ON SOME NONPARAMETRIC GENERALIZATIONS OF WILKS' TESTS FOR H_M, H_{VC} AND H_{MVC}, I*

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Summary

This paper is concerned with the nonparametric generalizations of the well-known likelihood ratio tests, proposed and studied by S. S. Wilks [12] (also see Votaw [11]), for testing the hypothesis of compound symmetry, i.e., equality of means (H_M), equality of variances (H_V), and equality of covariances (H_C) of a multinormal distribution. In this part of the paper, some nonparametric rank order tests are offered for testing the hypothesis H_M of equality of location parameters of a multivariate distribution of unspecified form. In the second part, the general problem of nonparametric tests for the hypotheses H_{VC} and H_{MVC} will be considered.

1. Introduction

Let $\boldsymbol{X}_\alpha=(X_{1\alpha}, \cdots, X_{p\alpha})$, $\alpha=1, \cdots, n$ be n independent and identically distributed (vector valued) random variables (i.i.d.r.v.), having a p ($\geqq 2$) variate continuous cumulative distribution function (cdf) $F(\boldsymbol{x})$, where $\boldsymbol{x}=(x_1, \cdots, x_p)$. When $F(\boldsymbol{x})$ is a multinormal cdf, it is completely specified by its mean vector $\boldsymbol{\xi}=(\xi_1, \cdots, \xi_p)$ and the dispersion matrix $\boldsymbol{\Sigma}=(\sigma_{ij})_{i,j=1,\cdots,p}$. It is also well-known that σ_{ii} ($i=1, \cdots, p$) are measures of dispersion of the p variates, and $\rho_{ij}=\sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$ ($i\neq j=1, \cdots, p$) are measures of their association. The hypothesis of compound symmetry (H_{MVC}) as sketched by Wilks [12], relates to

$$H_{MVC}=H_M \cap H_{VC} , \tag{1.1}$$

where

$$H_M: \quad \boldsymbol{\xi}=\xi(1, \cdots, 1) \text{ assuming } \boldsymbol{\Sigma}=\sigma^2(\delta_{ij}+(1-\delta_{ij})\rho) , \tag{1.2}$$

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(δ_{ij} being the Kronecker delta, $-1/(p-1)\leqq\rho<1$), and

$$H_{VC}: \quad \boldsymbol{\Sigma}=\sigma^2(\delta_{ij}+(1-\delta_{ij})\rho) . \tag{1.3}$$

Thus, H_M is really the hypothesis of equality of means assuming H_{VC} to be true. Wilks [12] has proposed and studied the likelihood ratio tests L_M, L_{VC} and L_{MVC} for testing the hypotheses H_M, H_{VC} and H_{MVC} respectively. Votaw [11] has extended these tests in the presence of an external criterion variable. The object of the present investigation is to propose and study some nonparametric generalizations of these tests. By analogy with the parametric case, let us define, in any convenient way, the location and scale parameters of x_i in $F(\boldsymbol{x})$ by μ_i and δ_i ($i=1, \cdots, p$), respectively. We then rewrite $F(\boldsymbol{x})$ as

$$F(\boldsymbol{x})=F_0([x_1-\mu_1]/\delta_1, \cdots, [x_p-\mu_p]/\delta_p) . \tag{1.4}$$

We also denote by $\mathscr{F}_0$ the class of all p-variate continuous cdf's $\{F(\boldsymbol{u})\}$, where $F(\boldsymbol{u})$ is a symmetric function of its arguments $\boldsymbol{u}=(u_1, \cdots, u_p)$. Now, in the nonparametric generalizations of H_{MVC}, H_M and H_{VC}, we proceed as follows:

$$H_M: \quad \mu_1= \cdots \mu_p , \quad \text{assuming} \quad \delta_1= \cdots =\delta_p \quad \text{and} \quad F_0\in\mathscr{F}_0 , \tag{1.5}$$

$$H_{VC}: \quad \delta_1= \cdots =\delta_p \quad \text{and} \quad F_0\in\mathscr{F}_0 , \tag{1.6}$$

$$H_{MVC}=H_M\cap H_{VC} \quad \text{i.e.,} \quad F\in\mathscr{F}_0 . \tag{1.7}$$

In this paper, we shall specifically consider nonparametric tests for the hypothesis H_M in (1.5), while in the second part, tests for the hypotheses in (1.6) and (1.7) will be considered.

2. Nonparametric generalizations of L_M test for H_M

Let us pool the n vector valued observations $\boldsymbol{X}_\alpha$, $\alpha=1, \cdots, n$ into a combined set of N $(=np)$ variables. We denote these N variables by

$$\boldsymbol{Z}_N=(Z_1, \cdots, Z_N) , \tag{2.1}$$

where we adopt the convention that

$$Z_{(\alpha-1)p+j}=X_{j\alpha} \qquad \alpha=1, \cdots, n, \quad j=1, \cdots, p . \tag{2.2}$$

We then arrange the N observations in (2.1) in order of magnitude, and denote them by

$$Z_{N,1}< \cdots <Z_{N,N} , \tag{2.3}$$

by virtue of the assumed continuity of $F(\boldsymbol{x})$, the possibility of ties in

(2.3) may be ignored in probability. Now, for any positive integer n, we define a sequence of rank functions (which depends on N $(=np)$ in an explicit manner) by

$$\boldsymbol{E}_N=(E_{N,1}, \cdots, E_{N,N}) , \tag{2.4}$$

where we adopt the Chernoff-Savage [2] convention, and define

$$E_{N,\alpha}=J_N(\alpha/(N+1)) \qquad 1\leqq\alpha\leqq N . \tag{2.5}$$

The function J_N need be defined only at $\alpha/(N+1)$ for $\alpha=1, \cdots, N$, but may have its domain of definition extended to (0, 1) by the convention in [2], [8]. For the ith variate, we define an indicator function

$$C^{(i)}_{N,\alpha}=\begin{cases} 1 & \text{if } Z_{N,\alpha} \text{ is an } X_{i\beta} \quad (\beta=1, \cdots, n), \\ 0 & \text{otherwise,} \end{cases} \tag{2.6}$$

for $\alpha=1, \cdots, N$ and $i=1, \cdots, p$. Thus, we have

$$\sum_{\alpha=1}^{N} C^{(i)}_{N,\alpha}=n , \qquad \sum_{\alpha=1}^{N} C^{(i)}_{N,\alpha}C^{(j)}_{N,\alpha}=n\delta_{ij} \qquad i, j=1, \cdots, p , \tag{2.7}$$

where δ_{ij} is the usual Kronecker delta, and finally

$$\sum_{i=1}^{p}\sum_{\alpha=1}^{N} C^{(i)}_{N,\alpha}=N . \tag{2.8}$$

Now, we consider a p-vector

$$\boldsymbol{T}_N=(T_{N,1}, \cdots, T_{N,p}) , \tag{2.9}$$

$$T_{N,i}=\frac{1}{n}\sum_{\alpha=1}^{N} C^{(i)}_{N,\alpha}E_{N,\alpha} \qquad i=1, \cdots, p . \tag{2.10}$$

It may be noted that by virtue of (2.8), we have

$$\frac{1}{p}\sum_{i=1}^{p} T_{N,i}=\frac{1}{N}\sum_{\alpha=1}^{N} E_{N,\alpha}=\bar{E}_N \tag{2.11}$$

where $\bar{E}_N$ is a non-stochastic constant depending only on $\boldsymbol{E}_N$. Thus, $\boldsymbol{T}_N$ can contain at most $(p-1)$ linearly independent quantities. Our proposed test is based on the stochastic vector $\boldsymbol{T}_N$. It may be noted that the null hypothesis H_M in (1.5) implies that $F(\boldsymbol{x})$ is a symmetric function of its arguments. Thus, the problem reduces to testing the interchangeability of the p variates $x_1, \cdots, x_p$ in $F(\boldsymbol{x})$. In H_M, we shall be particularly interested in the set of alternatives that $\mu_1, \cdots, \mu_p$ in (1.4) are not all equal. To develop a strictly distribution-free test, we shall extend the idea of bivariate interchangeability, derived by the author in an earlier paper [9], to the p ($\geqq 2$) variate case, and consider an analogous permutation procedure.

3. Permutationally distribution-free test for H_M

With reference to the order statistic (2.3), let us denote the rank of $X_{i\alpha}$ by $R_{i\alpha}$ for $i=1, \cdots, p$, $\alpha=1, \cdots, n$. Then, the rank p-tuplet corresponding to the vector $\boldsymbol{X}_\alpha$ is denoted by

$$\boldsymbol{R}_\alpha=(R_{1\alpha}, \cdots, R_{p\alpha}) , \qquad \alpha=1, \cdots, n . \tag{3.1}$$

We now consider the *collection (rank) matrix*, which we define as

$$\boldsymbol{R}_N^{p\times n}=(\boldsymbol{R}_1', \cdots, \boldsymbol{R}_n')=\begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & & \vdots \\ R_{p1} & R_{p2} & \cdots & R_{pn} \end{pmatrix} . \tag{3.2}$$

The N elements of $\boldsymbol{R}_N$ are the N natural integers $(1, \cdots, N)$, permuted in some way. The matrix $\boldsymbol{R}_N$ consists of n random rank p-tuplets which constitute the n columns of it; naturally, $\boldsymbol{R}_N$ is a stochastic matrix. Two such collection matrices, say, $\boldsymbol{R}_N$ and $\boldsymbol{R}_N^*$, are said to be equivalent when it is possible to arrive at $\boldsymbol{R}_N$ from $\boldsymbol{R}_N^*$ by a number of inversions of the columns of the latter. This implies that if instead of taking the observations $\boldsymbol{X}_\alpha$ in natural order ($\alpha=1, \cdots, n$), we take in any other order, say, $\boldsymbol{X}_{i_1}, \cdots, \boldsymbol{X}_{i_n}$, where $(i_1, \cdots, i_n)$ is a permutation of $(1, \cdots, n)$, the two collection matrices will be equivalent. Thus, the total number of non-equivalent realizations that $\boldsymbol{R}_N$ may have is equal to $(np)!/n!$. The set of all these realizations of $\boldsymbol{R}_N$ is denoted by $\mathfrak{R}_N$, so that $\boldsymbol{R}_N \in \mathfrak{R}_N$. Now, there are p elements in each column of $\boldsymbol{R}_N$. These p elements can be permuted among them in $p!$ ways. Thus, any given $\boldsymbol{R}_N$ may be used to derive a set of $(p!)^n$ realizations of such collection matrices, simply by permuting the elements within each column of it. This set of $(p!)^n$ realizations corresponding to the given $\boldsymbol{R}_N$ is denoted by $S(\boldsymbol{R}_N)$, and is termed the *permutation set* of R_N. Thus, $S(\boldsymbol{R}_N)$ is a subset of $\mathfrak{R}_N$, and the total number of non-equivalent subsets $S(\boldsymbol{R}_N)$ in $\mathfrak{R}_N$ is evidently $(np)!/\{n!(p!)^n\}$. Consequently, for any $\boldsymbol{Z}_N$ in (2.1),

$$\boldsymbol{R}_N \in S(\boldsymbol{R}_N) \subset \mathfrak{R}_N . \tag{3.3}$$

The probability distribution of $\boldsymbol{R}_N$ over $\mathfrak{R}_N$ (defined on an additive class of subsets A_N of $\mathfrak{R}_N$), will evidently depend on the cdf F, even when H_M holds. However, if H_M in (1.5) holds, then given $\boldsymbol{X}_\alpha$, all possible permutations of $(X_{1\alpha}, \cdots, X_{p\alpha})$ in the p places of the vector, will be conditionally equally likely, each having the permutational probability

$1/p!$. Thus, conditionally on $\boldsymbol{R}_\alpha$ in (3.1), under H_M in (1.5), the $p!$ possible permutations of the p rank elements $(R_{1\alpha}, \cdots, R_{p\alpha})$ among themselves, will be equally likely, each having the same conditional probability $1/p!$. Since $\{\boldsymbol{X}_\alpha, \alpha=1, \cdots, n\}$ are mutually stochastically independent, this implies that given $\boldsymbol{R}_N$ in (3.2), we may have $(p!)^n$ possible realizations derived from it, and under H_M, these $(p!)^n$ realizations are equally (conditionally) likely. Now, this set of $(p!)^n$ realizations of $\boldsymbol{R}_N$ is nothing but $S(\boldsymbol{R}_N)$. Hence, we may put the same statement in an alternative way. Corresponding to the permutation set $S(\boldsymbol{R}_N)$ being held fixed, there will be a set of $(p!)^n$ possible realizations $\{\boldsymbol{R}_N\}$, which are conditionally equally likely, viz.,

$$\text{(3.4)} \qquad \mathrm{P}\{\boldsymbol{R}_N \mid S(\boldsymbol{R}_N), H_M\} = (p!)^{-n},$$

for any $S(\boldsymbol{R}_N)$. Thus, if we now correspond the rank function $E_{N,\alpha}$ to the rank α for $\alpha=1, \cdots, N$, it follows that for each $\boldsymbol{R}_N$ there will be a matrix whose elements will be $E_{N,\alpha}$ instead of α, in (3.2). Thus, for each $\boldsymbol{R}_N$ we will have a value of $\boldsymbol{T}_N$ defined in (2.9) and (2.10). Hence, corresponding to the set $S(\boldsymbol{R}_N)$, we will have a set of $(p!)^n$ values of $\boldsymbol{T}_N$, which we denote by $\boldsymbol{T}_N[S(\boldsymbol{R}_N)]$. Consequently, from (3.4) we get that conditionally on the set $\boldsymbol{T}_N[S(\boldsymbol{R}_N)]$, the permutation distribution of $\boldsymbol{T}_N$ (over the $(p!)^n$ equally likely realizations) would be uniform under H_M in (1.5). Let us denote this permutational probability measure by $\mathcal{P}_n$, and consider a test function $\phi(\boldsymbol{Z}_N)$, which with each observed $\boldsymbol{Z}_N$ (in (2.1)) associates a probability of rejecting H_M in (1.5) with the aid of the completely specified probability measure $\mathcal{P}_n$. Thus, we can always select $\phi(\boldsymbol{Z}_N)$, in such a manner that

$$\text{(3.5)} \qquad \mathrm{E}\{\phi(\boldsymbol{Z}_N) \mid \mathcal{P}_n\} = \varepsilon \qquad 0<\varepsilon<1,$$

where ε is the preassigned level of significance of the test. (3.5) implies that $\mathrm{E}\{\phi(\boldsymbol{Z}_N) \mid H_M\} = \varepsilon$, hence, $\phi(\boldsymbol{Z}_N)$ is a distribution-free similar test of size ε.

Now, for the convenience in actual practice, we would prefer to use a single valued test statistic (say W_N), which may be used to specify the test function $\phi(\boldsymbol{Z}_N)$ in a precise way. We shall see in the next section that the permutation distribution of $\boldsymbol{T}_N$ (under the probability measure $\mathcal{P}_n$) has asymptotically a multinormal form. This suggests that an appropriate (though may not be optimum) way of arriving at a suitable test statistic may be to consider the quadratic form associated with this multinormal (permutation) distribution. It is easily shown that

$$\text{(3.6)} \qquad \mathrm{E}(T_{N,i} \mid \mathcal{P}_n) = \bar{E}_N \qquad i=1, \cdots, p,$$

where $\bar{E}_N$ is defined in (2.11). Also, it can be easily shown that

$$\text{(3.7)} \qquad \operatorname{Cov}(T_{N,i}, T_{N,j} \mid \mathcal{P}_N) = \frac{1}{n} \frac{(\delta_{ij}p-1)}{(p-1)} \sigma_N^2(\boldsymbol{R}_N) \qquad i, j=1, \cdots, p ,$$

where δ_{ij} is the usual Kronecker delta, and

$$\text{(3.8)} \qquad \sigma_N^2(\boldsymbol{R}_N) = \frac{1}{N} \sum_{\alpha=1}^{n} \sum_{i=1}^{p} (E_{N,R_{i\alpha}} - E_{N,R_{\cdot\alpha}})^2 ,$$

with $E_{N,R_{\cdot\alpha}}$ being defined as

$$\text{(3.9)} \qquad E_{N,R_{\cdot\alpha}} = \frac{1}{p} \sum_{i=1}^{p} E_{N,R_{i\alpha}} \qquad \alpha=1, \cdots, n .$$

Thus, $\sigma_N^2(\boldsymbol{R}_N)$ depends upon the collection matrix, but remains invariant under $S(\boldsymbol{R}_N)$. Thus, if we work with the inverse of the (permutational) covariance matrix of $T_{N,i}$, $i=1, \cdots, p-1$, and consider the associated quadratic form then by using (2.11), the same is shown to reduce to the following simple form

$$\text{(3.10)} \qquad W_N = n[(p-1)/p] \sum_{i=1}^{p} (T_{N,i} - \bar{E}_N)^2 / \sigma_N^2(\boldsymbol{R}_N) .$$

Now, under H_M, $\boldsymbol{T}_N$ will have the location vector $\bar{E}_N(1, \cdots, 1)$ (permutationally) and hence, it can be shown that if $\sigma_N^2(\boldsymbol{R}_N)$ is finite and non-zero, then under the permutational probability measure $\mathcal{P}_n$, W_N will have $(p!)^n$ possible realizations, which are equally likely. On the other-hand, if H_M does not hold and the p variates have locations, not all equal, then at least one of $T_{N,i}$ will be stochastically different from $\bar{E}_N$ (this will be made clear in a later section), and hence W_N, being a positive semi-definite quadratic form in $\boldsymbol{T}_N$, will be stochastically larger. Thus, it appears reasonable to base our permutation test on the following rejection rule:

$$\text{(3.11)} \qquad \phi(\boldsymbol{Z}_N) = \begin{cases} 1 & \text{if} \quad W_N > W_{N,\varepsilon}(\boldsymbol{R}_N) \\ \gamma_N(\boldsymbol{R}_N) & \text{if} \quad W_N = W_{N,\varepsilon}(\boldsymbol{R}_N) \\ 0 & \text{if} \quad W_N < W_{N,\varepsilon}(\boldsymbol{R}_N) , \end{cases}$$

where $W_{N,\varepsilon}(\boldsymbol{R}_N)$ and $\gamma_N(\boldsymbol{R}_N)$ are so chosen that

$$\text{(3.12)} \qquad \mathrm{E}\{\phi(\boldsymbol{Z}_N) \mid \mathcal{P}_n\} = \varepsilon .$$

Thus, if in actual practice n is not large, we can consider the set $\boldsymbol{T}_N[S(\boldsymbol{R}_N)]$ of $(p!)^n$ values of $\boldsymbol{T}_N$ (and hence, of W_N), which will provide us with the permutational distribution function of W_N, and the same may be used to find out $W_{N,\varepsilon}(\boldsymbol{R}_N)$ and $\gamma_N(\boldsymbol{R}_N)$. This test will naturally be a strictly distribution free similar size ε test. However, if n is not very small, the labor involved in this procedure increases tremendously. To obviate this drawback, we shall now consider the asymptotic permutation test and also show how the same is asymptotically equivalent to some unconditional test for H_M which may be based on the same rank order statistic (vector) $\boldsymbol{T}_N$.

4. Asymptotic permutation distribution of W_N

As in the case of the study of the asymptotic theory of rank order tests for various other problems of statistical inference ([2], [3], [7], [8], [9], [10]), we shall impose certain regularity conditions on $\boldsymbol{E}_N$ in (2.4) as well as on $F(\boldsymbol{x})$. Let us define

$$(4.1) \qquad F_{N[i]}(x)=\frac{1}{n}[\text{Number of } X_{i\alpha}\leqq x] \qquad i=1,\cdots,p\,,$$

$$(4.2) \qquad H_N(x)=\frac{1}{p}\sum_{i=1}^{p}F_{N[i]}(x)\,,$$

$$(4.3) \qquad F_{N[i,j]}(x,y)=\frac{1}{n}[\text{Number of } (X_{i\alpha},X_{j\alpha})\leqq(x,y)] \qquad i\neq j=1,\cdots,p\,,$$

$$(4.4) \qquad H_N^*(x,y)=\binom{p}{2}^{-1}\sum_{1\leqq i<j\leqq p}F_{N[i,j]}(x,y)\,.$$

Again, let $F_{[i]}(x)$ and $F_{[i,j]}(x,y)$ be respectively the marginal cdf of $X_{i\alpha}$ and $(X_{i\alpha},X_{j\alpha})$, for $i\neq j=1,\cdots,p$, and we define

$$(4.5) \qquad H(x)=\frac{1}{p}\sum_{i=1}^{p}F_{[i]}(x)\,,$$

$$(4.6) \qquad H^*(x,y)=\binom{p}{2}^{-1}\sum_{1\leqq i<j\leqq p}F_{[i,j]}(x,y)\,.$$

Then, we define J_N as in (2.5) and assume that

(4.7) (c.1) $\lim_{N=\infty}J_N(H)=J(H)$ exists for all $0<H<1$ and is not a constant,

$$(4.8)\ (\text{c.2}) \qquad \frac{1}{N}\sum_{\alpha=1}^{N}\left[J_N\left(\frac{\alpha}{N+1}\right)-J\left(\frac{\alpha}{N+1}\right)\right]=o(N^{-1/2})\,,$$

$$(4.9) \qquad \int_{-\infty}^{\infty}\left[J_N\left(\frac{N}{N+1}H_N(x)\right)-J\left(\frac{N}{N+1}H_N(x)\right)\right]dF_{N[i]}(x)=o_p(N^{-1/2}) \qquad i=1,\cdots,p\,.$$

(c.3) $J(H)$ is absolutely continuous in H: $0<H<1$, and

(4.10) $$|J^{(r)}(H)| = \left| \frac{d^r}{dH^r} J(H) \right| \leqq K[H(1-H)]^{-r-1/2+\delta} ,$$

for $r=0$, 1, and some $\delta>0$, where K is a constant.

For the permutation distribution theory, we require two more mild regularity conditions for the existence and convergence of $\sigma_N^2(\boldsymbol{R}_N)$ in (3.8). These, we state below.

(4.11) (c.4) $$\frac{1}{N}\sum_{\alpha=1}^{N}\left[J_N^2\left(\frac{\alpha}{N+1}\right)-J^2\left(\frac{\alpha}{N+1}\right)\right]=o(1) ,$$

(4.12) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[J_N\left(\frac{N}{N+1}H_N(x)\right)J_N\left(\frac{N}{N+1}H_N(y)\right)\right.$$
$$\left.-J\left(\frac{N}{N+1}H_N(x)\right)J\left(\frac{N}{N+1}H_N(y)\right)\right] dF_{N[i,j]}(x, y)=o_p(1) ,$$
$$i\neq j=1, \cdots, p .$$

Finally, we define

(4.13) $$\nu_{ij}(F)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(H(x))J(H(y))\, dF_{[i,j]}(x, y) \qquad i, j=1, \cdots, p ,$$

(4.14) $$\boldsymbol{\nu}(F)=(\nu_{ij}(F))_{i,j=1,\cdots,p}$$

(4.15) (c.5) $$\text{Rank of } \boldsymbol{\nu}(F)\geqq 2 .$$

It may be noted that for testing the hypothesis H_M we shall consider the class of rank order tests for which $J(H)$ is monotonic in H: $0<H<1$ (this point will be made clear at a later stage) and hence, it can be shown that if the scatter of $\boldsymbol{x}$ in $F(\boldsymbol{x})$ is not confined to any one-dimensional space on the p-dimensional Euclidean space, then (c.5) holds.

LEMMA 4.1. *Let us define*

(4.16) $$A^2=\int_0^1 J^2(u)\, du ,$$

and

(4.17) $$\bar{\nu}=\binom{p}{2}^{-1}\sum_{1\leq i<j\leq p}\nu_{ij}(F) .$$

Then, if (c.5) *holds,*

(4.18) $$A^2-\bar{\nu}>0 .$$

PROOF. It follows from (4.13) that

(4.19) $$\sum_{i=1}^{p}\nu_{ii}(F)=\sum_{i=1}^{p}\int_{-\infty}^{\infty} J^2(H(x))\, dF_{[i]}(x)=p\int_{-\infty}^{\infty} J^2(H(x))\, dH(x)=pA^2 .$$

Thus, we get from (4.16), (4.17) and (4.19) that

$$A^2-\bar{\nu}=\frac{1}{p}\sum_{i=1}^{p}\nu_{ii}(F)-\frac{1}{p(p-1)}\sum_{i\neq j=1}^{p}\nu_{ij}(F) \tag{4.20}$$
$$=\frac{1}{p-1}\sum_{i=1}^{p}\nu_{ii}(F)-\frac{1}{p(p-1)}\sum_{i=1}^{p}\sum_{j=1}^{p}\nu_{ij}(F)$$
$$=\frac{p}{p-1}\left[\frac{1}{p}\sum_{i=1}^{p}\nu_{ii}(F)-\frac{1}{p^2}\sum_{i=1}^{p}\sum_{j=1}^{p}\nu_{ij}(F)\right].$$

Now,

$$\left|\frac{1}{p^2}\sum_{i=1}^{p}\sum_{j=1}^{p}\nu_{ij}(F)\right|\leqq\left\{\frac{1}{p}\sum_{i=1}^{p}[\nu_{ii}(F)]^{1/2}\right\}^2,$$

the equality sign holds only when $\nu_{ij}(F)=[\nu_{ii}(F)\cdot\nu_{jj}(F)]^{1/2}$, for all $i, j=1, \cdots, p$. Then, the condition (c.5) implies that there is at least one (i, j) $(i\neq j=1, \cdots, p)$ for which

$$|\nu_{ij}(F)|<[\nu_{ii}(F)\nu_{jj}(F)]^{1/2}.$$

Thus, under (c.5), we have

$$\left|\frac{1}{p^2}\sum_{i=1}^{p}\sum_{j=1}^{p}\nu_{ij}(F)\right|<\left(\frac{1}{p}\sum_{i=1}^{p}[\nu_{ii}(F)]^{1/2}\right)^2. \tag{4.21}$$

Again, by elementary inequality relating to moments, we have

$$\left(\frac{1}{p}\sum_{i=1}^{p}[\nu_{ii}(F)]^{1/2}\right)^2\leqq\frac{1}{p}\sum_{i=1}^{p}\nu_{ii}(F), \tag{4.22}$$

where the equality sign holds only when $\nu_{11}(F)=\cdots=\nu_{pp}(F)$. Thus, from (4.20), (4.21) and (4.22), we get that under (c.5), the right-hand side of (4.20) will be strictly positive. Hence, the lemma.

LEMMA 4.2. *Under conditions* (c.1) *through* (c.5), $\sigma_N^2(\boldsymbol{R}_N)$ *defined in* (3.8), *converges in probability to* $[(p-1)/p][A^2-\bar{\nu}]>0$, *where* A^2 *and* $\bar{\nu}$ *are defined in* (4.16) *and* (4.17), *respectively.*

PROOF. We may rewrite $\sigma_N^2(\boldsymbol{R}_N)$ in (3.8) as

$$\frac{p-1}{p}\frac{1}{N}\sum_{\alpha=1}^{N}E_{N,\alpha}^2-\frac{1}{p^2}\sum_{i\neq j=1}^{p}\left\{\frac{1}{n}\sum_{\alpha=1}^{n}E_{N,R_{i\alpha}}E_{N,R_{j\alpha}}\right\}. \tag{4.23}$$

Now, by virtue of condition (c.4), it can be easily shown that

$$\frac{1}{N}\sum_{\alpha=1}^{N}E_{N,\alpha}^2\longrightarrow\int_0^1 J^2(u)\,du=A^2. \tag{4.24}$$

Again, $\frac{1}{n}\sum_{\alpha=1}^{n} E_{N,R_{i\alpha}}\cdot E_{N,R_{j\alpha}}$ may also be written (after using (4.12)) as

$$\text{(4.25)}\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J\left(\frac{N}{N+1}H_N(x)\right)J\left(\frac{N}{N+1}H_N(y)\right)dF_{N[i,j]}(x,y)+o_p(1)\ ,$$

for $i\neq j=1,\cdots,p$. It may be noted now that $F_{N[i]}$ $(i=1,\cdots,p)$ are the sample empirical cdf's based on n i.i.d.r.v. Hence, on using the well-known result on Kolmogorov-Smirnov statistic, we have

$$\text{(4.26)}\quad n^{1/2}[\operatorname*{Sup}_x |F_{N[i]}(x)-F_{[i]}(x)|]\ \text{bounded in probability}\ ,\qquad i=1,\cdots,p.$$

Consequently on using (4.2) and (4.5), we get that

$$\text{(4.27)}\qquad \operatorname*{Sup}_x N^{1/2}\left|\frac{N}{N+1}H_N(x)-H(x)\right| \leqq p^{-1/2}\sum_{i=1}^{p}\left\{\operatorname*{Sup}_x n^{1/2}\left|\frac{pn}{pn+1}F_{N[i]}(x)-F_{[i]}(x)\right|\right\}$$

is also bounded in probability (by Poincare's theorem on total probability.). Again, by elementary algebra, we have

$$\text{(4.28)}\qquad 0\leqq dF_{N[i]}(x)\leqq pdH_N(x)\ ,\qquad 0\leqq d[1-F_{N[i]}(x)]\leqq pd[1-H_N(x)]\ ,\quad i=1,\cdots,p.$$

Hence, proceeding precisely on the same line as the proof of theorem 4.2 of Puri and Sen [8], and omitting the details of the derivation, we will arrive at the stochastic convergence of (4.25) to ν_{ij}, defined in (4.13), for all $i,j=1,\cdots,p$. Consequently, from (4.23) and (4.24) and the convergence of (4.25) to ν_{ij}, we arrive at the following

$$\text{(4.29)}\qquad \sigma_N^2(\boldsymbol{R}_N)\xrightarrow{P}\frac{p-1}{p}A^2-\frac{1}{p^2}\sum_{i\neq j=1}^{p}\nu_{ij} =\frac{p-1}{p}A^2-\frac{p-1}{p}\bar{\nu}=\frac{p-1}{p}[A^2-\bar{\nu}]>0\ ,$$

where $\bar{\nu}$ is defined in (4.17) and where by lemma 4.1, $A^2-\bar{\nu}>0$.

Hence, the theorem.

THEOREM 4.3. *If conditions* (c.1) *through* (c.5) *hold then under the permutational probability measure* $\mathcal{P}_n$, *the statistic* W_N, *in* (3.10), *has asymptotically, in probability, a chi-square distribution with* $(p-1)$ *degrees of freedom.*

(It may be noted that the permutation distribution of W_N is essentially a conditional distribution, depending on $\boldsymbol{Z}_N$ in (2.1). Hence, the implication of the above theorem is that it holds, in probability, i.e., for almost all $\boldsymbol{Z}_N$.)

PROOF. Let us first prove that under $\mathcal{P}_n$, $\{n^{1/2}(T_{N,i}-\bar{E}_N),\ i=1, \cdots, p\}$ has asymptotically, in probability, a multinormal distribution of rank $(p-1)$. As in (2.11), we have shown that $\sum_{i=1}^{p}(T_{N,i}-\bar{E}_N)=0$, it follows that the rank of $\boldsymbol{T}_N$ may be at most equal to $p-1$. So, if we can show that for any non-null real $\boldsymbol{\delta}=(\delta_1, \cdots, \delta_{p-1})$, $\sum_{i=1}^{p-1}\delta_i n^{1/2}(T_{N,i}-\bar{E}_N)$ has a non-degenerate and asymptotically normal (permutation) distribution, our desired result will follow. Now, using (2.11), we can also write

$$n^{1/2}\sum_{i=1}^{p-1}\delta_i(T_{N,i}-\bar{E}_N)=n^{1/2}\sum_{i=1}^{p}\delta_i' T_{N,i}\ , \tag{4.30}$$

where

$$\sum_{i=1}^{p}\delta_i'=0 \text{ and at least one of } (\delta_1', \cdots, \delta_p')\neq 0\ . \tag{4.31}$$

Thus, it is sufficient to show that $n^{1/2}$ times any arbitrary contrast in $\boldsymbol{T}_N$, has asymptotically a normal (permutation) distribution. Now, using (2.9), (2.10) and condition (c.2), we may write (4.30) as

$$n^{-1/2}\sum_{\alpha=1}^{n}\sum_{i=1}^{p}\delta_i' J\left(\frac{R_{i\alpha}}{N+1}\right)+o_p(1)\ . \tag{4.32}$$

Let us then define

$$Y_{N,\alpha}(\boldsymbol{R}_N)=\sum_{i=1}^{p}\delta_i' J\left(\frac{R_{i\alpha}}{N+1}\right), \qquad \alpha=1, \cdots, n\ . \tag{4.33}$$

It thus follows from (4.32) and (4.33) that we are only to show that $n^{-1/2}\sum_{\alpha=1}^{n}Y_{N,\alpha}(\boldsymbol{R}_N)$ has asymptotically (under $\mathcal{P}_n$) a non-degenerate normal distribution, in probability. Now, under $\mathcal{P}_n$, there are $p!$ equally likely permutations of $(R_{1\alpha}, \cdots, R_{p\alpha})$ among themselves, and hence, $Y_{N,\alpha}(\boldsymbol{R}_N)$ can have only $p!$ equally likely permuted values, each with probability $1/p!$. Thus,

$$\mathrm{E}\{Y_{N,\alpha}(\boldsymbol{R}_N)\mid\mathcal{P}_n\}=\sum_{i=1}^{p}\delta_i'\left(\frac{1}{p}\sum_{\alpha=1}^{p}J\left(\frac{R_{i\alpha}}{N+1}\right)\right)=0\ , \tag{4.34}$$

$$\mathrm{E}\{Y_{N,\alpha}^2(R_N)\mid\mathcal{P}_n\}=\sum_{i=1}^{p}(\delta_i')^2\left\{\frac{1}{p-1}\sum_{j=1}^{p}\left[J\left(\frac{R_{j\alpha}}{N+1}\right)-\frac{1}{p}\sum_{l=1}^{p}J\left(\frac{R_{l\alpha}}{N+1}\right)\right]^2\right\}, \tag{4.35}$$

for $\alpha=1, \cdots, n$. Since,for each α, the $p!$ permutations have nothing to do with the permutations for other α' $(\alpha\neq\alpha'=1,\cdots, n)$, $\{Y_{N,\alpha}(\boldsymbol{R}_N), \alpha=1,$

$\cdots, n\}$ are (under $\mathcal{P}_n$) stochastically independent. To prove the central limit theorem for $\{Y_{N,\alpha}(\boldsymbol{R}_N), \alpha=1, \cdots, n\}$ (under $\mathcal{P}_n$), we shall now use the Berry-Esseen theorem (cf. Loeve [6], p. 288), which may be stated as follows:

Let $\{W_i\}$ be any sequence of independent random variables with means $\{\mu_i\}$, variances $\{\sigma_i^2\}$ and absolute third order moments $\{\beta_i\}$; let then

$$S_n^2=\sum_1^n \sigma_i^2\,, \qquad \rho_n=\sum_1^n \beta_i\,.$$

Also, let $G_n(x)$ be the cdf of $\sum_{i=1}^n (W_i-\mu_i)/S_n$, and $\Phi(x)$ be the standardized normal cdf. Then there exists a finite constant c ($<\infty$), such that for all x

$$|G_n(x)-\Phi(x)|<c\rho_n S_n^{-3}\,, \qquad c<\infty\,. \tag{4.36}$$

(It may be noted that instead of a single sequence of stochastic variables, we may have a double sequence $\{W_{n,i}\}$ with means $\{\mu_{n,i}\}$, variances $\{\sigma_{n,i}^2\}$ etc., and the theorem also applies to this situation.). Now, from (4.35) we get, following precisely on the same line as in theorem 4.2, that

$$\begin{aligned}\frac{1}{n}S_n^2&=\frac{1}{n}\sum_{\alpha=1}^n \mathrm{V}(Y_{n,\alpha}(\boldsymbol{R}_N)\,|\,\mathcal{P}_n)\\ &=\sum_{i=1}^p(\delta_i')^2\cdot\frac{p}{p-1}\left\{\frac{1}{N}\sum_{\alpha=1}^n\sum_{j=1}^p\left[J\left(\frac{R_{j\alpha}}{N+1}\right)-\frac{1}{p}\sum_{l=1}^b J\left(\frac{R_{l\alpha}}{N+1}\right)\right]^2\right\}\\ &\xrightarrow{P}\sum_{i=1}^p(\delta_i')^2\cdot[A^2-\bar{\nu}]>0\,,\end{aligned} \tag{4.37}$$

by lemma 4.1. Again,

$$\begin{aligned}\frac{1}{n}\rho_n&=\frac{1}{n}\sum_{\alpha=1}^n \mathrm{E}\{|Y_{N,\alpha}(\boldsymbol{R}_N)|^3\,|\,\mathcal{P}_n\}\\ &\leqq\left\{\sum_{i=1}^p|\delta_i'|\left[\operatorname*{Max}_{1\leqq\alpha\leqq N}J\left(\frac{\alpha}{N+1}\right)\right]\right\}\frac{1}{n}\sum_{\alpha=1}^N \mathrm{E}\{|Y_{N,\alpha}(\boldsymbol{R}_N)|^2\,|\,\mathcal{P}_n\}\\ &\leqq\left(\sum_{i=1}^p|\delta_i'|\right)KN^{1/2-\delta}\cdot S_n^2/n, \quad \text{by condition (c.3)}\,.\end{aligned} \tag{4.38}$$

Consequently, from (4.37) and (4.38), we get that

$$\rho_n S_n^{-3}\leqq KN^{-\delta}\sum_{i=1}^p|\delta_i'|\Big/\left(\frac{1}{n}S_n^2\right)^{1/2}=o_p(N^{-\delta})\,. \tag{4.39}$$

Hence, from (4.6), we may conclude that $n^{1/2}\sum_{i=1}^p \delta_i' T_{N,i}$ has asymptotically, in probability, a normal (permutation) distribution, for any $(\delta_1', \cdots, \delta_p')$ satisfying (4.31). This proves that $n^{1/2}[(T_{N,i}-\bar{E}_N), i=1, \cdots, p-1]$ has asymptotically, in probability, a $p-1$ variate normal distribution (under $\mathcal{P}_n$). Now by considering the exponent of this asymptotic multinormal distribution and using some well-known results on the limiting distribution of continuous functions of random variables, it can be shown by some routine analysis that under the permutational probability measure $\mathcal{P}_n$, the statistic W_N, in (3.10), has asymptotically, in probability, a χ^2 distribution with $(p-1)$ d.f.

Let us now denote by $\chi^2_{p-1,\varepsilon}$ the upper $100\varepsilon\%$ point of the chi-square distribution with $p-1$ d.f. Then from (3.11), (3.12) and theorem 4.3, we readily arrive at the following theorem.

THEOREM 4.4. *Under the permutational probability measure* $\mathcal{P}_n$,

$$W_{N,\varepsilon} \xrightarrow{P} \chi^2_{p-1,\varepsilon} \quad \textit{and} \quad \gamma_N(\boldsymbol{R}_N) \xrightarrow{P} 0 .$$

By virtue of theorem 4.4, the exact permutation test in (3.11) reduces asymptotically to

$$\phi(\boldsymbol{Z}_N)=\begin{cases} 1 & \text{if } W_N \geqq \chi^2_{p-1,\varepsilon} \\ 0 & \text{otherwise} . \end{cases} \tag{4.40}$$

In the sequal, (4.40) will be termed the *large sample permutation test* while (3.11) as the *exact permutation test*. For the study of the asymptotic properties of the proposed permutation tests, it appears that we may use (4.40) instead of (3.11). Now, the study of the asymptotic properties will require the knowledge of the asymptotic form of the unconditional distribution of W_N, which we shall proceed to consider in the next section.

5. Asymptotic multinormality of the standardized form of T_N

We adopt here precisely the same notations as in the beginning of section 4, and with the help of (2.5), (4.1) and (4.2), we rewrite $T_{N,i}$ as

$$T_{N,i}=\int_{-\infty}^{\infty} J_N\left(\frac{N}{N+1} H_N(x)\right) dF_{N[i]}(x) , \qquad i=1, \cdots, p . \tag{5.1}$$

Apparently, (5.1) has the same form as that of Chernoff-Savage [2] type of rank order statistics related to the multisample case, considered by Puri [7] (also [8]). However, in their case, all the samples are stochastically independent, while in our case, it is a p-variate sample. Let us then introduce the following definitions.

(5.2) $$\mu_{N,i}=\int_{-\infty}^{\infty} J(H(x))\,dF_{[i]}(x)\,, \qquad i=1,\cdots,p\,,$$

(5.3) $$\beta_{ii.kl}=\iint_{-\infty<x<y<\infty} F_{[i]}(x)[1-F_{[i]}(y)]J'(H(x))J'(H(y))\,dF_{[k]}(x)\,dF_{[l]}(y) + \iint_{-\infty<x<y<\infty} F_{[i]}(x)[1-F_{[i]}(y)]J'(H(x))J'(H(y))\,dF_{[l]}(x)\,dF_{[k]}(y)\,,$$

(5.4) $$\beta_{\substack{ij.kl\\ i\neq j}}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F_{[i,j]}(x,y) - F_{[i]}(x)F_{[j]}(y)]J'(H(x))J'(H(y))\,dF_{[k]}(x)\,dF_{[l]}(y)$$

for $i\neq j=1,\cdots,p$, $k,l=1,\cdots,p$. Finally, let

(5.5) $$\beta^*_{ij}=\frac{1}{p}\left\{\sum_{k=1}^{p}\sum_{l=1}^{p}[\beta_{kl.ij}+\beta_{ij.kl}-\beta_{kj.il}-\beta_{il.kj}]\right\}\,, \qquad i,j=1,\cdots,p,$$

and

(5.6) $$\boldsymbol{\beta}^*=((\beta^*_{ij}))_{i,j=1,\cdots,p}\,.$$

THEOREM 5.1. *If the conditions* (c.1), (c.2) *and* (c.3) *of section* 4 *hold, then the random vector* $N^{1/2}[(T_{N,i}-\mu_{N,i}),\ i=1,\cdots,p]$ *has asymptotically a multinormal distribution with a null mean vector and a dispersion matrix* $\boldsymbol{\beta}^*$.

(It may be noted that by virtue of (2.11), the above multinormal distribution will be essentially singular having a rank less than or equal to $p-1$.)

PROOF. We proceed precisely on the same line as in the proof of theorem 5.1 of [8] and write

(5.7) $$T_{N,i}=\mu_{N,i}+B^{(i)}_{1,N}+B^{(i)}_{2,N}+\sum_{l=1}^{4}C^{(i)}_{l,N}\,, \qquad i=1,\cdots,p\,,$$

where

(5.8) $$B^{(i)}_{1,N}=\int J(H)\,d[F_{N[i]}(x)-F_{[i]}(x)]\,,$$

(5.9) $$B^{(i)}_{2,N}=\int [H_N(x)-H(x)]J'(H(x))\,dF_{[i]}(x)\,,$$

(5.10) $$C^{(i)}_{1,N}=\frac{-1}{N+1}\int H_N(x)J'(H(x))\,dF_{N[i]}(x)\,,$$

(5.11) $$C^{(i)}_{2,N}=\int [H_N(x)-H(x)]J'(H(x))\,d[F_{N[i]}(x)-F_{[i]}(x)]\,,$$

$$C_{3,N}^{(i)}=\int\Big[J\Big(\frac{N}{N+1}H_N(x)\Big)-J(H(x)) \\ -\Big(\frac{N}{N+1}H_N(x)-H(x)\Big)J'(H(x))\Big]\,dF_{N[i]}(X)\ , \tag{5.12}$$

and

$$C_{4,N}^{(i)}=\int\Big[J_N\Big(\frac{N}{N+1}H_N(x)\Big)-J\Big(\frac{N}{N+1}H_N(x)\Big)\Big]\,dF_{N[i]}(x)\ , \tag{5.13}$$

in all expressions the range of integration is over $-\infty$ to ∞. Now, by condition (c.2), (5.13) will be $o_p(n^{-1/2})$, and precisely on the same line as in the treatment of C_{13N} of Chernoff and Savage ([2], p. 988) it is easily seen that $C_{1,N}^{(i)}$ is also $o_p(n^{-1/2})$, uniformly in $i=1,\cdots,p$. Further, it has been shown by the present author [9] that in the case of $p=2$, $C_{2,N}^{(i)}$ and $C_{3,N}^{(i)}$ (for $i=1$) are both $o_p(n^{-1/2})$. Essentially, the same argument holds for the general case of $p\geqq 2$, and hence, avoiding the details of these, we may write

$$N^{1/2}|(T_{N,i}-\mu_{N,i})-(B_{1,N}^{(i)}+B_{2,N}^{(i)})|=o_p(1)\ , \tag{5.14}$$

for all $i=1,\cdots,p$. Consequently, it is sufficient to prove that $\{N^{1/2}(B_{1,N}^{(i)}+B_{2,N}^{(i)}),\ i=1,\cdots,p\}$ has asymptotically a multinormal distribution. Now, by partial integration of (5.8), we readily arrive at the following expression, through a few simple steps.

$$B_{1,N}^{(i)}+B_{2,N}^{(i)}=\frac{1}{p}\sum_{k=1}^{p}\Big\{\frac{1}{n}\sum_{\alpha=1}^{n}[B_{i:k}(X_{i\alpha})-B_{k:i}(X_{k\alpha})]\Big\}\ , \tag{5.15}$$

where

$$B_{k:q}(X_{k\alpha})=\int_{-\infty}^{\infty}[F_{1[k]}^{(\alpha)}(x)-F_{[k]}(x)]J'(H(x))\,dF_{[q]}(x)\ , \tag{5.16}$$

$$F_{1[k]}^{(\alpha)}(x)=\begin{cases}0 & \text{if } x<X_{k\alpha}\\ 1 & \text{if } x\geqq X_{k\alpha}\ ,\end{cases} \tag{5.17}$$

for $k,q=1,\cdots,p\,;\ \alpha=1,\cdots,n$.

We shall now show that for any non-null real p-vector $\boldsymbol{\delta}=(\delta_1,\cdots,\delta_p)$, the random variable

$$N^{1/2}\sum_{i=1}^{p}\delta_i[B_{1,N}^{(i)}+B_{2,N}^{(i)}] \tag{5.18}$$

has asymptotically a normal distribution. Now, we may also rewrite (5.18) as

$$N^{1/2}\sum_{k=1}^{p}\sum_{q=1}^{p}\delta_{kq}^{*}\Big[\frac{1}{n}\sum_{\alpha=1}^{n}B_{k:q}(X_{k\alpha})\Big]\ ,$$

where $\boldsymbol{\delta}^*=(\delta_{11}^*, \cdots, \delta_{pp}^*)$ is also non-null and real. Now, if we write

$$(5.19)\qquad B(\boldsymbol{X}_\alpha;\, \boldsymbol{\delta}^*)=\sum_{k=1}^{p}\sum_{q=1}^{p} \delta_{kq}^* B_{k:q}(X_{k\alpha}) , \qquad \alpha=1, \cdots, n ,$$

it follows from the discussion above that (5.18) may also be written as

$$(5.20)\qquad N^{1/2}\left\{\frac{1}{n}\sum_{\alpha=1}^{n} B(\boldsymbol{X}_\alpha;\, \boldsymbol{\delta}^*)\right\} ,$$

which aparts from the factor $N^{1/2}$ is the average of n independent and identically distributed random variables $\{B(\boldsymbol{X}_\alpha : \boldsymbol{\delta}^*), \alpha=1, \cdots, n\}$. Hence, to apply the classical central limit theorem under Lindeberg condition, it is sufficient to show that $B(\boldsymbol{X}_\alpha, \boldsymbol{\delta}^*)$ has finite first and second order moments. We shall prove a slightly stronger result that for any η: $0<\eta<\delta$ (defined in (4.10),), $\mathrm{E}\{|B(\boldsymbol{X}_\alpha, \delta^*)|^{2+\eta}\}<\infty$, uniformly in $F_{[1]}, \cdots, F_{[p]}$. Now, using (5.19) and some well-known inequalities, we have

$$(5.21)\qquad \mathrm{E}\,|B(\boldsymbol{X}_\alpha, \delta^*)|^{2+\eta} \leqq p^{2(1+\eta)} \sum_{k=1}^{p}\sum_{q=1}^{p} |\delta_{kq}^*|^{2+\eta}\cdot \mathrm{E}\,|B_{k:q}(X_{k\alpha})|^{2+\eta} ,$$

and proceeding precisely on the same line as in the case of univariate several sample observations (for instance, see [7], section 5) we can easily prove that for all $0<\eta<\delta$,

$$(5.22)\qquad \mathrm{E}\,|B_{k:q}(X_{k\alpha})|^{2+\eta} \leqq K \iint_{0\leqq u<v\leqq 1} u(1-v)\,|J'(u)|\,|J'(v)|\,du\,dv<\infty ,$$

by condition (c.3) in (4.10). Thus, from (5.21) and (5.22), we conclude that $B(\boldsymbol{X}_\alpha;\, \boldsymbol{\delta}^*)$ has a finite moment of the order $2+\eta$, $\eta>0$, and this in turn, implies that the first two moments of the same are finite. Hence, we arrive at the asymptotic normality of the variable in (5.18), and this implies the asymptotic normality of the joint distribution of $N^{1/2}(T_{N,i}-\mu_{N,i})$, $i=1, \cdots, p$. Again, from (5.16), we get by an application of Fubini's theorem that

$$(5.23)\qquad \mathrm{Cov}\,(B_{i:j}(X_{i\alpha}),\, B_{k:q}(X_{k\beta}))=\begin{cases} 0 & \text{if } \alpha\neq\beta \\ \beta_{ik.jq} & \text{if } \alpha=\beta , \end{cases}$$

(where $\beta_{ij.kl}$ is defined in (5.3) and (5.4),) for $i, j, k, q=1, \cdots, p$; $\alpha, \beta= 1, \cdots, n$. From (5.15), (5.16) and (5.23), we readily arrive at

$$(5.24)\qquad \lim_{N=\infty}\{N\,\mathrm{Cov}\,(T_{N,i},\, T_{N,j})\}=\beta_{ij}^* , \qquad i, j=1, \cdots, p ,$$

where $\boldsymbol{\beta}^*=(\beta_{ij}^*)$ is defined in (5.6).

Hence, the theorem.

It has already been pointed out earlier that the asymptotic multinormal distribution, derived in theorem 5.1, is singular and is of rank at most equal to $p-1$. If the null hypothesis (1.5) holds, and we define $H(x)$, $H^*(x, y)$, A^2 and $\bar{\nu}$ as in (4.5), (4.6), (4.16) and (4.17), respectively, then it will readily follow from (5.24), (5.3) and (5.4) (though a few simple steps) that

$$\lim_{N=\infty} \{N \operatorname{Cov}(T_{N,i}, T_{N,j} \mid H_M \text{ in } (1.5))\} = (\delta_{ij}p-1)(A^2-\bar{\nu}) ,$$

for $i, j=1, \cdots, p$, where δ_{ij} is the usual Kronecker delta. Consequently, with the help of lemma 4.1, we readily arrive at the following.

COROLLARY 5.1.1. *If H_M in (1.5) holds and the conditions of theorem 5.1 holds, then under (c.5) in (4.5), $[N^{1/2}(T_{N,i}-\mu), i=1, \cdots, p]$ has a singular multinormal distribution of rank $p-1$, $\left(\text{where } \mu=\int_0^1 J(u)\,du\right)$.*

We shall now consider the usual type of Pitman's translation alternatives, and for this, we replace the parent cdf $F(\boldsymbol{x})$ by a sequence of cdf's $F_{(N)}(\boldsymbol{x})$, such that the marginal cdf's of $\{F_{(N)}(x)\}$ satisfy the sequence of alternatives $\{H_N\}$, where

$$H_N: \quad F_{[i](N)}(x)=H(x+N^{-1/2}\theta_i) , \qquad i=1, \cdots, p , \tag{5.25}$$

where $H(x)$ is assumed to be an absolutely continuous (univariate) cdf having a continuous density function $h(x)$, and where the assumption of equality of scales and symmetry in (1.5) are also assumed to hold for the sequence of cdf's $\{F_{(N)}(\boldsymbol{x})\}$. Let us then define

$$\zeta(H)=\int_{-\infty}^{\infty} \frac{d}{dx} J(H(x))\,dH(x) . \tag{5.26}$$

Then, it is easy to verify that

$$\lim_{N=\infty} [N^{1/2} \mathrm{E}\{(T_{N,i}-\mu) \mid H_N\}]=\theta_i\zeta(H) , \qquad i=1, \cdots, p, \tag{5.27}$$

$$\lim_{N=\infty} [\operatorname{Cov}(T_{N,i}, T_{N,j} \mid H_N)]=(\delta_{ij}p-1)(A^2-\bar{\nu}) , \qquad i, j=1, \cdots, p. \tag{5.28}$$

Consequently, it follows from theorem 5.1, that under $\{H_N\}$, $\{N^{1/2}(T_{N,i}-\mu), i=1, \cdots, p-1\}$ has asymptotically a $(p-1)$ variate normal distribution with a mean vector $\zeta(H)(\theta_1, \cdots, \theta_p)$ and a dispersion matrix

$$(\delta_{ij}p-1)(A^2-\bar{\nu}) . \tag{5.29}$$

Hence, it readily follows that under $\{H_N\}$

$$W_N^* = \frac{n}{A^2-\bar{\nu}} \sum_{i=1}^{p} (T_{N,i}-\bar{E}_N)^2 \tag{5.30}$$

has asymptotically a non-central χ^2 distribution with $(p-1)$ d.f. and the noncentrality parameter

$$\Delta_W = [\{\zeta(H)\}^2/(A^2-\bar{\nu})]\left\{\frac{1}{p}\sum_{i=1}^{p}(\theta_i-\bar{\theta})^2\right\}, \tag{5.31}$$

where $\bar{\theta}=\frac{1}{p}\sum_{i=1}^{p}\theta_i$.

Now, from (3.10), theorem 4.2 and (5.30), we readily get that under $\{H_N\}$, $W_N \overset{P}{\sim} W_N^*$, where $\overset{P}{\sim}$ means asymptotically equivalent, in probability. Hence, we arrive at the following.

THEOREM 5.2. *Under the sequence of alternative hypotheses* $\{H_N\}$ *in* (5.25), *the statistic* W_N *in* (3.10), *has asymptotically a non-central* χ^2 *distribution with* $(p-1)$ *d.f. and the noncentrality parameter* Δ_W *defined in* (5.31).

At this stage, we may consider also some asymptotically distribution free tests for H_M in (1.5). This may be formulated as follows. Let S^2 be some consistent estimator of $A^2-\bar{\nu}$, in the sense that

$$S^2 \xrightarrow{P} A^2-\bar{\nu} \quad \text{for all} \quad F_0 \in \mathscr{F}_0\,, \quad \text{in (1.4)}\,. \tag{5.32}$$

Then, it follows from (5.30) and a well-known limit theorem on the distribution of rational function of random variables that under $\{H_N\}$ in (5.25)

$$\hat{W}_N = \frac{n}{S^2}\sum_{i=1}^{p}(T_{N,i}-\bar{E}_N)^2 \overset{P}{\sim} W_N^*\,. \tag{5.33}$$

Hence, the test based on $\hat{W}_N$ will be asymptotically a distribution-free test for H_M in (1.5). It further follows from theorem 5.2, that the test based on W_N in (3.10) will be asymptotically power equivalent to the one based on $\hat{W}_N$, for any sequence of alternatives of the type $\{H_N\}$ in (5.25).

Thus, the permutation test based on W_N in (3.10) also appears to be power equivalent (asymptotically) to unconditional tests based on stochastically equivalent statistics. Since, we have shown that the permutation test has the advantage of being exactly distribution-free even for small sample sizes, we may advocate the unrestricted use of the same for all sample sizes.

6. Asymptotic efficiency of rank order tests

We shall now consider the asymptotic relative efficiency (A.R.E.) of our proposed rank order tests with respect to the likelihood ratio (L_M-) test, considered by Wilks [12]. It is easily seen that under the sequence of alternatives $\{H_N\}$ in (5.25), Wilks' L_M statistic has asymptotically (actually, $-2\log_e L_M$) a noncentral chi-square distribution with $(p-1)$ d.f. and the noncentrality parameter

$$\Delta_L = \frac{1}{\sigma^2(1-\bar{\rho})}\left\{\frac{1}{p}\sum_{i=1}^{p}(\theta_i-\bar{\theta})^2\right\}, \tag{6.1}$$

where $\bar{\rho}$ is the average (over the $\binom{p}{2}$ possible pairs) correlation coefficient between X_i and X_j, $i\neq j=1, \cdots, p$, and σ^2, the common variance.

Let us now write $A^2-\bar{\nu}$ (in (5.31)) in the form $A^2(1-\bar{\rho}_\nu)$, where $\bar{\rho}_\nu=\bar{\nu}/A^2$ is the average *score-correlation* of the p-variates. Then from theorem 5.2 and (6.1), we get that the A.R.E. of the W-test with respect to the L_M-test is given by

$$\begin{aligned} e(W, L_M) &= [\zeta(H)]^2\sigma^2(1-\bar{\rho})/A^2(1-\bar{\rho}_\nu) \\ &= \left[\frac{\zeta(H)\cdot\sigma}{A}\right]^2\left(\frac{1-\bar{\rho}}{1-\bar{\rho}_\nu}\right). \end{aligned} \tag{6.2}$$

The first factor on the right-hand side of (6.2) is solely dependent on the marginal distribution $H(x)$ in (4.5), while the second factor depends on the joint distribution $F(\mathbf{x})$, through the bivariate marginal cdf $H^*(x, y)$ in (4.6). Various bounds for the first factor are available in the literature for various common types of $J(u)$: $0<u<1$, and for an excellent account of these the reader is referred to Hodges and Lehmann [4], [5], and Chernoff and Savage [2]. The bounds for the second factor will depend in a quite involved manner on the parent cdf $F(x)$. Before we take up this discussion, we like to consider specifically two important rank order tests, namely, the rank-sum test and the normal score test. For the rank-sum test the scores $E_{N,i}$, $i=1, \cdots, N$ are the N natural numbers, i.e., $E_{N,i}=i$, $i=1, \cdots, N$. In this case, the first factor on the right-hand side of (6.2) reduces to

$$12\sigma^2\left[\int_{-\infty}^{\infty} f^2(x)\,dx\right]^2, \tag{6.3}$$

and it is well-known (cf. [4]) that (6.3) has (i) the value $3/\pi$ for normal cdf, and (ii) for any continuous cdf this can not be less than $108/125=0.864$. The second factor on the right-hand side of (6.2) reduces to

$$(1-\bar{\rho})/(1-\rho_g), \tag{6.4}$$

where ρ_g is the grade correlation. For normal cdf, $\rho_g=(6/\pi)\sin^{-1}(\rho/2)$, and hence, for normal cdf, (6.2) reduces to

$$(3/\pi)[(1-(6/\pi)\sin^{-1}(\rho/2))\,/\,(1-\rho)]^{-1}\,. \tag{6.5}$$

This is a sole function of the correlation coefficient ρ, and its bounds can easily be found. Since, these efficiency-bounds are already studied by Bickel [1], (in connection with multivariate one sample location problem), we shall not enter into the detailed discussion of them. The author is not aware of any distribution-free bound for (6.4), and hence, nothing can be said about the bounds for the A.R.E. in (6.2) for rank-sum test and arbitrary cdf $F(x)$. For the normal score test, $E_{N,i}$ is the expected value of the ith smallest observation in a random sample of size N drawn from a standard normal cdf, for $i=1,\cdots,N$. In this case, it follows from the results of Chernoff and Savage [2] (see also [5]) that the first factor on the right-hand side of (6.2) is always greater than or equal to unity; it becomes equal to unity only when the parent cdf is normal. But the second factor is again unknown, and its bounds are quite involved. In fact, if the parent cdf is multinormal, this factor becomes equal to unity, as a result, the A.R.E. also becomes equal to one. On the otherhand, for arbitrary cdf, the A.R.E. may not be proved to be greater than or equal to unity. If, however, the cdf $F(x)$ is singular (i.e., $\rho=-1/(p-1)$) then it can be easily shown that the second factor on the right-hand side of (6.2) is always at least as large as unity, and hence, (6.2) is also greater than or equal to unity. In actual practice, often (cf. [10]) the singularity of $F(x)$ (in the above sense) can be shown to be true, and in such cases, the normal score test can be justified to be asymptotically at least as efficient as the L_M test. For the details of this, the reader is referred to [10].

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