BAN LINEAR ESTIMATES OF THE PARAMETERS OF THE NORMAL DISTRIBUTION FROM CENSORED SAMPLES*

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1. Introduction

The present paper is concerned with the linear estimation of the parameters of the normal distribution on the basis of a complete or a Type II (singly or doubly) censored sample whose observations are order statistics. The exact maximum likelihood estimates are difficult to compute and usually biased. For this reason some authors, for example Gupta [7], have suggested methods to simplify the computations and others, like Saw [16], have succeeded in reducing the bias. Lloyd (1952) has proposed best linear estimates, but their coefficients involve heavy computations. Approximation of such estimates based on a complete sample has been considered by Jung [11] and Blom [3].

The linear estimates obtained by us belong to the class of the restricted Best Asymptotically Normal (BAN) estimates (Neyman, [13]) based on the corresponding complete or censored sample. The technique we use is to linearize the partial derivatives (with respect to the parameters) of the likelihood function based on the complete or censored sample. The coefficients of the estimates are explicit functions of either (i) the expected values of the order statistics (section 2.2, the estimates are strictly unbiased for any sample size) or (ii) population quantiles (section 2.3, asymptotically unbiased) from the standard normal distribution. The generalization of the above estimates to multiple censoring is indicated (section 2.4). The expressions of our estimates in (i) are the same as those of Plackett [14] where he considered double censoring only. However we do not agree his derivation on grounds discussed in section 3. Using different techniques, asymptotically unbiased estimates analogous to our estimates in (ii) have been studied by Weiss [18] for double censoring and by Chernoff, Gastwirth and Johns [4] for complete

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samples or multiple censoring. A simple estimate of the population mean proposed by Dixon [6] is also investigated (section 4).

2. The estimates

2.1. *Notations and definitions*

Let an ordered random sample of size n from the normal distribution with unknown true mean μ_0 and unknown true standard deviation σ_0 be denoted by

$$x_1 < x_2 < \cdots < x_n . \tag{2.1}$$

For the estimation of μ_0 and σ_0, the following uncensored portion of (2.1) is available.

$$x_u < x_{u+1} < \cdots < x_v , \tag{2.2}$$

where u and v are determined by two fixed numbers α and β satisfying $0 \leqq \alpha < \beta \leqq 1$ such that $u=[n\alpha]+1$, $v=[n\beta]+1$; $[q]$ denotes the greatest integer $\leqq q$ (when $\beta=1$, we let $v=n$). (2.2) will be called a relevant sample under Type II censoring. It may be a complete sample ($\alpha=0$, $\beta=1$), a singly censored sample ($\alpha=0$, $\beta<1$ or $\alpha>0$, $\beta=1$), or a doubly censored sample ($\alpha>0$, $\beta<1$).

Let F and f denote the distribution function and density function of $N(0, 1)$ and $y_i=(x_i-\mu_0)/\sigma_0$. Then y_i is the ith order statistic from $N(0, 1)$. The likelihood function of (2.2) is

$$L(x_u, \cdots, x_v, \mu, \sigma)=\frac{n!}{(u-1)!\,(n-v)!}\left[F\left(\frac{x_u-\mu}{\sigma}\right)\right]^{u-1}\prod_{i=u}^{v}\frac{1}{\sigma}f\left(\frac{x_i-\mu}{\sigma}\right) \cdot\left[1-F\left(\frac{x_v-\mu}{\sigma}\right)\right]^{n-v} ,$$

where μ and σ are values in some neighborhoods of μ_0 and σ_0, respectively. Now let us define L_1 and L_2 as follows

$$\begin{aligned} L_1(y_u, \cdots, y_v)&=\left[\sigma\frac{\partial \log L(x_u, \cdots, x_v; \mu, \sigma)}{\partial\mu}\right]_{u=\mu_0, \sigma=\sigma_0} \\ &=-(u-1)\gamma_1(y_u)+\sum y_i+(n-v)\gamma_1(-y_v) , \end{aligned} \tag{2.3a}$$

and

$$\begin{aligned} L_2(y_u, \cdots, y_v)&=\left[\sigma\frac{\partial \log L(x_u, \cdots, x_v; \mu, \sigma)}{\partial\sigma}\right]_{u=\mu_0, \sigma=\sigma_0} \\ &=-(u-1)\gamma_2(y_u)-(v-u+1)+\sum y_i^2-(n-v)\gamma_2(-y_v) , \end{aligned} \tag{2.3b}$$

where $\gamma_1(y)=f(y)/F(y)$ and $\gamma_2(y)=y\gamma_1(y)$. Throughout this paper, the notation $\sum$, without any affix means that the summation is from $i=u$ to v.

For $0 \leqq \alpha < \beta \leqq 1$, let η and ξ be the numbers satisfying $F(\eta)=\alpha$ and $F(\xi)=\beta$ ($\eta=-\infty$ when $\alpha=0$ and $\xi=\infty$ when $\beta=1$). The matrix of the asymptotic Cramér-Rao lower bound for the relevant sample (2.2) is $\|n\sigma^{-2}K_{rt}(\alpha, \beta)\|^{-1}$, $r, t=1, 2$, where

$$K_{11}(\alpha, \beta)=(\beta-\alpha)+[\eta f(\eta)-\xi f(\xi)]+\frac{1}{\alpha}f^2(\eta)+\frac{1}{1-\beta}f^2(\xi) ,$$

$$K_{12}(\alpha, \beta)=K_{21}(\alpha, \beta)=[f(\eta)-f(\xi)]+[\eta^2 f(\eta)-\xi^2 f(\xi)]+\frac{1}{\alpha}\eta f^2(\eta)+\frac{1}{1-\beta}\xi f^2(\xi) ,$$

$$K_{22}(\alpha, \beta)=2(\beta-\alpha)+[\eta f(\eta)-\xi f(\xi)]+[\eta^3 f(\eta)-\xi^3 f(\xi)]+\frac{1}{\alpha}\eta^2 f^2(\eta) + \frac{1}{1-\beta}\xi^2 f^2(\xi) .$$

If two estimates T_{1n} and T_{2n} based on (2.2) are such that $(\sqrt{n}(T_{1n}-\mu_0), \sqrt{n}(T_{2n}-\sigma_0))$ converges in distribution to $N((0, 0), \sigma_0^2\|K_{rt}(\alpha, \beta)\|^{-1})$, then they are said to belong to the restricted class of BAN estimates based on the corresponding relevant sample.

2.2. *The estimates*

Let $\mu_i=\mathrm{E}(y_i)$ be the expected values of y_i and $\gamma_t'(s)=[d\gamma_t(y)/dy]_{y=s}$, $t=1, 2$. Also let

$$S_{1u}=1-(u-1)\gamma_1'(\mu_u) , \qquad S_{1v}=1-(n-v)\gamma_1'(-\mu_v) ,$$

$$S_{2u}=2\mu_u-(u-1)\gamma_2'(\mu_u) , \qquad S_{2v}=2\mu_v+(n-v)\gamma_2'(-\mu_v) ,$$

$$S_{1i}=1 \text{ and } S_{2i}=2\mu_i , \qquad i=u+1, \cdots, v-1 .$$

We first state our theorem 1 and then prove several lemmas.

THEOREM 1. *Let $\hat{\mu}$ and $\hat{\sigma}$ be the estimates obtained by solving the equations*

$$\sum\left(\frac{x_i-\mu}{\sigma}-\mu_i\right)S_{1i}=0 , \qquad \sum\left(\frac{x_i-\mu}{\sigma}-\mu_i\right)S_{2i}=0$$

which are based on the relevant sample (2.2), *then $\hat{\mu}$ and $\hat{\sigma}$ are strictly unbiased linear estimates of μ_0 and σ_0, respectively, and they belong to the restricted class of BAN estimates.*

LEMMA 1. *Let ξ_i be the number satisfying $F(\xi_i)=i/(n+1)$ and*

$\tilde{R}_n(\lambda)=\sum_{i=1}^{n}(y_i-\xi_i)^2/n^\lambda$, *then* $\lim_{n\to\infty}\mathrm{E}(\tilde{R}_n(\lambda))=0.$

PROOF. The authors [2] have shown that $\xi_i\leqq \mathrm{E}(y_i)$ if $i\geqq(n+1)/2$ and $\xi_i>\mathrm{E}(y_i)$ if $i<(n+1)/2$. So

$$
\begin{aligned}
(2.4)\qquad 0\leqq \mathrm{E}(\tilde{R}_n(\lambda))\leqq n^{\lambda_0}[1-(\textstyle\sum \xi_i^2/n)]&=n^{\lambda_0}\int_0^1 y^2\,dF-\left[\int_{1/(n+1)}^{n/(n+1)} y^2\,dF+\frac{\xi_n^2}{n+1}\right.\\
&\qquad\left.+\frac{n-1}{12(n+1)^3}\cdot\frac{d^2(y^2)}{dF^2}\bigg|_{F=\theta_n}\right]\\
&<2n^{\lambda_0}\int_{n/(n+1)}^1 y^2\,dF\,,
\end{aligned}
$$

where $\lambda_0=1-\lambda$, $1/(n+1)<\theta_n<n/(n+1)$. The equality in (2.4) follows from the trapezoidal rule, and the last inequality follows from the fact that $[y(F)]^2$ is convex and $\xi_n^2>0$. Since $n=F(\xi_n)/(1-F(\xi_n))$ and $\lim_{n\to\infty}F(\xi_n)=1$, by applying L'Hopital's rule to the last term of (2.4) we have

$$
(2.5)\qquad 0\leqq\lim_{n\to\infty}\xi_n^2[1-F(\xi_n)]^\lambda/\lambda_0\leqq\lim_{n\to\infty}\left[\int_{\xi_n}^\infty y^{2/\lambda}\,dF(y)\right]^\lambda\Big/\lambda_0=0\ .
$$

LEMMA 2. (Sen [17]) $\lim_{n\to\infty}n\mathrm{E}(y_u-\xi_u)^2=\dfrac{\alpha(1-\alpha)}{f^2(\eta)}$, $\lim_{n\to\infty}n\mathrm{E}(y_v-\xi_v)^2=\dfrac{\beta(1-\beta)}{f^2(\xi)}$.

LEMMA 3. $\lim_{n\to\infty}L_t(\mu_u,\cdots,\mu_v)/\sqrt{n}=0,\quad t=1,2.$

PROOF. Take the Taylor's expansions of the left-hand terms in (2.3a, b) about $\mu_u,\cdots,\mu_v$:

$$
(2.6a)\qquad \frac{1}{\sqrt{n}}L_1(y_u,\cdots,y_v)=\frac{1}{\sqrt{n}}L_1(\mu_u,\cdots,\mu_v)+\frac{1}{\sqrt{n}}\sum(y_i-\mu_i)S_{1i}+R_{1n}+R_{1n}^*\,,
$$

$$
(2.6b)\qquad \frac{1}{\sqrt{n}}L_2(y_u,\cdots,y_v)=\frac{1}{\sqrt{n}}L_2(\mu_u,\cdots,\mu_v)+\frac{1}{\sqrt{n}}\sum(y_i-\mu_i)S_{2i}+R_{2n}+R_{2n}^*+R_n\,,
$$

where

$$
R_{1n}=\frac{u-1}{2\sqrt{n}}(y_u-\mu_u)^2\gamma_{1,u}''\,,\qquad R_{1n}^*=\frac{n-v}{2\sqrt{n}}(y_v-\mu_v)^2\gamma_{1,v}''\,,
$$

$$R_{2n}=\frac{u-1}{2\sqrt{n}}(y_u-\mu_u)^2\gamma''_{2,u}\ , \qquad R^*_{2n}=\frac{n-v}{2\sqrt{n}}(y_v-\mu_v)^2\gamma''_{2,v}\ ,$$

$$R_n=\frac{1}{\sqrt{n}}\sum(y_i-\mu_i)^2\ ,$$

and $\gamma''_{t,u}$, $\gamma''_{t,v}$, $t=1, 2$, are the second derivatives of γ_t evaluated at some intermediate values. Here we note that for singly censored cases ($u=1$ and $v<n$, or $u>1$ and $v=n$) either $R_{tn}=0$ or $R^*_{tn}=0$. For the complete sample case both $R_{tn}=0$ and $R^*_{tn}=0$.

It can be seen that γ''_t are bounded. Therefore by lemma 2

$$\lim_{n\to\infty}\mathrm{E}(R_{tn})=0 \quad \text{and} \quad \lim_{n\to\infty}\mathrm{E}(R^*_{tn})=0\ . \tag{2.7}$$

Since $\mathrm{E}(y_i-\mu_i)^2\leqq\mathrm{E}(y_i-\xi_i)^2$, $R_n\leqq\tilde{R}(1/2)$. Then by lemma 1 we have

$$\lim_{n\to\infty}\mathrm{E}(R_n)=0\ . \tag{2.8}$$

An immediate extension to two parameter case of the expression (3.31) in Halperin's paper [8] shows that for censored cases

$$\mathrm{E}(L_t(y_u, \cdots, y_v))=0\ , \qquad t=1, 2\ . \tag{2.9}$$

For the complete sample case let $y^*_1, y^*_2, \cdots, y^*_n$ be the independent observations corresponding to $y_1<y_2<\cdots<y_n$. From (2.3a, b) we have

$$L_1=\sum y_i=\sum y^*_i\ , \qquad L_2=\sum(-1+y^2_i)=\sum(-1+y^{*2}_i)\ . \tag{2.10}$$

So

$$\mathrm{E}(L_t(y_1, \cdots, y_n))=0\ . \tag{2.11}$$

Then the lemma follows from taking expectations on both sides of (2.6a, b) and then applying (2.9), (2.7) and (2.8) for censored cases, or lemma 1 and (2.11) for complete sample case.

LEMMA 4. *$(\sum(y_i-\mu_i)S_{1i}/\sqrt{n},\ \sum(y_i-\mu_i)S_{2i}/\sqrt{n})$ converges in distribution to $N((0, 0), \|K_{rt}(\alpha, \beta)\|)$, $0\leqq\alpha<\beta\leqq1$.*

PROOF. For the censored cases, Halperin [8] has indicated that

$$(L_1(y_u, \cdots, y_v)/\sqrt{n}\ , \quad L_2(y_u, \cdots, y_v)/\sqrt{n})$$

converges in distribution to

$$N((0, 0), \|K_{rt}(\alpha, \beta)\|)\ . \tag{2.12a}$$

For the complete sample case we have by (2.10) and the two dimensional central limit theorem that

(2.12b) $(L_1/\sqrt{n}, L_2/\sqrt{n})$ converges in distribution to $N((0, 0), \|K_{rt}(0, 1)\|)$.

From (2.7) and (2.8) we see that R_{tn}, R^*_{tn}, and R_n converge in probability to zero. These convergences and lemma 3 enable us to apply the convergence theorem, which is an obvious extension of 'a convergence theorem' by Cramér ([5], p. 254) to a sequence of two dimensional random variables, to (2.12a or b). Then the lemma is proved.

LEMMA 5. *For* $0 \leqq \alpha < \beta \leqq 1$,

$$\lim_{n\to\infty} \sum S_{1i}/n = K_{11}(\alpha, \beta), \qquad \lim_{n\to\infty} \sum \mu_i S_{1i}/n = K_{12}(\alpha, \beta),$$

$$\lim_{n\to\infty} \sum S_{2i}/n = K_{21}(\alpha, \beta), \qquad \lim_{n\to\infty} \sum \mu_i S_{2i}/n = K_{22}(\alpha, \beta).$$

PROOF. The lemma follows from the theorem 1 and lemmas 5 and 8 given by Hoeffding [10].

PROOF OF THEOREM 1. The explicit expressions of the solutions of the equations in this theorem are

$$\hat{\mu} = \frac{(\sum \mu_i S_{2i})(\sum x_i S_{1i}) - (\sum \mu_i S_{1i})(\sum x_i S_{2i})}{\Delta} \tag{2.13a}$$

$$\hat{\sigma} = \frac{(\sum S_{1i})(\sum x_i S_{2i}) - (\sum S_{2i})(\sum x_i S_{1i})}{\Delta} \tag{2.13b}$$

where $\Delta = (\sum S_{1i})(\sum \mu_i S_{2i}) - (\sum S_{2i})(\sum \mu_i S_{1i})$. Since $x_i = \mu_0 + \sigma_0 y_i$, it can be readily verified that $\hat{\mu}$ and $\hat{\sigma}$ are strictly unbiased.

In matrix notation we have

$$\sqrt{n} \begin{Vmatrix} \hat{\mu} - \mu_0 \\ \hat{\sigma} - \sigma_0 \end{Vmatrix} = \sigma_0 Z_{2\times 2} \begin{Vmatrix} \sum (y_i - \mu_i) S_{1i}/\sqrt{n} \\ \sum (y_i - \mu_i) S_{2i}/\sqrt{n} \end{Vmatrix},$$

where

$$Z_{2\times 2} = \begin{Vmatrix} (\sum \mu_i S_{2i}/n) / (\Delta/n^2) & -(\sum \mu_i S_{1i}/n) / (\Delta/n^2) \\ -(\sum S_{2i}/n) / (\Delta/n^2) & (\sum S_{1i}/n) / (\Delta/n^2) \end{Vmatrix}.$$

From lemma 5 we have

$$\lim_{n\to\infty} Z_{2\times 2} = \|K_{rt}(\alpha, \beta)\|^{-1}.$$

Then it follows from lemma 4 that $(\sqrt{n}(\hat{\mu} - \mu_0), \sqrt{n}(\hat{\sigma} - \sigma_0))$ converges in distribution to $N((0, 0), \sigma_0^2 \|K_{rt}(\alpha, \beta)\|^{-1})$.

An immediate consequence of theorem 1 is that the best linear estimates based on the relevant sample (2.2) are asymptotically efficient.

The values of μ_i have been computed by Harter [9] for all i when

$n=2(1)100(25)250(50)400$. So $\hat{\mu}$ and $\hat{\sigma}$ may be useful when the coefficients of the best linear estimates are not available.

Table 1 gives the coefficients of $\hat{\mu}$ and $\hat{\sigma}$ for some (u, v) when $n=15$. The coefficients of the corresponding best linear estimates (Sarhan and Greenberg, [15]), denoted by μ^* and σ^*, are also given for references. Table 2 gives the efficiences of $\hat{\mu}$ and $\hat{\sigma}$ relative to μ^* and σ^* for some (u, v) when $n=15$. For the case $u=1$, $v=n$ when $n=2(1)8$, 10, 12, $\hat{\sigma}$ has the efficiency never falling below .9989 relative to σ^* (while $\hat{\mu}$ is the

Table 1

(u, v)		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
(1,5)	$\hat{\mu}$	−.3044	−.1578	−.0675	.0025	1.5273					
	$\hat{\sigma}$	−.5229	−.3329	−.2159	−.1252	1.1969					
	μ^*	−.3217	−.1364	−.0560	.0043	1.5097					
	σ^*	−.5459	−.3054	−.2002	−.1211	1.1722					
(1,10)	$\hat{\mu}$	.0071	.0276	.0403	.0501	.0585	.0661	.0733	.0803	.0872	1.5094
	$\hat{\sigma}$	−.2271	−.1573	−.1143	−.0810	−.0525	−.0267	−.0023	.0213	.0450	.5948
	μ^*	.0030	.0305	.0425	.0516	.0593	.0663	.0727	.0789	.0849	.5104
	σ^*	−.2414	−.1481	−.1071	−.0760	−.0496	−.0258	−.0035	.0180	.0393	.5940
(2,5)	$\hat{\mu}$		−.6126	−.1362	−.0416	1.7904					
	$\hat{\sigma}$		−1.1157	−.3318	−.1994	1.6469					
	μ^*		−.6260	−.1163	−.0356	1.7779					
	σ^*		−1.1359	−.3024	−.1890	1.6273					
(6,10)	$\hat{\mu}$						.3754	.0831	.0831	.0831	.3754
	$\hat{\sigma}$						−1.4571	−.0692	.0000	.0692	1.4571
	μ^*						.3769	.0820	.0821	.0820	.3769
	σ^*						−1.4613	−.0604	.0000	.0604	1.4614

Coefficients of $\hat{\mu}$, $\hat{\sigma}$ (2.15a, b) and the best linear estimates μ^*, σ^* when $n=15$.

Table 2

(u, v)	(1,5)	(1,10)	(1,15)	(2,5)	(4,9)	(6,10)
$\frac{v(\mu^*)}{v(\hat{\mu})}$	.9995	.9988	1.0000	.9996	.9989	.9995
$\frac{v(\sigma^*)}{v(\hat{\sigma})}$	.9994	.9955	.9990	.9996	.9972	.9997
JE	.9988	.9939	.9990	.9992	.9966	1.0000

Efficiencies of the linear estimates $\hat{\mu}$ and $\hat{\sigma}$ relative to the best linear estimates μ^* and σ^* from a normal distribution when $n=15$.

$$JE=\frac{v(\mu^*)\,v(\sigma^*)-\mathrm{Cov}\,(\mu^*, \sigma^*)}{v(\hat{\mu})\,v(\hat{\sigma})-\mathrm{Cov}\,(\hat{\mu}, \hat{\sigma})}$$

well-known sample mean $\sum_{i=1}^{n} x_i/n$, Ali and Chan, [1]).

2.3. *Alternative estimates*

Let ξ_i be the number satisfying $F(\xi_i)=i/(n+1)$. Also let

$$\tilde{S}_{1u}=1-(u-1)\gamma_1'(\xi_u)\,, \qquad \tilde{S}_{1v}=1-(n-v)\gamma_1'(-\xi_v)\,,$$

$$\tilde{S}_{2u}=2\xi_u-(u-1)\gamma_2'(\xi_u)\,, \qquad \tilde{S}_{2v}=2\xi_v+(n-v)\gamma_2'(-\xi_v)\,,$$

$$\tilde{S}_{1i}=1 \quad \text{and} \quad \tilde{S}_{2i}=2\xi_i\,, \qquad i=u+1,\cdots,v-1\,.$$

Now we state theorem 2 and prove the following lemmas.

THEOREM 2. *Let $\tilde{\mu}$ and $\tilde{\sigma}$ be the estimates obtained by solving the equations*

$$\sum\left(\frac{x_i-\mu}{\sigma}-\xi_i\right)\tilde{S}_{1i}=0\,, \qquad \sum\left(\frac{x_i-\mu}{\sigma}-\xi_i\right)\tilde{S}_{2i}=0$$

which are based on the relevant sample (2.2), *then $\tilde{\mu}$ and $\tilde{\sigma}$ are asymptotically unbiased and belong to the restricted class of BAN estimates.*

LEMMA 6. $\lim_{n\to\infty} L_t(\xi_u,\cdots,\xi_v)/\sqrt{n}=0, \quad t=1,2.$

PROOF. Let us consider the case $u=1$, $v<n$ and $t=2$ (when $t=1$, the consideration is similar). Since

$$\frac{L_2(\xi_1,\cdots,\xi_v)}{\sqrt{n}}=\sqrt{n}\left[-\frac{v}{n}+\frac{\sum\xi_i^2}{n}+\frac{(n-v)}{n}\cdot\frac{\xi_v f(\xi_v)}{1-F(\xi_v)}\right],$$

it is sufficient to prove that

$$\lim_{n\to\infty}\sqrt{n}\left[\int_0^\beta y^2\,dF-\frac{\sum\xi_i^2}{n}\right]=0\,,$$

$$\lim_{n\to\infty}\sqrt{n}\left[\int_0^\beta y^2\,dF-\frac{v}{n}+\frac{(n-v)}{n}\cdot\frac{\xi_v f(\xi_v)}{1-F(\xi_v)}\right]=0\,.$$

The first limit can be proved by Euler-MacLaurin's sum formula (cf. Cramér, [5], p. 124) and the second limit by noting that $\int_0^\beta y^2\,dF=\beta-\xi f(\xi)$.

The cases $u>1$, $v=n$, $t=1,2$ and the cases $u>1$, $v<n$, $t=1,2$ can be similarly proved.

For the case $u=1$, $v=n$, we have by the symmetry of $N(0,1)$

$$L_1(\xi_1,\cdots,\xi_n)/\sqrt{n}=\sum\xi_i/\sqrt{n}=0$$

and from (2.4) and (2.5) that

$$\lim_{n\to\infty} L_2(\xi_1, \cdots, \xi_n)/\sqrt{n} = \lim_{n\to\infty} \sqrt{n}\left(1-\frac{\sum \xi_i^2}{n}\right)=0\,.$$

LEMMA 7. *For* $0 \leqq \alpha < \beta \leqq 1$,

$$\lim_{n\to\infty} \sum \tilde{S}_{1i}/n = K_{11}(\alpha, \beta)\,, \qquad \lim_{n\to\infty} \sum \xi_i \tilde{S}_{1i}/n = K_{12}(\alpha, \beta)\,,$$

$$\lim_{n\to\infty} \sum \tilde{S}_{2i}/n = K_{21}(\alpha, \beta)\,, \qquad \lim_{n\to\infty} \sum \xi_i \tilde{S}_{2i}/n = K_{22}(\alpha, \beta)\,.$$

PROOF. This lemma can be proved by using Euler-MacLaurin's sum formula.

PROOF OF THEOREM 2. First we expand (2.3a, b) in Taylor's expansion around ξ_i, $i=u, \cdots, v$. Then following the similar procedure as given in theorem 1 we apply lemmas 1, 2, 4, 6 and 7.

Since ξ_i can always be obtained from the normal distribution table, $\tilde{\mu}$ and $\tilde{\sigma}$ may be useful when the μ_i's are not available.

It is of interest to notice that if the $\sum \tilde{S}_{ti}$ and $\sum \xi_i \tilde{S}_{ti}$ in $\tilde{\mu}$ and $\tilde{\sigma}$ are replaced by the corresponding limits $nK_{rt}(\alpha, \beta)$ and the additional discrete weights given to the sample quantiles x_u and x_v by the corresponding limits, $\tilde{\mu}$ and $\tilde{\sigma}$ become the estimates studied by Weiss [18] and Chernoff, etc., [4].

2.4. *The generalization to multiple censoring*

We see that following the similar arguments, theorems 1 and 2 can be generalized to the case where only the order statistics lying between (and including) the sample quantiles $x_{[n\alpha_i]+1}$ and $x_{[n\beta_i]+1}$ are available, where

$$1 \leqq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_k < \beta_k \leqq 1\,.$$

3. A discussion on Plackett's derivation

In his exploratory paper [14] Plackett has shown that under mild conditions the asymptotically linearized maximum likelihood estimates of μ_0 and σ_0 are asymptotically normal and asymptotically efficient for the double censoring $\alpha>0$, $\beta<1$. The forms of his estimates are the same as the $\hat{\mu}$ and $\hat{\sigma}$ in (2.13a, b). However, the present authors do not agree with the following important step in his proof.

We feel that he has used sample quantile theory on all the order statistics with ranks lying between (and including) $u=[n\alpha]$ and $v=[n\beta]$. For example, the following statement has been given in his paper (p. 138), "Typically, $i=[np]$, and as $n\to\infty$, $(\sqrt{n}/z)((y_i-\mu)/\sigma-t)$ is asymptotically normal.... According to (9), (t_i-t) is $0(n^{-1})$...." Since in the quantile theory the proportions p should be fixed in advance, the number of the p's, say k, is a fixed number independent of n. Usually we only know the k x_i's with $i=[np]$, i.e., the sample quantiles with orders p, are asymptotically normal. But in his derivation (cf. inequality (52), p. 138) it appears to us that he has considered all the x_i's with $u \leqq i \leqq v$ as sample quantiles (or at least each x_i is asymptotically normal). But this is impossible because the number of x_i's with $u \leqq i \leqq v$ tends to ∞ with n.

4. A remark on Dixon's estimate

For the symmetric censoring $\alpha=1-\beta$, $\mu_u=-\mu_v$. Then the μ in (2.13a) becomes

$$\mu=\frac{\sum x_i/(v-u+1)+[(S_{1v}-1)/(v-u+1)](x_u+x_v)}{1+2[(S_{1v}-1)/(v-u+1)]}$$

where $S_{1v}-1=(n-v)\frac{f}{1-F}\left(-y+\frac{f}{1-F}\right)y-\mu_v$ (if σ_0 is a known quantity, the estimate of μ_0 obtained by solving the first equation of theorem 1 has the same form as the $\hat{\mu}$ if the censoring is symmetric.) Since

$$\begin{aligned} T(y)&=\frac{f}{1-F}\left(-y+\frac{f}{1-F}\right) \\ &=y^2\left\{-\frac{f}{y(1-F)}+\left[\frac{f}{y(1-F)}\right]^2\right\}=\frac{1-y^{-1}F(y)}{[1-y^{-2}+yR(y)]^2}\,, \end{aligned}$$

where $|R(y)|<y^{-3}$ (cf. Kendall and Stuart, [12], p. 137), we have $\lim_{y\to\infty} T(y)=1$. This implies that $(S_{1v}-1)/(v-u+1)$ is approximately equal to $(n-v)/(v-u+1)$ if μ_v is sufficiently large. In such case $\hat{\mu}$ reduces to Dixon's simple estimate $\hat{\mu}_D=(ux_u+x_{u+1}+\cdots+x_{v-1}+ux_v)/n$, $u=n-v+1$, which has an efficiency never falling below 99.912% relative to the corresponding best linear estimate for all u when $n \leqq 20$. It is of interest to note that $T(y)$ is still close to 1 even for moderate values of y, e.g., $T(0.6)=0.747$, $T(0.9)=.789$, $T(2.0)=0.886$, $T(5.0)=.967$.

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