ON EXACT PROBABILITIES OF SOME GENERALIZED KOLMOGOROV'S D-STATISTICS

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1. Introduction

The paper gives a unified computational method for exact probabilities of the most generalized form of D-statistic, which was proposed by Kolmogorov for non-parametric tests of fit. First, we give a historical survey of the subject and in section 2, we state goodness-of-fit D-tests (based on some general bounds) constructing general acceptance and confidence regions, sizes of which are calculated in a distribution-free way. The method is also applied to calculation of the exact power of tests for a certain continuous alternative. A computational method for functional $\alpha_n(\cdot,\cdot)$ defined by (2.5) is presented in section 3.

Let $F(X)$ be any one-dimensional continuous cumulative distribution function and $F_n(X)$ the empirical distribution based on a sample of size n from F. Kolmogorov [34] introduced the statistic

$$D_n = \sup_{-\infty<x<\infty} |F_n(X)-F(X)|$$

and obtained the asymptotic result

$$\lim_{n\to\infty} d_n\left(\frac{\lambda}{\sqrt{n}}\right) = \sum_{j=-\infty}^{\infty} (-1)^j e^{-2j^2\lambda^2} \tag{1.1}$$

for the probability (distribution of D_n)

$$d_n(\varepsilon) = \Pr\{D_n<\varepsilon\} . \tag{1.2}$$

Subsequently, Smirnov [51], Feller [24], Doob [21], Chung [13], Donsker [20] and others gave improved proofs of the result (1.1). Smirnov [51], [53] proposed the statistic

$$D_n^- = \sup_{-\infty<x<\infty} \{F_n(x)-F(x)\}$$

for one-sided tests and showed that

$$\lim_{n\to\infty} d_n^-\left(\frac{\lambda}{\sqrt{n}}\right)=1-e^{-2\lambda^2}, \tag{1.3}$$

where

$$d_n^-(\varepsilon)=\Pr\{D_n^-<\varepsilon\}. \tag{1.4}$$

Independently of them, Wald and Wolfowitz [61] obtained expressions for exact probabilities of more general statistics than D_n or D_n^-. Further, an alternative expression of the probability (1.4) for finite n was given by Birnbaum and Tingey [5]:

$$d_n^-(\varepsilon)=1-\sum_{j=0}^{[n(1-\varepsilon)]} b(n, j; \Delta_j)\frac{\Delta_o}{\Delta_j}, \tag{1.5}$$

where

$$b(n, j; \Delta)=\binom{n}{j}\Delta^j(1-\Delta)^{n-j},$$

$$\Delta_j=\varepsilon+\frac{j}{n}, \qquad j=0, 1, \cdots, n.$$

Chang Li-chien [8] showed that when δ is a multiple of $1/n$,

$$r_n^+(c; \delta)=\Pr\left\{\sup_{0<F(x)\leqq\delta}\frac{F_n(x)}{F(x)}<c\right\} \tag{1.6}$$

$$=\begin{cases}\displaystyle\sum_{r=0}^{[nc\delta]} b(n, r; \delta) \\ \qquad \displaystyle -\sum_{k=1}^{[nc\delta]}\frac{1}{k}b\left(n, k; \frac{k}{nc}\right)\sum_{r=k}^{[nc\delta]} b\left(n-k, r-k; \frac{nc\delta-k}{nc-\delta}\right), & 0<c\leqq 1/\delta \\ \displaystyle 1-\frac{1}{c}, & c>1/\delta \\ 0, & c\leqq 0.\end{cases}$$

Alternative forms of (1.6) were given by Ishii [27], [28] and Csörgo [15] etc. One of them is as follows:

$$\begin{aligned} r_n^-(c; \delta) &= \Pr\left\{\sup_{\delta\leqq F(x)\leqq 1}\frac{F_n(x)}{F(x)}<c\right\} \\ &= \Pr\left\{\sup_{\delta\leqq F(x)\leqq 1}\frac{F_n(x)-F(x)}{F(x)}<c-1\right\} \\ &= \sum_{j=[n(1-c\delta)]+1}^{n} b(n, j; \alpha_j)\frac{\alpha_0}{\alpha_j}, \end{aligned} \tag{1.7}$$

where

$$\alpha_j = \frac{j}{nc} + \frac{c-1}{c}\,, \qquad j=0, 1, \cdots, n\,.$$

The limiting version of the relation (1.7) was initially proved by Rényi [47]:

$$\lim_{n\to\infty} r_n^-\left(1+\frac{c}{\sqrt{n}};\,\delta\right) = \Phi(c\sqrt{\delta/(1-\delta)})\,,$$

where

$$\Phi(x) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^x e^{-t^2/2}\,dt\,, & x \geqq 0 \\ 0\,, & x \leqq 0\,. \end{cases}$$

Another extension of (1.5) was given by Dempster [19] and Dwass [22], independently:

$$d_n(\varepsilon, \delta) = \Pr\left\{\sup_{F(x)>0} \frac{F_n(x)-\delta}{F(x)} \leqq \frac{1-\delta}{1-\varepsilon}\right\} \tag{1.8}$$
$$= 1 - \sum_{j=0}^{[n(1-\delta)]} b(n, j;\, m_j)\frac{m_0}{m_j}$$

where

$$m_j = \varepsilon + \frac{1-\varepsilon}{1-\delta}\,\frac{j}{n}\,, \qquad j=0, 1, \cdots, n\,.$$

Clearly,

$$d_n(\varepsilon, \varepsilon) = d_n^-(\varepsilon)\,, \qquad d_n(\varepsilon, 0) = r_n^+\left(\frac{1}{1-\varepsilon};\,1\right) = r_n^-\left(\frac{1}{1-\varepsilon};\,0\right).$$

On the other hand, the probability (1.2) for finite n was given by Massey [42] (when ε is a multiple of $1/n$) in a recurrence formula. An explicit expression for (1.2) was obtained by Chang Li-chien [9]:

$$d_n(\varepsilon) = 1 - \frac{2n!}{n^n}\sum_{j=1}^{[(1+\varepsilon)/2\varepsilon]}\ \sum_{m_1+\cdots+m_j=n}\ \prod_{i=1}^{j-1} w_{m_i}(2n\varepsilon)[w_{m_j}^*(n\varepsilon)] \tag{1.9}$$
$$+ \frac{n!}{n^n}\sum_{j=1}^{[1/2\varepsilon]}\ \sum_{m_1+\cdots+m_j=n}\ \prod_{i=1}^{j-1} w_{m_i}(2n\varepsilon)[w_{m_j}^*(n\varepsilon) + \sum_{l_1+l_2=m_j} w_{l_1}(n\varepsilon)w_{l_2}^*(n\varepsilon)]$$

where

$$w_m(\beta) = \sum_{l=0}^{[m-\beta]} \frac{(\beta^2-l)(m-\beta-l)^{m-l}}{(\beta+l)^2(m-l)!}\,, \qquad m \geqq \beta\,,$$
$$w_m^*(\beta) = \sum_{l=0}^{[m-\beta]} \frac{(\beta^2-\beta l)(m-\beta-l)^{m-l}}{(\beta+l)^2(m-l)!}\,, \qquad m \geqq \beta\,,$$

$$w_m(\beta)=w_m^*(\beta)=0\ , \qquad m<\beta\ .$$

Blackman [6] and Kemperman [31] also gave the exact expression for the probability, $\Pr\{-\varepsilon_1<F_n(x)-F(x)<\varepsilon_2;\ -\infty<x<\infty\}$, which is more complicated than (1.9) and is omitted to write down here.

Using the elegant idea of Doob [21] and Donsker [20], Anderson-Darling [1] proposed a method of attacking the limiting distribution of a more general type of D_n:

$$d_n(\varepsilon;\psi)=\Pr\{\sup_{-\infty<x<\infty}(|F_n(x)-F(x)|\cdot\psi[F(x)])<\varepsilon\} \tag{1.10}$$

where $\psi(t)$ $(\geqq 0)$ is a preassigned weight function. They also gave the explicit limiting distributions for some special forms of ψ, which include Kolmogorov's result (1.1) and a result given by Maniya [40].

2. Goodness-of-fit D-tests

2.1. General acceptance region and confidence region

Let $\mathscr{B}$ be the class of all functions β's which are defined on the interval $[0,1]$ and satisfy the following conditions:

(i) For all $0\leqq t\leqq 1$, $t\leqq\beta(t)\leqq 1$.

(ii) β is monotone non-decreasing and continuous to the right.

Then for every positive integer n, we can define

$$\mu_j=\mu_j^{(n)}[\beta]\equiv\min_{0\leqq t\leqq 1}\left\{t;\ \beta(t)\geqq\frac{j}{n}\right\}\ ,\qquad j=1,2,\cdots,n\ . \tag{2.1}$$

For each $\beta\in\mathscr{B}$, we shall define a conjugate function $\bar\beta$ by

$$\bar\beta(t)=1-\beta(1-t)\ ,\qquad 0\leqq t\leqq 1\ .$$

Since $\bar\beta$ is continuous to the left, we can define

$$\nu_j=\nu_j^{(n)}[\bar\beta]\equiv\max_{0\leqq t\leqq 1}\left\{t;\ \bar\beta(t)\leqq\frac{j-1}{n}\right\}\ ,\qquad j=1,2,\cdots,n\ . \tag{2.2}$$

Noticing the fact that

$$\nu_j^{(n)}[\bar\beta]=1-\mu_{n-j+1}^{(n)}[\beta]\ ,$$

we can easily prove the following.

LEMMA. *For any continuous F and $\beta_1,\beta_2\in\mathscr{B}$ and for some x^*, the following two relations are equivalent to each other:*

(iii) $\mu_j^{(n)}[\beta_1]\leqq F(x^*)\leqq\nu_j^{(n)}[\bar\beta_2]$,

(iv) $\beta_1(F(x^*)) \geqq \frac{j}{n}$ *and* $\bar{\beta}_2(F(x^*)) \leqq \frac{j-1}{n}$.

Let Γ be the class of all continuous cumulative distribution functions on the real line $R=(-\infty, \infty)$. Let $F_n(x)$ be the empirical distribution function of $F \in \Gamma$;

$$F_n(x)=F_n(x : \boldsymbol{X}_n)=\frac{1}{n}\{\text{no. of } i : X_i \leqq x,\ i=1, 2, \cdots, n\} ,$$

where $X_1, X_2, \cdots, X_n$ are independent random variables, each distributed according to F. By Γ_n, we shall denote the set of such F_n's for all F in Γ and for all samples of size n. Using $\beta_1, \beta_2 \in \mathscr{B}$, we can define the following acceptance region for each $F \in \Gamma$:

$$\begin{aligned}(2.3) \qquad A_n(F)&=A_n(F : \beta_1, \bar{\beta}_2)\\ &=\{G_n \in \Gamma_n : \bar{\beta}_2(F(x)) \leqq G_n(x) \leqq \beta_1(F(x)),\ -\infty<x<\infty\} .\end{aligned}$$

In the next place, in order to construct the confidence region, we shall define the following two non-decreasing functions:

$$\begin{aligned}L^{(n)}(x)&=L^{(n)}(x; \beta, \boldsymbol{X}_n^*)\\ &=\begin{cases}0 , & x<X_1^*\\ \mu_i^{(n)}[\beta] , & X_i^* \leqq x<X_{i+1}^* , \quad i=1, 2, \cdots, n-1\\ \mu_n^{(n)}[\beta] , & x \geqq X_n^* ,\end{cases}\\ U^{(n)}(x)&=U^{(n)}(x; \bar{\beta}, \boldsymbol{X}_n^*)=1-L^{(n)}(1-x; \beta, \tilde{\boldsymbol{X}}_n^*) ,\end{aligned}$$

where $\boldsymbol{X}_n^*=(X_1^*, X_2^*, \cdots, X_n^*)$ is the order statistic of $(X_1, X_2, \cdots, X_n)$ and $\tilde{\boldsymbol{X}}_n^*=(1-X_n^*, \cdots, 1-X_1^*)$. Put

$$\begin{aligned}(2.4) \quad R(F_n)&=R(F_n; \beta_1, \bar{\beta}_2)\\ &=\{G \in \Gamma : L^{(n)}(x; \beta_1, \boldsymbol{X}_n^*) \leqq G(x) \leqq U^{(n)}(x; \bar{\beta}_2, \boldsymbol{X}_n^*),\ -\infty<x<\infty\} .\end{aligned}$$

Then we have

THEOREM 1. *For any $F \in \Gamma$, $A_n(F)$ and $R(F_n)$ given by* (2.3) *and* (2.4), *are respectively the acceptance and confidence regions of size α, where*

$$(2.5) \qquad \alpha=\alpha_n(\beta_1, \bar{\beta}_2)=\Pr\{\mu_j^{(n)}[\beta_1] \leqq U_j^* \leqq \nu_j^{(n)}[\bar{\beta}_2],\ j=1, 2, \cdots, n\}$$

and $(U_1^*, \cdots, U_n^*)$ is an ordered sample of size n from the uniform distribution U on $(0, 1)$.

PROOF. Since $F_n(x)$, $L^{(n)}(x)$ and $U^{(n)}(x)$ increase only at the points X_i^*, the event $F_n \in A_n(F)$ and the event $F \in R(F_n)$ are expressed as

(2.6) $\beta_1(F(X_j^*)) \geqq \frac{j}{n}, \quad \bar{\beta}_2(F(X_j^*)) \leqq \frac{j-1}{n}, \quad j=1, \cdots, n,$

and

(2.7) $\mu_j^{(n)}[\beta_1] \leqq F(X_j^*) \leqq \nu_j^{(n)}[\bar{\beta}_2], \quad j=1, \cdots, n,$

respectively. Because of the lemma stated above, two events (2.6) and (2.7) are identical to each other and the probability of (2.7) is given by (2.5). This completes the proof.

Remark. All of the probabilities stated in section 1, are special types of the expression (2.5). For example, putting

$$\beta^0(t) = \beta^0(t;\varepsilon) = \max(t+\varepsilon, 1)$$

$$\beta^+(t) = \beta^+(t;c,\delta) = \begin{cases} ct & 0 \leqq t < \delta \\ 1 & \delta \leqq t \leqq 1, \end{cases}$$

$$\beta^-(t) = \beta^-(t;c,\delta) = \begin{cases} c & 0 \leqq t \leqq \delta \\ \min(ct, 1) & \delta \leqq t \leqq 1, \end{cases}$$

$$\beta^*(t) = \beta^*(t, \varepsilon\delta) = \min\left(\delta + \frac{1-\delta}{1-\varepsilon}t, 1\right),$$

we have

$$d_n(\varepsilon) = \alpha_n(\beta^0, \bar{\beta}^0), \qquad d_n^-(\varepsilon) = \alpha_n(\beta^0, \mathbf{0})$$

$$r_n^{\pm}(c;\delta) = \alpha_n(\beta^{\pm}, \mathbf{0}), \qquad d_n(\varepsilon, \delta) = \alpha_n(\beta^*, \mathbf{0})$$

where $\mathbf{0}$ means the function which identically equals 0. Wald-Wolfowitz [61] also considered a similar general situation. However, their assumption on the continuity of β and the requirement (f) given there, are slightly restrictive, because such case excludes the type (1.6) and some of the others.

2.2. Exact power of some tests

Next we shall consider the following non-parametric goodness-of-fit test. Let

(2.8) $\begin{cases} \text{hypothesis } H: & F_0 \\ \text{alternative } K: & F_1 = \varphi F \end{cases}$

where $F_0 \in \Gamma$ and φ is a function on (0, 1) which satisfies

(2.9) $\varphi(0)=0 \qquad \varphi(1)=1,$

(2.10) $\varphi(t)$ is strictly increasing and continuous.

By the assumption (2.10), $F_1 \in \Gamma$ and φ^{-1} can be defined. Furthermore, for any $\beta_1, \bar{\beta}_2 \in \mathscr{B}$, we have $\beta_1\varphi^{-1}$, $\bar{\beta}_2\varphi^{-1} \in \mathscr{B}$ and

$$A_n(F_0;\, \beta_1, \bar{\beta}_2) = A_n(F_1;\, \beta_1\varphi^{-1}, \bar{\beta}_2\varphi^{-1})$$

for $F_1 = \varphi F_0$, because of the definition (2.3). Thus we have

THEOREM 2. *In the testing situation described by* (2.8)$\sim$(2.10), *power of the test whose acceptance region is* $A_n(F_0;\, \beta_1, \bar{\beta}_2)$, *is given by*

$$P_\varphi(\beta_1, \bar{\beta}_2) = 1 - \alpha_n(\beta_1\varphi^{-1}, \bar{\beta}_2\varphi^{-1}) ,$$

where functional $\alpha_n(\cdot\,,\cdot)$ *is defined by* (2.5).

Massey [43], [44] gave an asymptotic result on the lower bound for the power of some test based on $d_n(\varepsilon)$ and compared it with the χ^2-test. Birnbaum [2], [3] showed a sharp lower (and upper) bound for the power of various tests based on $d_n^-(\varepsilon)$. Van der Waerden [60] made the comparison of power between the classical most powerful test and the one-sided D-test concerning the mean of some normal distribution. Further systematic comparison between various tests of fit was made by Chapman [11] and general discussion about the asymptotic power of the general D-tests, was recently given by Quade [46].

3. Computation of the probability $\alpha_n(\beta_1, \bar{\beta}_2)$

3.1. One-sided case

First, we shall define the following polynomials for each $\boldsymbol{\mu}_n = (\mu_1, \mu_2, \cdots, \mu_n)$:

(3.1) $$Q_0 = 1 ,$$

(3.2) $$Q_k = Q_k(\mu_1, \cdots, \mu_k) = -\sum_{i=0}^{k-1} \binom{k}{i} \mu_k^{k-i} Q_i , \qquad k = 1, 2, \cdots, n .$$

For example,

(3.3)
$$\begin{aligned} Q_1 &= Q_1(\mu_1) = -\mu_1 \\ Q_2 &= Q_2(\mu_1, \mu_2) = -\mu_2^2 + 2\mu_2\mu_1 = \mu_1^2 - (\mu_2 - \mu_1)^2 \\ Q_3 &= -\mu_1^3 - (\mu_3 - \mu_1)^3 + 3\mu_3(\mu_2 - \mu_1)^2 \\ Q_4 &= \mu_1^4 - (\mu_4 - \mu_1)^3 + 2\mu_4[2\mu_1^3 + 2(\mu_3 - \mu_1)^3 - 3(2\mu_2 - \mu_4)(\mu_2 - \mu_1)^2] . \end{aligned}$$

We note that for any $t > 0$, $k = 1, 2, \cdots, n$,

(3.4) $$t^k Q_k(\mu_1/t, \cdots, \mu_k/t) = Q_k(\mu_1, \cdots, \mu_k) .$$

THEOREM 3. *For any* $\beta \in \mathscr{B}$, *we have*

$$\alpha_n(\beta, 0) = f_n(\boldsymbol{\mu}_n[\beta]) , \tag{3.5}$$

where

$$f_n(\boldsymbol{\mu}_n) = f_n(\mu_1, \cdots, \mu_n) = \sum_{k=0}^{n} \binom{n}{k} Q_k(\mu_1, \cdots, \mu_k) \tag{3.6}$$

and Q_k*'s are the polynomials given by* (3.1) *and* (3.2).

PROOF. Since by (2.2) $\boldsymbol{\nu}_n[0] = (1, 1, \cdots, 1)$, it is sufficient to prove that

$$\Pr\{U_j^* \geqq \mu_j,\ j=1, 2, \cdots, n\} = f_n(\mu_1, \mu_2, \cdots, \mu_n) . \tag{3.7}$$

For $n=1$, by the definitions (3.1)∼(3.3) and (3.6) we have

$$f_1(\mu_1) = 1 - \mu_1 .$$

This shows that the relation (3.7) holds for $n=1$. Next we assume that (3.7) is true for some n. Let $U_1^*, \cdots, U_{n+1}^*$ be the order statistics from U. It is easily seen that given U_{n+1}^*, the conditional distribution of

$$U_1^*/U_{n+1}^*,\ U_2^*/U_{n+1}^*,\ \cdots,\ U_n^*/U_{n+1}^*$$

is that of the order statistics of n independent random variables, each distributed as U. From this fact, noticing the relation (3.4) we can easily verify that

$$\begin{aligned} \Pr\{U_1^* > \mu_1, \cdots, U_n^* > \mu_n \mid U_{n+1}^* = t\} \\ = f_n(\mu_1/t, \cdots, \mu_n/t) \\ = \sum_{k=0}^{n} \binom{n}{k} t^{-k} Q_k(\mu_1, \cdots, \mu_k) , \end{aligned} \tag{3.8}$$

using the assumption (3.7) for n. Using the fact that the density function of U_{n+1}^* is $(n+1)t^n$ on $I=[0, 1]$, we have, from (3.8) and (3.2),

$$\begin{aligned} &\Pr\{U_j^* \geqq \mu_j,\ j=1, 2, \cdots, n+1\} \\ &= (n+1)\int_{\mu_{n+1}}^{1} \Pr\{U_j^* > \mu_j,\ j=1, 2, \cdots, n \mid U_{n+1}^* = t\} t^n\, dt \\ &= (n+1)\sum_{k=0}^{n} \binom{n}{k} Q_k \int_{\mu_{n+1}}^{1} t^{n-k}\, dt \\ &= \sum_{k=0}^{n} \binom{n+1}{k} Q_k - \sum_{k=0}^{n} \binom{n+1}{k} \mu_{n+1}^{n+1-k} Q_k = \sum_{k=0}^{n+1} \binom{n+1}{k} Q_k . \end{aligned}$$

This completes the proof.

When $\beta = \mathbf{1}$, where $\mathbf{1}$ stands for such $\beta(t)$ that takes identically 1 on

I, we have $\alpha_n(\mathbf{1}, \bar{\beta})=\alpha_n(\beta, \mathbf{0})$. Therefore, such a case can also be treated by means of theorem 2.

3.2. Two-sided case

This case can be also reduced to the computation of one-sided case in the following manner.

For any $\beta_1, \beta_2 \in \mathcal{B}$, we have $\mu_j=\mu_j^{(n)}[\beta_1]$, $\nu_j=\nu_j^{(n)}[\bar{\beta}_2]$, $j=1, 2, \cdots, n$ by means of (2.1) and (2.2). Let $U_1^*, U_2^*, \cdots, U_n^*$ be the order statistics from U and define

$$\begin{aligned} E&=E[\boldsymbol{\mu}_n]=[U_j^*\geqq\mu_j,\ j=1, 2, \cdots, n] \\ E_j&=E[\boldsymbol{\mu}_n; \nu_j]=[U_j^*>\nu_j]\cap E\ , \qquad j=1, 2, \cdots, n_1\ , \end{aligned} \tag{3.9}$$

where n_1 is the largest integer such that $\nu_j<1$ in (2.2). Using theorem 2, we have

$$\begin{aligned} &\Pr\{E\}=f_n(\boldsymbol{\mu}_n) \\ &\Pr\{E_j\}=f_n(\nu_j\cdot\boldsymbol{\mu}_n)\equiv b_j\ , \end{aligned} \tag{3.10}$$

where

$$\nu_j\cdot\boldsymbol{\mu}_n=(\mu_1, \cdots, \mu_{j-1}, \mu_j\vee\nu_j, \mu_{j+1}\vee\nu_j, \cdots, \mu_n\vee\nu_j)\ ,$$
$$a\vee b=\max[a, b]\ .$$

Since E_j is a proper subset of E, every

$$E_j'=E-E_j$$

is not empty (otherwise probability (2.5) becomes 0). Put

$$\begin{aligned} &F\equiv\bigcap_{j=1}^{n_1} E_j'=[\mu_j\leqq U_j^*\leqq\nu_j,\ j=1, 2, \cdots, n]\ , \\ &F_j=\begin{cases} E_1\ , & j=1 \\ \left[\bigcap_{i=1}^{j-1} E_i'\right]\cap E_j\ , & j=2, \cdots, n_1\ . \end{cases} \end{aligned} \tag{3.11}$$

Then, it is easily seen that $F_1, F_2, \cdots, F_{n_1}$ and F are mutually exclusive events and that

$$E_j\subset\bigcup_{i=1}^{j} F_i\ , \qquad E=\bigcup_{j=1}^{n_1} F_j\cup F\ .$$

Accordingly,

$$\Pr\{E_j\}=\sum_{i=1}^{j}\Pr\{F_i\}\Pr\{E_j\,|\,F_i\}\ , \qquad j=1, \cdots, n_1\ , \tag{3.12}$$

$$\text{(3.13)} \qquad \Pr\{F\} = \Pr\{E\} - \sum_{i=1}^{n_1} \Pr\{F_j\} .$$

Let us put

$$G_1 = [U_1^* > \nu_1] ,$$

$$G_i = [U_{i-1}^* < \nu_i < U_i^*] , \qquad i = 2, \cdots, n_1 .$$

Noticing that the order statistics $U_1^*, U_2^*, \cdots, U_n^*$ are drawn from the uniform distribution, it is easily seen that for $1 \leqq i \leqq j \leqq n_1$,

$$\text{(3.14)} \qquad \Pr\{E_j \mid F_i\} = \Pr\{E_j \mid G_i\} .$$

Furthermore, for $i = 1, \cdots, n_1$, putting

$$\mu_j[\nu_i] = \max\left\{0, \frac{\mu_{j+i-1} - \nu_i}{1 - \nu_i}\right\} , \qquad j = 1, \cdots, n-i+1 ,$$

$$\nu_j[\nu_i] = \max\left\{0, \frac{\nu_{j+i-1} - \nu_i}{1 - \nu_i}\right\} , \qquad j = 1, \cdots, n-i+1 ,$$

$$\boldsymbol{\mu}_{n-i+1}[\nu_i] = (\mu_1[\nu_i], \cdots, \mu_{n-i+1}[\nu_i]) ,$$

we have, for $1 \leqq i \leqq j \leqq n_1$,

$$\text{(3.15)} \qquad \Pr\{E_j \mid G_i\} = \Pr\{E[\boldsymbol{\mu}_{n-i+1}[\nu_i]; \nu_{j-i+1}[\nu_i]]\} ,$$

where $E[\cdot\, ; \cdot]$ has been defined in general by (3.9). Thus, combining (3.14), (3.15) with theorem 3, we obtain

$$\text{(3.16)} \qquad \Pr\{E_j \mid F_i\} = f_{n-i+1}(\nu_{j-i+1}[\nu_i] \cdot \boldsymbol{\mu}_{n-i+1}[\nu_i]) \equiv b_{j,i} .$$

Therefore, for $j = 1, 2, \cdots, n_1$, defining

$$\text{(3.17)} \qquad a_j = \sum_{i=1}^{j} \Pr\{F_i\} , \qquad c_{j,i} = \begin{cases} b_{j,i} - b_{j,i+1} & \text{for } 1 \leqq i \leqq j \\ b_{j,j} & \text{for } i = j , \end{cases}$$

we can write down the relations (3.12) as

$$\text{(3.18)} \qquad b_j = \sum_{i=1}^{j} c_{j,i} a_i \qquad j = 1, 2, \cdots, n_1 .$$

By means of the matrix notations

$$\boldsymbol{a} = (a_1, a_2, \cdots, a_{n_1})'$$

$$\boldsymbol{b} = (b_1, b_2, \cdots, b_{n_1})'$$

$$C=\begin{bmatrix} c_{11} & 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & 0 & \cdots & 0 \\ c_{31} & c_{32} & c_{33} & \cdots & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n_1,1} & c_{n_1,2} & c_{n_1,3} & \cdots & c_{n_1n_1} \end{bmatrix}, \tag{3.19}$$

the relations (3.18) are expressed as

$$\boldsymbol{b}=C\boldsymbol{a} .$$

Since the triangular matrix C is non-singular,

$$\boldsymbol{a}=C^{-1}\boldsymbol{b} .$$

Combining the relation (3.13) with the above consideration, we can obtain the following;

THEOREM 4. *For any pair* $\beta_1, \beta_2 \in \mathscr{B}$, *we have*

$$\alpha_n(\beta_1, \bar{\beta}_2)=f_n(\boldsymbol{\mu}_n)-\sum_{j=1}^{n_1} b_j c_j^* ,$$

where c_j^* *is the* (n_1, j)*th element of the inverse matrix* C^{-1} *of* C *given by* (3.19).

Wald-Wolfowitz [61] also gave a computational method for less general $\alpha_n(\beta_1, \bar{\beta}_2)$. But their method is not practical, since it needs computation of some definite integrals. While, they proposed an approximate formula of $\alpha_n(\beta_1, \bar{\beta}_2)$ using the one-sided case's $\alpha_n(\beta_1, \boldsymbol{0})$, $\alpha_n(\boldsymbol{1}, \bar{\beta}_2)$, which include no integration.

On the other hand, various approximations for finite n by their limiting versions were given by many authors ([35], [8], [9], [6], [17]). The general result by Darling [17] is the following:

For $a>0$, $b>0$, when $n\to\infty$,

$$\delta_n(a, b)=\delta(a, b)+\frac{1}{6\sqrt{n}}\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right)\delta(a, b)+O\left(\frac{1}{n}\right) ,$$

where

$$\delta_n(a, b)=\Pr\left\{-\frac{a}{\sqrt{n}}<F_n(x)-F(x)<\frac{b}{\sqrt{n}},\ \forall x\in R\right\} ,$$

$$\begin{aligned}\delta(a, b)&=\lim_{n\to\infty}\delta_n(a, b)\\ &=1+\sum_{j=1}^{\infty}\{2e^{-2j^2(a+b)^2}-e^{-2[ja+(j-1)b]^2}-e^{-2[(j-1)a+jb]^2}\} .\end{aligned}$$

Other approximations for the value $d_n^-(\varepsilon)$ were given by Butler and McCarty [7], Whittle [63] etc. For example, the inequalities by Whittle are :

$$d_n^-(\varepsilon)<1-(1-\varepsilon)^{2n\varepsilon+2}<1-(1-\varepsilon)^2 e^{-2n\varepsilon^2-n\varepsilon^3/(1-\varepsilon)} \qquad 0\leqq\varepsilon\leqq 1 ,$$

$$1-d_n^-(\varepsilon)<e^{-2n\varepsilon^2+3.83n[\varepsilon/(1-\varepsilon)]^3} \qquad 0\leqq\varepsilon\leqq 0.31 .$$

4. Complementary remarks

4.1. We are now preparing some numerical results [58], which show that our method described in section 3 is of a form simple enough for practical use. The author [57] also intends to summarize the statistics used in non-parametric tests of fit.

4.2. Smirnov [52] showed that the limiting distribution of the modified D-statistic

$$D_{mn}=\sup|F_1(x)-F_2(x)| , \tag{4.1}$$

for two-sample problem, coincides with that of D_n under the null hypothes. The exact distribution of (4.1) for finite sample size is found in [45], [36] etc. The extension to the multi-sample case was made by David [18], Kiefer [32] or others. Recently, Conover [14] has summarized results on this subject.

4.3. As a competitor of the D-statistic, Smirnov [49], [50] also considered an asymptotic expression for the distribution of the statistic

$$\omega_n^2=n\int_{-\infty}^{\infty}[F_n(x)-F(x)]^2 dF(x) . \tag{4.2}$$

Anderson-Darling [1] proposed a generalization of (4.2),

$$W_n^2=n\int_{-\infty}^{\infty}[F_n(x)-F(x)]^2\psi(F(x))\,dF(x) ,$$

for some special weight function ψ. But they have not given any asymptotic result. The exact distribution of (4.2) for $n=1, 2, 3$ was shown by Marshall [41]. Kuiper [39] modified the D-statistic as the statistic

$$V_n=\sup\{F_n(x)-F(x)\}-\inf\{F_n(x)-F(x)\} , \tag{4.3}$$

which is suitable for observations on a circle, and he gave some asymptotic result concerning the distribution of (4.3). The same type of statistic as (4.3) corresponding to the ω_n^2-statistic,

$$U_n^2=n\int_{-\infty}^{\infty}\left\{F_n(x)-F(x)-\int_{-\infty}^{\infty}[F_n(y)-F(y)]\,dF(y)\right\}^2 dF(x) \tag{4.4}$$

was introduced by Watson [62], who gave the asymptotic distribution of (4.4). Various results (exact distribution, moments, lower tails, etc.) for small sample size n, were recently given by Stephens [55], [56]. Further references of these subjects are seen in the expository papers [16], [26].

4.4. Other various statistics which are used for some goodness-of-fit tests and based on the empirical data alone, have been considered. Kac [29] showed that as $n \to \infty$, the random variable

$$M_n = \int_{F_n(x) > F(x)} dF(x)$$

has the asymptotic distribution U, the uniform distribution on I. Later on, the fact that M_n is distributed as U for each n, was proved by Gnedenko-Mihalevič [25], Birnbaum-Pyke [4], Dwass [22], Kuiper [38] or others. Birnbaum-Pyke [4] or Dwass [22] showed that the distribution of

$$T_n = \inf \{F(t);\ F_n(t) - F(t) = D_n^-\}$$

is also U. The asymptotic or exact analysis of the numbers of piercing points and non-negative jump points was investigated by Chang Li-chien [10], Cheng Ping [12] or others. Further systematic treatment of these subjects was made by Darling [17] and recently by Takács [59].

4.5. From a probability-theoretic point of view, it is interesting to note the limit theorems related to the D-statistic. As for this topic, see the recent expository paper [30].

4.6. There are many unsolved problems with respect to the D-statistic. For example, (i) extension to multi-variate case (see [33]), (ii) modification to discontinuous case (see [48]), (iii) selection of the optimal pair among the general bounds $(\beta_1, \bar{\beta}_2)$ for some test problem, (iv) proposition of another new statistic with practical utility and so on.

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