \sum_{-\infty}^{\infty} \eta'_m e^{i\{(2m+1)\pi/\rho\}t} \in \boldsymbol{P}^-(\sigma) .$$

Completely the same arguments are applied to the function $g(t)$ to obtain

(3.25)
$$\int_{-\infty}^{t} dt \int_{-\infty}^{t} g(t)e^{-A\tau}d\tau = (\lambda_2(t) + \mu_2(t))e^{(\alpha - A)t} , \qquad t=0$$

where

$$\lambda_2(t) \in \boldsymbol{P}^+(\rho) , \qquad \mu_2(t) \in \boldsymbol{P}^-(\rho) .$$

Now set,

(3.26)
$$f_1(t) = e^{At} \int_{-\infty}^{t} dt \int_{-\infty}^{t} f(\tau)e^{-A\tau}d\tau , \qquad t<0 ,$$

and

(3.27)
$$g_1(t) = e^{At} \int_{-\infty}^{t} dt \int_{-\infty}^{t} g(\tau)e^{-A\tau}d\tau , \qquad t<0 .$$

Since $f(t)e^{-At}$ and $g(t)^{-At}$ are left continuous, the functions $f_1(t)$ and $g_1(t)$ have second derivatives on the left. We have, for any negative A,

$$f_1(t+A) = e^{At} \int_{-\infty}^{t} dt \int_{-\infty}^{t} f(\tau + A)e^{-A\tau}d\tau ,$$

and

$$g_1(t+A) = e^{At} \int_{-\infty}^{t} dt \int_{-\infty}^{t} g(t+A)e^{-A\tau}d\tau .$$

Using (3.1) and (3.2) we obtain the relation,

(3.28) $f_1(t)=f_1(t+A_1)+\cdots+f_1(t+A_p)+g_1(t+A_{p+1})+\cdots+g_1(t+A_n)$,

(3.29) $g_1(t)=g_1(t+A_1)+\cdots+g_1(t+A_p)+f_1(t+A_{p+1})+\cdots+f_1(t+A_n)$.

On the other hand, from (3.24) and (3.25),

$$f_1(t)=(\lambda_1(t)+\mu_1(t))e^{\alpha t}\,, \tag{3.31}$$

$$g_1(t)=(\lambda_2(t)+\mu_2(t))e^{\alpha t}\,. \tag{3.32}$$

Substituting these expressions in (3.28) and (3.29) and considering the fact that $\lambda_1(t), \lambda_2(t) \in \boldsymbol{P}^+(\rho)$ and $\mu_1(t), \mu_2(t) \in \boldsymbol{P}^-(\rho)$, we obtain

$$\lambda_1(t)=\lambda_2(t) \quad \text{and} \quad \mu_1(t)=-\mu_2(t) \qquad \text{for all } t\,.$$

Thus (3.25) and (3.32) become respectively

$$\int_{-\infty}^{t} dt \int_{-\infty}^{t} g(\tau)e^{-\Lambda\tau}d\tau=(\lambda_1(t)-\mu_1(t))e^{(\alpha-\Lambda)t} \tag{3.33}$$

and

$$g_1(t)=(\lambda_1(t)-\mu_1(t))e^{\alpha t}\,. \tag{3.34}$$

From (3.31) and (3.34) we obtain

$$\lambda_1(t)=(f_1(t)+g_1(t))e^{-\alpha t}/2$$

and

$$\mu_1(t)=(f_1(t)-g_1(t))e^{-\alpha t}/2\,.$$

These equations show that $\lambda_1(t)$ and $\mu_1(t)$ have second derivatives on the left $\lambda_1^{-\prime\prime}(t)$ and $\mu_1^{-\prime\prime}(t)$. Differentiating (3.24) and (3.33), we obtain

$$f(t)=-(\lambda(t)+\mu(t))e^{\alpha t}\,, \qquad t<0 \tag{3.35}$$

and

$$g(t)=-(\lambda(t)-\mu(t))e^{\alpha t}\,, \qquad t<0 \tag{3.36}$$

where

$$\lambda(t)=-\{\lambda_1^{-\prime\prime}(t)+2(\alpha-\Lambda)\lambda_1'(t)+(\alpha-\Lambda)^2\lambda_1(t)\} \in \boldsymbol{P}^+(\rho)$$

and

$$\mu(t)=-\{\mu_1^{-\prime\prime}(t)+2(\alpha-\Lambda)\mu_1'(t)+(\alpha-\Lambda)^2\mu_1(t)\} \in \boldsymbol{P}^-(\rho)\,.$$

As in the case (i), (3.35) and (3.26) hold in fact for all t. (q.e.d.)

4. Proof of the theorems

Let $\varphi(t) \in T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$. Then for any positive integer m, $\varphi(t)$ can be expressed as a product of exactly n^m ch.f.'s of the form $\varphi(a_1^{k_1}\cdots a_p^{k_p}(-a_{p+1})^{k_{p+1}}\cdots(-a_n)^{k_n}t)$;

$$\varphi(t) = \prod_{\substack{k_i \geqq 0 \\ k_1+\cdots+k_n=m}} \varphi^{m!/k_1!\cdots k_n!}(a_1^{k_1}\cdots a_p^{k_p}(-a_{p+1})^{k_{p+1}}\cdots(-a_n)^{k_n}t)\,.$$

If $a=\max\{a_1, \cdots, a_n\}$, then for any t, $|a_1^{k_1}\cdots a_p^{k_p}(-a_{p+1})^{k_{p+1}}\cdots(-a_n)^{k_n}t| \leqq |a^m t| \to 0$ as $m \to 0$. $\varphi(t)$ is infinitely divisible ch.f. and is uniquely put in the form

$$\log\varphi(t) = i\gamma t - \frac{1}{2}\sigma^2 t^2 + \int_0^\infty h(t, x)dM(x) + \int_{-\infty}^0 h(t, x)dN(x)\,, \tag{4.1}$$

where

$$h(t, x) = e^{itx} - 1 - \frac{itx}{1+x^2}$$

and

$$M(x),\ -N(-x) \in \boldsymbol{M}\,.$$

Now for any $c>0$, we have,

$$\log\varphi(t) = i(c\gamma+\gamma(c))t - \frac{1}{2}c^2\sigma^2t^2 + \int_0^\infty h(t, x)dM\left(\frac{x}{c}\right) + \int_{-\infty}^0 h(t, x)dN\left(\frac{x}{c}\right)$$

and

$$\log\varphi(t) = -i(c\gamma+\gamma(c))t - \frac{1}{2}c^2\sigma^2t^2 + \int_0^\infty h(t, x)d\left(-N\left(-\frac{x}{c}\right)\right) + \int_{-\infty}^0 h(t, x)d\left(-M\left(-\frac{x}{c}\right)\right)$$

where

$$\gamma(c) = c\left\{\int_0^\infty \left(\frac{x}{1+(cx)^2} - \frac{x}{1+x^2}\right)dM(x) + \int_{-\infty}^0 \left(\frac{x}{1+(cx)^2} - \frac{x}{1+x^2}\right)dN(x)\right\}\,.$$

Substituting these expressions in (1.2) with $c=a_1, a_2, \cdots, a_n$ and considering the uniqueness of the representation (4.1), we obtain

(4.2) $$\sigma_1(1)\gamma+\sum_{1}^{p}\gamma(a_j)-\sum_{p+1}^{n}\gamma(a_j)=0\,,$$

(4.3) $$\sigma_0(2)\sigma^2=0\,,$$

(4.4) $$M(x)=M(x/a_1)+\cdots+M(x/a_p)-N(-x/a_{p+1})-\cdots-N(-x/a_n)$$

and

(4.5) $$N(x)=N(x/a_1)+\cdots+N(x/a_p)-M(-x/a_{p+1})-\cdots-M(-x/a_n)\,.$$

LEMMA 7. *Suppose $P(x)$ is a real valued monotone non-decreasing function defined on* $(0, \infty)$.

If $P(x)$ satisfies

(4.6) $$P(x)=P(x/a_1)+\cdots+P(x/a_n)$$

and if $P(x)$ is not identically zero, then

(i) $P(x)<0$ *for all* $x>0$, *and* $\lim_{x\to\infty} P(x)=0$,

(ii) $\int_0^1 x^\gamma dP(x)<\infty$ *if and only if* $\gamma>\alpha$,

and

(iii) $\int_1^\infty x^\beta dP(x)<\infty$ *if* $\beta<\alpha$.

PROOF. We can show by induction on m that we have for any positive m,

(4.7) $$P(x)=\sum_{\substack{k_1+\cdots+k_n=m\\ k_i\geq 0}}\frac{m!}{k_1!\cdots k_n!}P(x/a_1^{k_1}\cdots a_n^{k_n})\,.$$

(i) follows directly from (4.7) and the assumption that $P(x)$ is non-decreasing.

To prove (ii) it suffices to show that there exist positive constants C and D such that for all $x\geqq 1$,

(4.8) $$-C/x^\alpha\leqq P(x)\leqq -D/x^\alpha$$

holds. Let,

$$\xi(x)=-P\left(\frac{1}{x}\right)\Big/x^\alpha\,, \qquad (>0)\,.$$

Then (4.6) is equivalent to,

$$\xi(x)=a_1^\alpha\xi(a_1x)+\cdots+a_n^\alpha\xi(a_nx)\,.$$

Since $a_1^\alpha+\cdots+a_n^\alpha=1$, we can find for any given $x_0>1$, and for any positive integer k, a sequence $a_{i_1},\cdots,a_{i_k}$ such that

$$\xi(x_0)\leqq\xi(a_{i_1}\cdot\ \cdots\ \cdot a_{i_k}x_0)\ .$$

If $b=\min(a_1,\cdots,a_n)$, then $a_{i_1}\cdot\ \cdots\ \cdot a_{i_{k-1}}x_0\geqq 1/b$ implies $a_{i_1}\cdot\ \cdots\ \cdot a_{i_k}x_0\geqq a_{i_1}\cdot\ \cdots\ \cdot a_{i_{p-1}}bx_0\geqq 1$. Hence by suitable choice of k, we can make $a_{i_1}\cdot\ \cdots\ \cdot a_{i_k}x_0$ lie in the interval $[1, 1/b]$. This means that

$$\sup_{x\geqq 1}\xi(x)=\sup_{1/b\geqq x\geqq 1}\xi(x)\ .$$

Putting the right-hand side C, we see that this implies

$$-C/x^\alpha\leqq P(x)\ ,\qquad x\leqq 1\ .$$

Suppose $C=\infty$ (or $=0$), then there exists a sequence $\{x_m\}_{m=1,2,\ldots}$ in the interval $[1, 1/b]$, such that

$$\lim_{m\to\infty}\xi(x_m)=-\lim_{m\to\infty}P\left(\frac{1}{x_m}\right)\Big/x_m^\alpha=\infty\ \ (\text{or }=0)\ .$$

But this implies $\lim_{m\to\infty}P\left(\frac{1}{x_m}\right)=-\infty$ (or $=0$) which is impossible. Another inequality is proved in the same way.

To prove (iii) let the sequence $\{C_m\}$ of positive numbers be defined as follows: $C_0=1$, $C_m=C_{m-1}/a_j$ if $P(C_{m-1}/a_k)<a_k^\alpha P(C_{m-1})$, $k=1,\cdots,j-1$, and if $P(C_{m-1}/a_j)\geqq a_j^\alpha P(C_{m-1})$. Then the sequence $\{C_m\}$ is strictly monotone and $C_m\to\infty$ as $m\to\infty$. Moreover, if $C_{k-1}=C_{k-2}/a_j$,

$$\begin{aligned}&(P(C_k)-P(C_{k-1}))C_k^\beta\leqq -P(C_{k-1})C_k^\beta\leqq -b^{-\beta}P(C_{k-1})C_{k-1}^\beta\\&=-b^{-\beta}P(C_{k-2}/a_j)C_{k-2}^\beta a_j^{-\beta}\leqq -b^{-\beta}P(C_{k-2})C_{k-2}^\beta a_j^{\alpha-\beta}\\&\leqq -b^{-\beta}P(C_{k-2})C_{k-2}^\beta a^{\alpha-\beta}\leqq -b^{-\beta}P(1)(a^{\alpha-\beta})^{k-1}\ .\end{aligned}$$

Hence if $\beta<\alpha$, we have

$$\int_1^{C_m}x^\beta dP(x)\leqq -b^{-\beta}P(1)\sum_{k=1}^m(a^{\alpha-\beta})^{k-1}<-P(1)/b^\beta(1-a^{\alpha-\beta})<\infty\ .$$

(q.e.d.)

Now set $P(x)=M(x)-N(-x)$. This function belongs to $\boldsymbol{M}$ and satisfies the equation (4.6).

If $\alpha\geqq 2$ we have by lemma 7 that $P(x)\equiv 0$ or $M(x)\equiv 0$ and $N(x)\equiv 0$, and in this case (4.1) is reduced to $\varphi(t)=\exp\left\{i\gamma t-\frac{1}{2}\sigma^2t^2\right\}$.

In what follows we assume that $0<\alpha<2$. Then $\sigma_0(2)\neq 0$, and (4.3) implies $\sigma^2=0$.

Functions $f(t)=M(e^{-t})$ and $g(t)=-N(-e^{-t})$ satisfy the assumptions of Theorem 5, and (4.4) and (4.5) are equivalent to (3.1) and (3.3) respectively.

PROOF OF THEOREM 1. It follows from Theorem 5 that $f(t)=-\lambda e^{\alpha t}$, $g(t)=-\mu e^{\alpha t}$ or equivalently,

$$M(x)=-\lambda/x^{\alpha}\,, \qquad x>0\,,$$

$$N(x)=\mu/|x|^{\alpha}\,, \qquad x<0\,,$$

where

$$\lambda=\mu \qquad \text{if} \quad p<n\,.$$

If $\alpha\neq 1$, (4.1) becomes*

$$\varphi(t)=\exp\left\{i\gamma t-c|t|^{\alpha}\left(1+\beta\frac{t}{|t|}\tan\frac{\pi}{2}\alpha\right)\right\}\,.$$

Substituting this in (1.2) we obtain

$$\gamma=0 \qquad \text{if} \quad p=n$$

and

$$\beta=0 \qquad \text{if} \quad p<n\,.$$

If $\alpha=1$, we have $\lambda=\mu$ irrespective of p. In fact, if $p=n$, then $\sigma_0(1)=0$ and (4.2) becomes $\sum_1^n \gamma(a_j)=0$, which implies $\lambda=\mu$. Hence $\varphi(t)$ is the ch.f. of the Cauchy distribution. (q.e.d.)

PROOF OF THEOREM 2.

Necessity. Using Theorem 5, we obtain $f(t)=g(t)=-\lambda(t)e^{\alpha t}$ or equivalently,

(4.9) $$M(x)=-\lambda(-\log x)/x^{\alpha}\,, \qquad x>0$$

and

(4.10) $$N(x)=\lambda(-\log|x|)/|x|^{\alpha} \qquad x<0\,,$$

where $\lambda(t)\in \boldsymbol{P}^{+}(\rho)$, and (4.1) is reduced to (2.1). Considering (1.2), we obtain (2.2).

Sufficiency. If $M(x)$ satisfies the conditions of the Theorem, then we have

$$M(x/a_k)=a_k^{\alpha}M(x)\,, \qquad k=1,2,\cdots,n\,.$$

Hence lemma 7 is applied to $M(x)$ to show that $M(x)\in \boldsymbol{M}$. The complex valued function $\varphi(t)$ defined by (2.1) is then an infinitely divisible ch.f. If, further, (2.2) is satisfied, $\varphi(t)$ satisfies the equation (1.2), proving the sufficiency of the Theorem. (q.e.d.)

* See [2].

PROOF OF THEOREMS 3 AND 4.

Necessity. As we have already shown, ch.f. $\varphi(t) \in T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$ with $\alpha<2$, can be put in the form

$$\log \varphi(t)=i\gamma t+\int_0^\infty h(t, x)\, dM(x) +\int_{-\infty}^0 h(t, x)\, dN(x) \ , \tag{4.11}$$

where $M(x)$ and $N(x)$ are monotone non-decreasing functions defined respectively on $(0, \infty)$ and $(-\infty, 0)$, and satisfy the relation (4.4) and (4.5). It follows from Theorem 5 that $M(x)$ and $N(x)$ can be written in the forms (2.8) and (2.9) under the assumptions of Theorem 3, and in the forms (2.10) and (2.11) under the assumptions of Theorem 4. We consider three cases separately.

(i) case $1<\alpha<2$

Lemma 7 implies in this case that

$$0\leqq\int_0^\infty \left(x-\frac{x}{1+x^2}\right) dP(x)<\infty \ ,$$

or

$$0\leqq\int_0^\infty \left(x-\frac{x}{1+x^2}\right) dM(x)\equiv\gamma_1<\infty \ , \tag{4.12}$$

and

$$0\geqq\int_{-\infty}^0 \left(x-\frac{x}{1+x^2}\right) dN(x)\equiv\gamma_2>-\infty \tag{4.13}$$

and (4.11) becomes (2.3) with $\beta=\gamma+\gamma_1+\gamma_2$.

Substituting (2.3) in (1.2), we obtain (2.4) if $p<n$, and $\beta=0$ if $p=n$.

(ii) case $\alpha=1$

Necessity has already been shown.

(iii) case $0<\alpha<1$

Lemma 7 implies in this case that

$$0\leqq\int_0^\infty \frac{x}{1+x^2}\, dP(x)<\infty \ ,$$

or

$$0\leqq\int_0^\infty \frac{x}{1+x^2}\, dM(x)\equiv\gamma_3<\infty \tag{4.14}$$

and

(4.15) $$0 \geqq \int_{-\infty}^{0} \frac{x}{1+x^2}\, dN(x) \equiv \gamma_4 > -\infty \ ,$$

and (4.11) becomes

$$\log \varphi(t) = i\beta t + \int_0^{\infty} (e^{itx}-1)\, dM(x) + \int_{\infty}^{0} (e^{itx}-1)\, dN(x) \ ,$$

where $\beta = \gamma + \gamma_3 + \gamma_4$.

Substituting in (1.2), we obtain $\beta = 0$.

Sufficiency. Suppose that $\varphi(t)$, $M(x)$ and $N(x)$ satisfy the conditions of either Theorem 3 or Theorem 4. Then $M(x)$ and $N(x)$ are related to each other by equations (4.4) and (4.5), from which we conclude that $\varphi(t)$ formally satisfies the equation (1.2). We have only to prove that $\varphi(t)$ is a ch.f. If follows from lemma 7 that $M(x)$ and $-N(-x)$ belong to $\boldsymbol{M}$, and that (4.12), (4.13) (if $1<\alpha<2$) and (4.14), (4.15) (if $0<\alpha<1$) hold. Hence in any case $\varphi(t)$ can be rewritten as (4.1) with $\sigma^2=0$, showing that $\varphi(t)$ is an infinitely divisible ch.f. (q.e.d.)

5. Examples

Example 1. Let X and Y be independent and identically distributed random variables, and let a and b be real constants whose absolute values are 2/3 and 1/3 respectively. If the distribution of $aX+bY$ is identical with that of X, then the distribution is the Cauchy distribution.

Example 2. When $p<n$, $\varphi(t) \in T_\alpha(a_1, \cdots, a_p, -a_{p+1}, \cdots, -a_n)$ is symmetric except for a factor e^{irt} in many cases but not always. We shall give a simple example. Equation $x^3-(1-x)^2=0$ has unique real zero a_0 between 0 and 1 exclusive. Let $0<\alpha<2$ and set $a_1=a_0^{1/\alpha}$, and $a_2=(1-a_0)^{1/\alpha}$.

We can easily show that $(a_1, a_2) \in C_2^1(-\log a_0/2\alpha)$.

Let $n \geqq 1$ be any integer and let $\lambda_1, \cdots, \lambda_n$; $\mu_1, \cdots, \mu_n$ be any real numbers, while λ be a positive number. Set,

$$M(x) = \left\{-\lambda + \sum_1^n \lambda_k \cos\left(\frac{4\alpha k\pi}{\log a_0} \log x\right) + \sum_1^n \mu_k \cos\left(\frac{2\alpha(2k-1)\pi}{\log a_0} \log x\right)\right\} \Big/ x^\alpha \ , \quad x>0$$

and

$$N(x) = \left\{\lambda - \sum_1^n \lambda_k \cos\left(\frac{4\alpha k\pi}{\log a_0} \log |x|\right) + \sum_1^n \mu_k \cos\left(\frac{2\alpha(2k-1)\pi}{\log a_0} \log |x|\right)\right\} \Big/ |x|^\alpha \ .$$

Then for sufficiently large λ, $M(x)$ and $N(x)$ satisfy the conditions of Theorem 3, and

$$\varphi_0(t)=\int_0^{\infty}(e^{itx}-1)\,dM(x)+\int_{-\infty}^{0}(e^{itx}-1)\,dN(x)$$

is a ch.f. belonging to $T_\alpha(a_1, -a_2)$ if $\alpha<1$. It is symmetric if and only if $\mu_1=\cdots=\mu_n=0$.

When $\lambda_1=\cdots=\lambda_n=0$, we have

$$|\varphi_0(t)|^2=e^{-2c|t|^\alpha}\,,$$

where

$$c=2\lambda\alpha\int_0^{\infty}(\cos x-1)x^{-\alpha-1}\,dx>0$$

or

$$|\varphi_0(t)|=e^{-c|t|^\alpha}\,.$$

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