ON THE NULL-DISTRIBUTION OF THE F-STATISTICS FOR TESTING A 'PARTIAL' NULL-HYPOTHESIS IN A RANDOMIZED PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN WITH m ASSOCIATE CLASSES UNDER THE NEYMAN MODEL

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0. Summary and Introduction

In a previous paper [1], it has been shown that for a partially balanced incomplete block design with two associate classes the null-distribution of the F-statistic under the 'total' null-hypothesis (i.e., treatment-effects being all equal to zero) can be approximated by the familiar central F-distribution even under the Neyman model (i.e., an intra-block analysis model with both the unit errors and the technical errors), if it is randomized. As was announced in that paper, the approximate distributions of the F-statistic under the 'partial' null-hypothesis have been left to further discussion. In the present article, the authors are concerned with this problem.

They set forth the problem for a partially balanced incomplete block design with m associate classes and consider the null-distribution of the F-statistic for testing a 'partial' null-hypothesis, so that it includes the 'total' null-hypothesis as a special case and they reached the conclusion that the null-distribution of the F-statistic can be approximated, after the randomization, by a certain central F-distribution with appropriate degrees of freedom, if certain uniformity conditions are imposed on the unit errors and the number b of the blocks is sufficiently large.

In section 1 the spectral decomposition of the matrix NN', where N being the incidence matrix of the design under consideration, is given and this is useful for the later discussions.

The null-distribution of the F-statistic for testing a partial null-hypothesis before the randomization under the Neyman model is presented in section 2, and this turns out to be a non-central F-distribution whose non-centrality parameter depends upon the quantities $\bar{\theta}$ and $\bar{\bar{\theta}}$ both being the quadratic forms of the unit errors.

In section 3, the means, variances and the covariance of $\bar{\theta}$ and $\bar{\bar{\theta}}$ with respect to the permutation due to the randomization are calculated, and in section 4, it is shown that the permutation distribution of $(\bar{\theta}, \bar{\bar{\theta}})$ can be approximated by a certain two-dimensional continuous distribution, if the number of the blocks is sufficiently large and certain uniformity conditions on the within-block variances of the unit errors are satisfied.

Finally in section 5, it is shown that the null-distribution of the F-statistic after the randomization can be approximated by a central F-distribution with appropriate degrees of freedom, provided the two conditions mentioned above are satisfied.

1. Spectral decomposition of the matrix NN'

We shall be concerned with a partially balanced incomplete block design with m associate classes which has v treatments with the association, b blocks of size k each, r replications of each treatment, and the number of incidence of any pair of treatments λ_u if they are u th associates.

As for the definition of a partially balanced incomplete block design with m associate classes and related notations, references should be made to [2] and [3].

Let the association matrices be $\boldsymbol{A}_0=\boldsymbol{I}_v, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_m$ and let their regular representations be $\mathcal{P}_0=\boldsymbol{I}_m, \mathcal{P}_1, \cdots, \mathcal{P}_m$ respectively, where

$$\mathcal{P}_u=\begin{Vmatrix} p_{0u}^0 & p_{0u}^1 & \cdots & p_{0u}^m \\ p_{1u}^0 & p_{1u}^1 & \cdots & p_{1u}^m \\ \cdots & \cdots & \cdots & \cdots \\ p_{mu}^0 & p_{mu}^1 & \cdots & p_{mu}^m \end{Vmatrix}, \qquad u=0, 1, \cdots, m.$$

Let the characteristic roots of $\mathcal{P}_u$ be $z_{0u}=n_u, z_{1u}, \cdots, z_{mu}$, then there exists a non-singular matrix

$$\boldsymbol{C}=\begin{Vmatrix} c_{00} & c_{01} & \cdots & c_{0m} \\ c_{10} & c_{11} & \cdots & c_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m0} & c_{m1} & \cdots & c_{mm} \end{Vmatrix}$$

such that

$$\boldsymbol{C}\mathcal{P}_u\boldsymbol{C}^{-1}=\begin{Vmatrix} z_{0u} & & & 0 \\ & z_{1u} & & \\ & & \ddots & \\ 0 & & & z_{mu} \end{Vmatrix}, \qquad u=0, 1, \cdots, m \tag{1.1}$$

simultaneously. It should be noted that

$$n_u > z_{iu}, \qquad i=1,\cdots,m; \quad u=1,\cdots,m,$$

and by the relations

$$\sum_{k=0}^{m} p_{uu'}^{k} z_{ik} = z_{iu} z_{iu'},$$

one may put

$$c_{ui} = z_{ui}/n_i, \qquad i, u = 0, 1, \cdots, m. \tag{1.2}$$

$m+1$ orthogonal idempotents of the association algebra generated by the association matrices over the field of all real numbers are given by

$$A_u^{\sharp} = \left(\sum_{i=0}^{m} c_{ui} z_{ui}\right)^{-1} \sum_{i=0}^{m} c_{ui} A_i, \qquad u=0, 1, \cdots, m \tag{1.3}$$

with respective ranks $\alpha_0=1, \alpha_1 \cdots, \alpha_m$. By taking the trace of the matrix $A_u^{\sharp} A_{u'}^{\sharp}$, we get the relations

$$\sum_{i=0}^{m} \frac{z_{ui} z_{u'i}}{n_i} = \delta_{uu'} \frac{v}{\alpha_u}, \qquad u, u'=0, 1, \cdots, m, \tag{1.4}$$

where $\delta_{uu'}$ denotes the Kronecker delta. It is also noted that

$$A_0^{\sharp} = \frac{1}{v} G_v \quad \text{and} \quad \sum_{u=0}^{m} A_u^{\sharp} = I_v. \tag{1.5}$$

If we denote the incidence matrix of the design under consideration by N, then the spectral decomposition of the matrix NN' is given by

$$NN' = \sum_{u=0}^{m} \rho_u A_u^{\sharp}, \tag{1.6}$$

where $\rho_0 = rk, \rho_1, \cdots, \rho_m$ are the characteristic roots of NN' with multiplicities $\alpha_0, \alpha_1, \cdots, \alpha_m$ respectively and are given by

$$\rho_u = \sum_{i=0}^{m} \lambda_i z_{ui}, \qquad u=0, 1, \cdots, m, \tag{1.7}$$

where we have put $\lambda_0 = r$.

2. The null-distribution of the F-statistic before the randomization for testing a partial null-hypothesis under the Neyman model

As for the notations being used in this section, references should be made to [2] and [3].

Let the incidence matrices of treatments and blocks be Φ and Ψ respectively, then the Neyman model assuming no interaction between the treatment and the block is given by

$$x=\gamma I+\Phi\tau+\Psi\beta+\pi+e, \tag{2.1}$$

where $x'=(x_1,\cdots,x_n)$ is the observation vector, $\tau'=(\tau_1,\cdots,\tau_v)$ and $\beta'=(\beta_1,\cdots,\beta_b)$ are treatment-effects and block-effects being subjected to the restrictions

$$\tau_1+\cdots+\tau_v=0 \quad \text{and} \quad \beta_1+\cdots+\beta_b=0$$

respectively, and $\pi'=(\pi_1,\cdots,\pi_n)$ stands for the unit errors being subjected to the restriction

$$\Psi'\pi=0.$$

Finally, $e'=(e_1,\cdots,e_n)$ is the technical error vector being distributed as $N(0',\sigma^2 I)$.

Sums of squares due to treatments adjusted and errors are given by

$$\begin{aligned} S_t^2&=x'(V_1^\sharp+\cdots+V_m^\sharp)x, \\ S_e^2&=x'\left(I-\frac{1}{k}B-V_1^\sharp-\cdots-V_m^\sharp\right)x, \end{aligned} \tag{2.2}$$

respectively, where

$$V_u^\sharp=c_u\left(I-\frac{1}{k}B\right)T_u^\sharp\left(I-\frac{1}{k}B\right), \qquad u=1,\cdots,m \tag{2.3}$$

with

$$T_u^\sharp=\Phi A_u^\sharp\Phi', \qquad B=\Psi\Psi'$$

and

$$c_u=\frac{k}{rk-\rho_u}, \qquad u=1,\cdots,m. \tag{2.4}$$

Now, we are interested in testing a partial null-hypothesis that some of the hypotheses $A_u^\sharp\tau=0$, $u=1,\cdots,m$ are true. We can take, without any loss of generality, the null-hypothesis

$$H_{0(h)}:\ A_u^\sharp\tau=0, \qquad u=1,\cdots,h, \tag{2.5}$$

where h is a positive integer not greater than m. Clearly, this hypothesis is equivalent to $\sum_{u=1}^{h} A_u^\sharp\tau=0$, and when $h=m$ this reduces to the total null-hypothesis H_0: $\tau=0$.

To test the null-hypothesis $H_{0(h)}$, we consider the partial sum of squares

$$S_{t(h)}^2=x'(V_1^\sharp+\cdots+V_h^\sharp)x \tag{2.6}$$

instead of S_t^2 given by (2.2). Then it follows from (1.5), (2.1) and (2.3), that

$$S_{t(h)}^2=\boldsymbol{\pi}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^\sharp+\cdots+c_h\boldsymbol{A}_h^\sharp)\boldsymbol{\Phi}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{\pi}$$
$$+2\boldsymbol{\pi}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^\sharp+\cdots+c_h\boldsymbol{A}_h^\sharp)\boldsymbol{\Phi}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{e}$$
$$+\boldsymbol{e}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^\sharp+\cdots+c_h\boldsymbol{A}_h^\sharp)\boldsymbol{\Phi}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{e}, \tag{2.7}$$

provided the null-hypothesis $H_{0(h)}$ is true.

Hence the null-distribution of the variate

$$\chi_1^2=S_{t(h)}^2/\sigma^2$$

before the randomization is the non-central chi-square distribution of the degrees of freedom

$$\bar{\alpha}=\alpha_1+\cdots+\alpha_h \tag{2.8}$$

with the non-centrality parameter

$$\bar{K}_1=\boldsymbol{\pi}'\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^\sharp+\cdots+c_h\boldsymbol{A}_h^\sharp)\boldsymbol{\Phi}'\boldsymbol{\pi}/\sigma^2. \tag{2.9}$$

Whence its probability element is given by

$$\exp\Big(-\frac{\bar{K}_1}{2}\Big)\sum_{\mu=0}^{\infty}\frac{(K_1/2)^\mu}{\mu!}\frac{(\chi_1^2/2)^{\bar{\alpha}/2+\mu-1}}{\Gamma(\bar{\alpha}/2+\mu)}\exp\Big(-\frac{\chi_1^2}{2}\Big)d\Big(\frac{\chi_1^2}{2}\Big). \tag{2.10}$$

The sum of squares due to error, S_e^2, given by (2.2), becomes

$$S_e^2=\boldsymbol{\pi}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)[\boldsymbol{I}-\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^\sharp+\cdots+c_m\boldsymbol{A}_m^\sharp)\boldsymbol{\Phi}']\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{\pi}$$
$$+2\boldsymbol{\pi}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)[\boldsymbol{I}-\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^\sharp+\cdots+c_m\boldsymbol{A}_m^\sharp)\boldsymbol{\Phi}']\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{e}$$
$$+\boldsymbol{e}'\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)[\boldsymbol{I}-\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^\sharp+\cdots+c_m\boldsymbol{A}_m^\sharp)\boldsymbol{\Phi}']\Big(\boldsymbol{I}-\frac{1}{k}\boldsymbol{B}\Big)\boldsymbol{e} \tag{2.11}$$

independently of the null-hypothesis.

The distribution of the variate

$$\chi_2^2=S_e^2/\sigma^2$$

before the randomization is the non-central chi-square distribution of degrees of freedom $n-b-v+1$ with the non-centrality parameter

$$\begin{aligned} K_2&=\boldsymbol{\pi}[\boldsymbol{I}-\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^\sharp+\cdots+c_m\boldsymbol{A}_m^\sharp)\boldsymbol{\Phi}']\boldsymbol{\pi}/\sigma^2\\ &=\Delta/\sigma^2-\bar{K}_1-\bar{\bar{K}}_1, \end{aligned} \tag{2.12}$$

where $\Delta=\boldsymbol{\pi}'\boldsymbol{\pi}$ and

$$\bar{\bar{K}}_1=\boldsymbol{\pi}'\boldsymbol{\Phi}(c_{h+1}\boldsymbol{A}^{*}_{h+1}+\cdots+c_m\boldsymbol{A}^{*}_{m})\boldsymbol{\Phi}\boldsymbol{\pi}/\sigma^2. \tag{2.13}$$

The probability element of the variate χ_2^2 is given by

$$\exp\left(-\frac{K_2}{2}\right)\sum_{\nu=0}^{\infty}\frac{(K_2/2)^{\nu}}{\nu!}\frac{(\chi_2^2/2)^{(n-b-v+1)/2+\nu-1}}{\Gamma((n-b-v+1)/2+\nu)}\exp\left(-\frac{\chi_2^2}{2}\right)d\left(\frac{\chi_2^2}{2}\right). \tag{2.14}$$

Since χ_1^2 and χ_2^2 are mutually independent in the stochastic sense, the null-distribution of the F-statistic

$$F=\frac{n-b-v+1}{\bar{\alpha}}\frac{S^2_{t(h)}}{S_e^2} \tag{2.15}$$

before the randomization is the non-central F-distribution, whose probability element is given by

$$\begin{aligned}&\exp(-(\bar{K}_1+K_2)/2)\sum_{\mu,\nu=0}^{\infty}\frac{(\bar{K}_1/2)^{\mu}(K_2/2)^{\nu}}{\mu!\nu!}\frac{\Gamma((n-b-\bar{\bar{\alpha}})/2+\mu+\nu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2+\nu)}\\&\cdot\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2+\mu-1}\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-(n-b-\bar{\alpha})/2-\mu-\nu}\\&\cdot d\left(\frac{\bar{\alpha}}{n-b-v+1}F\right),\end{aligned} \tag{2.16}$$

where

$$\bar{\bar{\alpha}}=\alpha_{h+1}+\cdots+\alpha_m. \tag{2.17}$$

If we put

$$\bar{\theta}=\Delta^{-1}\boldsymbol{\pi}'\boldsymbol{\Phi}(c_1\boldsymbol{A}^{*}_1+\cdots+c_h\boldsymbol{A}^{*}_h)\boldsymbol{\Phi}'\boldsymbol{\pi}, \tag{2.18}$$

and

$$\bar{\bar{\theta}}=\Delta^{-1}\boldsymbol{\pi}'\boldsymbol{\Phi}(c_{h+1}\boldsymbol{A}^{*}_{h+1}+\cdots+c_m\boldsymbol{A}^{*}_m)\boldsymbol{\Phi}'\boldsymbol{\pi}, \tag{2.19}$$

then the probability element given by (2.16) may be rewritten as

$$\begin{aligned}&\exp(-\Delta/2\sigma^2)\sum_{l=0}^{\infty}\frac{(\Delta/2\sigma^2)^l}{l!}\sum_{\mu+\nu+\gamma=l}\frac{l!}{\mu!\nu!\gamma!}\bar{\theta}^{\mu}\bar{\bar{\theta}}^{\gamma}(1-\bar{\theta}-\bar{\bar{\theta}})^{\nu}\\&\cdot\frac{\Gamma((n-b-\bar{\bar{\alpha}})/2+\mu+\nu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2+\nu)}\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2+\mu-1}\\&\cdot\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-(n-b-\bar{\bar{\alpha}})/2-\mu-\nu}d\left(\frac{\bar{\alpha}}{n-b-v+1}F\right).\end{aligned} \tag{2.20}$$

The null-distribution of the F-statistic after the randomization should be obtained as follows:

$$(2.21)\qquad \frac{\Gamma((n-b-\bar{\bar{\alpha}})/2)}{\Gamma(\bar{\alpha}/2)((n-b-v+1)/2)}\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2-1}$$

$$\cdot\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-(n-b-\bar{\bar{\alpha}})/2}d\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)$$

$$\cdot\ \exp(-\Delta/2\sigma^2)\sum_{l=0}^{\infty}\frac{(\Delta/2\sigma^2)^l}{l!}\sum_{\mu+\nu+\gamma=l}\frac{l!}{\mu!\nu!\gamma!}\mathcal{E}[\bar{\theta}^{\mu}\bar{\bar{\theta}}^{\gamma}(1-\bar{\theta}-\bar{\bar{\theta}})^{\nu}]$$

$$\cdot\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-(\mu+\nu)}\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\mu}$$

$$\cdot\frac{\Gamma((n-b-\bar{\alpha})/2+\mu+\nu)\Gamma(\bar{\alpha}/2)\Gamma((n-b-v+1)/2)}{\Gamma(\bar{\alpha}/2+\mu)((n-b-v+1)/2+\nu)((n-b-\bar{\alpha})/2)}$$

where the operator $\mathcal{E}$ stands for the expectation with respect to the permutation distribution of $(\bar{\theta}, \bar{\bar{\theta}})$ due to randomization. Thus our task has been reduced to the calculations of the expected value $\mathcal{E}[\bar{\theta}^{\mu}\bar{\bar{\theta}}^{\gamma}(1-\bar{\theta}-\bar{\bar{\theta}})^{\nu}]$ for $\mu+\nu+\gamma=l$.

3. The calculations of the means and variances of the quantities $\bar{\theta}$ and $\bar{\bar{\theta}}$ with respect to the permutation distribution due to the randomization

3.1. *Necessary notations*

Let us put

$$(3.1)\qquad \begin{aligned}\bar{\boldsymbol{T}}^{*}&=c_1\boldsymbol{T}_1^{*}+\cdots+c_h\boldsymbol{T}_h^{*}=\boldsymbol{\Phi}(c_1\boldsymbol{A}_1^{*}+\cdots+c_h\boldsymbol{A}_h^{*})\boldsymbol{\Phi}',\\ \bar{\bar{\boldsymbol{T}}}^{*}&=c_{h+1}\boldsymbol{T}_{h+1}^{*}+\cdots+c_m\boldsymbol{T}_m^{*}=\boldsymbol{\Phi}(c_{h+1}\boldsymbol{A}_{h+1}^{*}+\cdots+c_m\boldsymbol{A}_m^{*})\boldsymbol{\Phi}',\end{aligned}$$

then, by (1.3), we get

$$(3.2)\qquad \begin{aligned}\bar{\boldsymbol{T}}^{*}&=\bar{\mu}_0\boldsymbol{T}_0+\bar{\mu}_1\boldsymbol{T}_1+\cdots+\bar{\mu}_m\boldsymbol{T}_m,\\ \bar{\bar{\boldsymbol{T}}}^{*}&=\bar{\bar{\mu}}_0\boldsymbol{T}_0+\bar{\bar{\mu}}_1\boldsymbol{T}_1+\cdots+\bar{\bar{\mu}}_m\boldsymbol{T}_m,\end{aligned}$$

where

$$(3.3)\qquad \bar{\mu}_u=\sum_{i=1}^{h}\mu_{iu}\quad\text{and}\quad \bar{\bar{\mu}}_u=\sum_{i=h+1}^{m}\mu_{iu},\qquad u=0,1,\cdots,m,$$

with

$$\mu_{iu}=\alpha_i c_i z_{iu}/(vn_i), \qquad i=1,\cdots,m, \qquad u=0,1,\cdots,m.$$

Numbering the whole units from 1 through n in such a way the ith unit in the pth block bears the number $f=(p-1)k+i$, let us put

$$\boldsymbol{T}_u=\|\boldsymbol{T}_{pq}^{(u)}\|_{(p,q=1,\cdots,b)}, \quad \boldsymbol{T}_{pq}^{(u)}=\|t_{ij}^{(u)pq}\| \qquad (i,j=1,\cdots,k),$$

where

$$t_{ij}^{(u)pq}=\begin{cases}1, & \text{if the } f=(p-1)k+i\text{th and } f'=(q-1)k+j\text{th units receive treatments which are } u\text{th associates,}\\ 0, & \text{otherwise.}\end{cases}$$

Clearly

$$t_{ij}^{(u)pq}=t_{ji}^{(u)pq}, \quad t_{ij}^{(0)pp}=\delta_{ij} \quad \text{and} \quad t_{ij}^{(0)pq}+t_{ij}^{(1)pq}+\cdots+t_{ij}^{(m)pq}=1.$$

If we put

$$\bar{\boldsymbol{T}}^*=\|\bar{\boldsymbol{T}}_{pq}^*\|, \qquad \bar{\boldsymbol{T}}_{pq}^*=\|\bar{t}_{ij}^{*pq}\|$$

and

$$\bar{\bar{\boldsymbol{T}}}^*=\|\bar{\bar{\boldsymbol{T}}}_{pq}^*\|, \qquad \bar{\bar{\boldsymbol{T}}}_{pq}^*=\|\bar{\bar{t}}_{ij}^{*pq}\|,$$

then

$$\begin{aligned}\bar{t}_{ij}^{*pq}&=\bar{\mu}_0 t_{ij}^{(0)pq}+\bar{\mu}_1 t_{ij}^{(1)pq}+\cdots+\bar{\mu}_m t_{ij}^{(m)pq},\\ \bar{\bar{t}}_{ij}^{*pq}&=\bar{\bar{\mu}}_0 t_{ij}^{(0)pq}+\bar{\bar{\mu}}_1 t_{ij}^{(1)pq}+\cdots+\bar{\bar{\mu}}_m t_{ij}^{(m)pq}.\end{aligned} \tag{3.4}$$

Let us put $\pi_f=\pi_i^{(p)}$ if $f=(p-1)k+i$,

$$\begin{aligned}&\boldsymbol{\pi}^{(p)\prime}=(\pi_1^{(p)},\cdots,\pi_k^{(p)}),\\ &\Delta_p=\boldsymbol{\pi}^{(p)\prime}\boldsymbol{\pi}^{(p)},\\ &\Gamma_p=\textstyle\sum_{i=1}^k \pi_i^{(p)^4}, \qquad p=1,\cdots,b.\end{aligned} \tag{3.5}$$

Other notations which are necessary for the calculations in this section are listed in the following for the case when $k\geqq 4$.

(i) $\lambda_{pp}^{(1)\alpha\alpha}=\sum_{i\neq j} t_{ij}^{(\alpha)pp}, \qquad \alpha=0,1,\cdots,m,$

(ii) $\lambda_{pp}^{(2)\alpha\beta}=\sum_{i\neq j\neq l} t_{ij}^{(\alpha)pp}t_{il}^{(\beta)pp}, \qquad \alpha,\beta=0,1,\cdots,m,$

(iii) $\lambda_{pp}^{(3)\alpha\beta}=\sum_{i\neq j\neq l\neq s} t_{ij}^{(\alpha)pp}t_{ls}^{(\beta)pp}, \qquad \alpha,\beta=0,1,\cdots,m,$

(iv) $\lambda_{pq}^{(4)\alpha\alpha}=\sum_{i,j} t_{ij}^{(\alpha)pq}=\sum_i t_{ii}^{(\alpha)pq}+\sum_{i\neq j} t_{ij}^{(\alpha)pq}, \qquad \alpha=0,1,\cdots,m,$

(v) $\lambda_{pq}^{(5)\alpha\beta}=\sum_{i\neq j} t_{ii}^{(\alpha)pq}t_{jj}^{(\beta)pq}+\sum_{i\neq j} t_{ij}^{(\alpha)pq}t_{ji}^{(\beta)pq}+2\sum_{i\neq j\neq l} t_{ii}^{(\alpha)pq}t_{jl}^{(\beta)pq}$
$+2\sum_{i\neq j\neq l} t_{ij}^{(\alpha)pq}t_{li}^{(\beta)pq}+\sum_{i\neq j\neq l\neq s} t_{ij}^{(\alpha)pq}t_{ls}^{(\beta)pq}, \qquad \alpha, \beta=0, 1, \cdots, m,$

(vi) $\lambda_{pq}^{(6)\alpha\beta}=2\sum_{i\neq j} t_{ij}^{(\alpha)pq}t_{il}^{(\beta)pq}+\sum_{i\neq j\neq l} t_{ij}^{(\alpha)pq}t_{il}^{(\beta)pq}, \qquad \alpha, \beta=0, 1, \cdots, m.$

3.2. *The mean value of $\bar{\theta}$ and $\bar{\bar{\theta}}$*

First, we calculate the mean value $\mathcal{E}(\bar{\theta})$ of $\bar{\theta}$ with respect to the permutation distribution due to the randomization.

Put,

$$\Delta\bar{\theta}=\bar{\Theta}=\boldsymbol{\pi}'\bar{\boldsymbol{T}}^{*}\boldsymbol{\pi}=\boldsymbol{\pi}'(\bar{\mu}_0\boldsymbol{T}_0+\bar{\mu}_1\boldsymbol{T}_1+\cdots+\bar{\mu}_n\boldsymbol{T}_m)\boldsymbol{\pi}.$$

Then, an analogous calculation to that given in [1] leads us to the following

$$\mathcal{E}(\bar{\theta})=\frac{1}{\Delta k(k-1)}\sum_{p=1}^{b}\sum_{i=0}^{m}(\bar{\mu}_0-\bar{\mu}_i)\lambda_{pp}^{(1)ii}\Delta_p.$$

In a completely similar manner one obtains

$$\mathcal{E}(\bar{\bar{\theta}})=\frac{1}{\Delta k(k-1)}\sum_{p=1}^{b}\sum_{i=0}^{m}(\bar{\bar{\mu}}_0-\bar{\bar{\mu}}_i)\lambda_{pp}^{(1)ii}\Delta_p. \tag{3.6}$$

3.3. *The variances of $\bar{\theta}$ and $\bar{\bar{\theta}}$*

Since the variance of $\bar{\bar{\theta}}$ can be obtained by an analogous way to that of $\bar{\theta}$, we present only the calculation of the variance of $\bar{\theta}$. The calculation goes parallel to that given in [1].

Now,

$$\mathcal{E}(\bar{\Theta}^2)=\mathcal{E}(\boldsymbol{\pi}'\boldsymbol{U}_\sigma'\bar{\boldsymbol{T}}^{*}\boldsymbol{U}_\sigma\boldsymbol{\pi})^2$$
$$=\mathcal{E}(\sum_{p,q}\boldsymbol{\pi}^{(p)'}\boldsymbol{S}_{\sigma_q}'\bar{\boldsymbol{T}}_{pq}^{*}\boldsymbol{S}_{\sigma_q}\boldsymbol{\pi}^{(q)})^2.$$

Expanding the last expression we obtain

$$\mathcal{E}(\bar{\Theta}^2)=\mathcal{E}(\bar{A})+\mathcal{E}(\bar{B})+2\mathcal{E}(\bar{C}), \tag{3.7}$$

where

$$\bar{A}=\sum_{p}\boldsymbol{\pi}^{(p)'}\boldsymbol{S}_{\sigma_p}'\bar{\boldsymbol{T}}_{pp}^{*}\boldsymbol{S}_{\sigma_p}\boldsymbol{\pi}^{(p)}\boldsymbol{\pi}^{(p)'}\boldsymbol{S}_{\sigma_p}'\bar{\boldsymbol{T}}_{pp}^{*}\boldsymbol{S}_{\sigma_p}\boldsymbol{\pi}^{(p)},$$

$$\bar{B}=\sum_{p\neq q}\boldsymbol{\pi}^{(p)'}\boldsymbol{S}_{\sigma_p}'\bar{\boldsymbol{T}}_{pp}^{*}\boldsymbol{S}_{\sigma_p}\boldsymbol{\pi}^{(p)}\boldsymbol{\pi}^{(q)'}\boldsymbol{S}_{\sigma_p}'\bar{\boldsymbol{T}}_{qq}^{*}\boldsymbol{S}_{\sigma_p}\boldsymbol{\pi}^{(q)},$$

$$\bar{C}=\sum_{p\neq q}\boldsymbol{\pi}^{(p)'}\boldsymbol{S}_{\sigma_q}'\bar{\boldsymbol{T}}_{pq}^{*}\boldsymbol{S}_{\sigma_q}\boldsymbol{\pi}^{(q)}\boldsymbol{\pi}^{(p)'}\boldsymbol{S}_{\sigma_q}'\bar{\boldsymbol{T}}_{pq}^{*}\boldsymbol{S}_{\sigma_q}\boldsymbol{\pi}^{(q)}.$$

It can be shown that

$$\mathcal{E}(\bar{A})=\sum_p\Big[\frac{1}{k(k-1)}\sum_i(\bar{\mu}_0-\bar{\mu}_i)^2\lambda_{pp}^{(1)ii}\Delta_p^2+(\Delta_p^2-2\Gamma_p)$$
$$\cdot\Big\{\frac{1}{k(k-1)}\sum_i\bar{\mu}_i^2\lambda_{pp}^{(1)ii}-\frac{4}{k(k-1)(k-2)}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pp}^{(2)ij}$$
$$+\frac{3}{k(k-1)(k-2)(k-3)}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pp}^{(3)ij}\Big\}\Big].$$

$$\mathcal{E}(\bar{B})=\mathcal{E}^2(\bar{\Theta})-\sum_p\Delta_p\Big[\frac{1}{k(k-1)}\sum_i(\bar{\mu}_0-\bar{\mu}_i)^2\lambda_{pp}^{(1)ii}$$
$$-\frac{1}{k(k-1)}\sum_i\mu_i^2\lambda_{pp}^{(1)ii}+\frac{2}{k^2(k-1)^2}\sum_i\mu_i^2\lambda_{pp}^{(1)ii}$$
$$+\frac{4}{k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pp}^{(2)ij}+\frac{1}{k^2(k-1)}\sum_{i,j}\mu_i\mu_j\lambda_{pp}^{(3)ij}\Big].$$

$$\mathcal{E}(\bar{C})=\sum_{p\neq q}\Delta_p\Delta_q\Big[\frac{1}{k^2}\sum_i\bar{\mu}_i^2\lambda_{pq}^{(4)ii}+\frac{1}{k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pq}^{(5)ij}$$
$$-\frac{2}{k^2(k-1)}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pq}^{(6)ij}\Big].$$

Thus we obtain

$$\text{(3.8)}\qquad \mathrm{Var}(\bar{\theta})=\Delta^{-2}\,\mathrm{Var}(\bar{\Theta})=\Delta^{-2}[\mathcal{E}(\bar{\Theta}^2)-\mathcal{E}^2(\bar{\Theta})]$$
$$=\frac{1}{\Delta^2}\sum_p(\Delta_p^2-2\Gamma_p)\Big[\frac{1}{k(k-1)}\sum_i\bar{\mu}_i^2\lambda_{pp}^{(1)ii}$$
$$-\frac{4}{k(k-1)(k-2)}\sum_{i,j}\mu_i\mu_j\lambda_{pp}^{(2)ij}$$
$$+\frac{3}{k(k-1)(k-2)(k-3)}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pp}^{(3)ij}\Big]$$
$$+\frac{1}{\Delta^2}\sum_p\Delta_p^2\Big[\frac{1}{k(k-1)}\sum_i\bar{\mu}_i^2\lambda_{pp}^{(1)ii}-\frac{2}{k^2(k-1)^2}\sum_i\bar{\mu}_i^2\lambda_{pp}^{(1)ii}$$
$$-\frac{4}{k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pp}^{(2)ij}-\frac{1}{k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pp}^{(3)ij}\Big]$$
$$+\frac{1}{\Delta^2}\sum_{p\neq q}\Delta_p\Delta_q\Big[\frac{2}{k^2}\sum_i\bar{\mu}_i^2\lambda_{pq}^{(4)ii}+\frac{2}{k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pq}^{(5)ij}$$
$$-\frac{4}{k^2(k-1)}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\lambda_{pq}^{(6)ij}\Big].$$

The variance of $\bar{\bar{\theta}}$ may be calculated in a similar manner and is given by

$$\text{(3.9)}\qquad \mathrm{Var}(\bar{\bar{\theta}})=\frac{1}{\Delta^2}\sum_p(\Delta_p^2-2\Gamma_p)\Big[\frac{1}{k(k-1)}\sum_i\bar{\bar{\mu}}_i^2\lambda_{pp}^{(1)ii}$$

$$-\frac{4}{k(k-1)(k-2)}\sum_{i,j}\bar{\bar{\mu}}_i\bar{\bar{\mu}}_j\lambda_{pp}^{(2)ij}$$
$$+\frac{3}{k(k-1)(k-2)(k-3)}\sum_{i,j}\bar{\bar{\mu}}_i\bar{\bar{\mu}}_j\lambda_{pp}^{(3)ij}\Big]$$
$$+\frac{1}{\Delta^2}\sum_p \Delta_p^2\Big[\frac{1}{k(k-1)}\sum_i\bar{\bar{\mu}}_i^2\lambda_{pp}^{(1)ii}-\frac{2}{k^2(k-1)^2}\sum_i\bar{\bar{\mu}}_i^2\lambda_{pp}^{(1)ii}$$
$$-\frac{4}{k^2(k-1)^2}\sum_{i,j}\bar{\bar{\mu}}_i\bar{\bar{\mu}}_j\lambda_{pp}^{(2)ij}-\frac{1}{k^2(k-1)^2}\sum_{i,j}\bar{\bar{\mu}}_i\bar{\bar{\mu}}_j\lambda_{pp}^{(3)ij}\Big]$$
$$+\frac{1}{\Delta^2}\sum_{p\neq q}\Delta_p\Delta_q\Big[\frac{2}{k^2}\sum_i\bar{\bar{\mu}}_i^2\lambda_{pq}^{(4)ii}+\frac{2}{k^2(k-1)^2}\sum_{i,j}\bar{\bar{\mu}}_i\bar{\bar{\mu}}_j\lambda_{pq}^{(5)ij}$$
$$-\frac{1}{k^2(k-1)}\sum_{i,j}\bar{\bar{\mu}}_i\bar{\bar{\mu}}_j\lambda_{pq}^{(6)ij}\Big].$$

This is obtained from (3.8) by replacing $\bar{\mu}$'s by $\bar{\bar{\mu}}$'s throughout.

3.4. *The covariance between $\bar{\theta}$ and $\bar{\bar{\theta}}$*

We calculate the covariance between $\bar{\theta}$ and $\bar{\bar{\theta}}$ from the relation

$$\text{(3.10)}\qquad \operatorname{Cov}(\bar{\theta},\bar{\bar{\theta}})=\frac{1}{2}[\operatorname{Var}(\bar{\theta}+\bar{\bar{\theta}})-\operatorname{Var}(\bar{\theta})-\operatorname{Var}(\bar{\bar{\theta}})].$$

Since

$$\bar{\theta}+\bar{\bar{\theta}}=\Delta^{-1}\boldsymbol{\pi}'\boldsymbol{\Phi}(c_1A_1^*+\cdots+c_mA_m^*)\boldsymbol{\Phi}'\boldsymbol{\pi},$$

$\operatorname{Var}(\bar{\theta}+\bar{\bar{\theta}})$ can be obtained from (3.8) replacing $\bar{\mu}$'s by $\mu=\bar{\mu}+\bar{\bar{\mu}}$'s throughout.

Thus one gets

$$\text{(3.11)}\qquad \operatorname{Var}(\bar{\theta}+\bar{\bar{\theta}})=\frac{1}{\Delta^2}\sum_p(\Delta_p^2-2\Gamma_p)\Big[\frac{1}{k(k-1)}\sum_i\mu_i^2\lambda_{pp}^{(1)ii}$$
$$-\frac{4}{k(k-1)(k-2)}\sum_{i,j}\mu_i\mu_j\lambda_{pp}^{(2)ij}$$
$$+\frac{3}{k(k-1)(k-2)(k-3)}\sum_{i,j}\mu_i\mu_j\lambda_{pp}^{(3)ij}\Big]$$
$$+\frac{1}{\Delta^2}\sum_p\Delta_p^2\Big[\frac{1}{k(k-1)}\sum_i\mu_i^2\lambda_{pp}^{(1)ii}-\frac{2}{k^2(k-1)^2}\sum_i\mu_i^2\lambda_{pp}^{(1)ii}$$
$$-\frac{4}{k^2(k-1)^2}\sum_{i,j}\mu_i\mu_j\lambda_{pp}^{(2)ij}-\frac{1}{k^2(k-1)^2}\sum_{i,j}\mu_i\mu_j\lambda_{pp}^{(3)ij}\Big]$$
$$+\frac{1}{\Delta^2}\sum_{p\neq q}\Delta_p\Delta_q\Big[\frac{2}{k^2}\sum_i\mu_i^2\lambda_{pq}^{(4)ii}+\frac{2}{k^2(k-1)^2}\sum_{i,j}\mu_i\mu_j\lambda_{pq}^{(5)ij}$$
$$-\frac{4}{k^2(k-1)}\sum_{i,j}\mu_i\mu_j\lambda_{pq}^{(6)ij}\Big].$$

By using the relations

$$\mu_i^2-\bar{\mu}_i^2-\bar{\bar{\mu}}_i^2=2\bar{\mu}_i\bar{\bar{\mu}}_i, \qquad \mu_i\mu_j-\bar{\mu}_i\bar{\mu}_j-\bar{\bar{\mu}}_i\bar{\bar{\mu}}_j=\bar{\mu}_i\bar{\bar{\mu}}_j+\bar{\bar{\mu}}_i\bar{\mu}_j,$$

$$\lambda_{pp}^{(2)ij}=\lambda_{pp}^{(2)ji}, \quad \lambda_{pp}^{(3)ij}=\lambda_{pp}^{(3)ji}, \quad \lambda_{pq}^{(5)ij}=\lambda_{qp}^{(5)ji}, \quad \lambda_{pq}^{(6)ij}=\lambda_{qp}^{(6)ji},$$

one can obtain from (3.8), (3.9), (3.10) and (3.11) the following result:

$$\begin{aligned}(3.12)\quad \mathrm{Cov}(\bar{\theta},\bar{\bar{\theta}})=&\frac{1}{\Delta^2}\sum_p(\Delta_p^2-2\Gamma_p)\Big[\frac{1}{k(k-1)}\sum_i\bar{\mu}_i\bar{\bar{\mu}}_i\lambda_{pp}^{(1)ii}\\ &-\frac{4}{k(k-1)(k-2)}\sum_{i,j}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_{pp}^{(2)ij}\\ &+\frac{3}{k(k-1)(k-2)(k-3)}\sum_{i,j}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_{pp}^{(3)ij}\Big]\\ &+\frac{1}{\Delta^2}\sum_p\Delta_p^2\Big[\frac{1}{k(k-1)}\sum_i\bar{\mu}_i\bar{\bar{\mu}}_i\lambda_{pp}^{(1)ii}-\frac{2}{k^2(k-1)^2}\sum_i\bar{\mu}_i\bar{\bar{\mu}}_i\lambda_{pp}^{(1)ii}\\ &-\frac{4}{k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_{pp}^{(2)ij}-\frac{1}{k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_{pp}^{(3)ij}\Big]\\ &+\frac{1}{\Delta^2}\sum_{p\neq q}\Delta_p\Delta_q\Big[\frac{1}{k^2}\sum_i\bar{\mu}_i\bar{\bar{\mu}}_i\lambda_{pq}^{(4)ii}+\frac{2}{k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_{pq}^{(5)ij}\\ &-\frac{4}{k^2(k-1)}\sum_{i,j}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_{pq}^{(6)ij}\Big].\end{aligned}$$

4. An approximation to the permutation distribution of $(\bar{\theta},\bar{\bar{\theta}})$ by a certain two-dimensional continuous distribution when the number of b of blocks is sufficiently large and certain uniformity conditions are imposed on the unit errors

In this section, we assume that the following uniformity conditions on the unit errors are satisfied:

$$(4.1)\qquad \Delta_p=\Delta_0 \quad\text{and}\quad \Gamma_p=\Gamma_0 \qquad \text{for } p=1,2,\cdots,b.$$

Under such conditions, the means and variances of $\bar{\theta}$ and $\bar{\bar{\theta}}$ are expressed as follows:

$$(4.2)\qquad \mathcal{E}(\bar{\theta})=\frac{1}{bk(k-1)}\sum_i(\bar{\mu}_0-\bar{\mu}_i)\sum_p\lambda_{pp}^{(1)ii},$$

$$(4.3)\qquad \mathcal{E}(\bar{\bar{\theta}})=\frac{1}{bk(k-1)}\sum_i(\bar{\bar{\mu}}_0-\bar{\bar{\mu}}_i)\sum_p\lambda_{pp}^{(1)ii},$$

$$(4.4)\qquad \mathrm{Var}(\bar{\theta})=\Big(1-\frac{2\Gamma_0}{\Delta_0^2}\Big)\Big[\frac{1}{b^2k(k-1)}\sum_i\bar{\mu}_i^2\sum_p\lambda_{pp}^{(1)ii}$$

$$-\frac{4}{b^2k(k-1)(k-2)}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\sum_p \lambda_{pp}^{(2)ij}$$
$$\left.+\frac{3}{b^2k(k-1)(k-2)(k-3)}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\sum_p \lambda_{pp}^{(3)ij}\right]$$
$$+\frac{1}{b^2k(k-1)}\sum_i\bar{\mu}_i^2\sum_p\lambda_{pp}^{(1)ii}-\frac{2}{b^2k^2(k-1)^2}\sum_i\bar{\mu}_i^2\sum_p\lambda_{pp}^{(1)ii}$$
$$-\frac{4}{b^2k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\sum_p\lambda_{pp}^{(2)ij}-\frac{1}{b^2k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\sum_p\lambda_{pp}^{(3)ij}$$
$$+\frac{2}{b^2k^2}\sum_i\bar{\mu}_i^2\sum_{p\neq q}\lambda_{pq}^{(4)ii}+\frac{2}{b^2k^2(k-1)^2}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\sum_{p\neq q}\lambda_{pq}^{(5)ij}$$
$$-\frac{4}{b^2k^2(k-1)}\sum_{i,j}\bar{\mu}_i\bar{\mu}_j\sum_{p\neq q}\lambda_{pq}^{(6)ij},$$

and

(4.5) $\mathrm{Var}(\bar{\bar{\theta}})$ is obtained from $\mathrm{Var}(\bar{\theta})$ given in (4.4) by replacing all $\bar{\mu}$'s by $\bar{\bar{\mu}}$'s throughout.

Quite analogously to the calculations presented in [1], it can be shown that

$$\text{(i)}\quad \sum_p \lambda_{pp}^{(1)ii}=(1-\delta_{0i})\lambda_i n_i v,\qquad i=0,1,\cdots,m,$$

$$\text{(ii)}\quad \sum_{p\neq q}\lambda_{pq}^{(4)ii}=(r^2-\lambda_i)n_i v,\qquad i=0,1,\cdots,m,$$

$$\text{(4.6)}\quad \text{(iii)}\quad \sum_{p\neq q}\lambda_{pq}^{(5)ij}=\sum_{\alpha,\beta=1}^{m}\lambda_\alpha\lambda_\beta\sum_{\gamma=0}^{m}p_{\alpha\beta}^{\gamma}p_{ij}^{\gamma}n_j v-\delta_{ij}\lambda_i n_i v-2\lambda_{ij}^{(1)}-\lambda_{ij}^{(2)}$$
$$-\delta_{0i}(1-\delta_{0j})(k-2)\lambda_j n_j v-\delta_{0j}(k-2)\lambda_i n_i v,$$

$$\text{(iv)}\quad \sum_{p\neq q}\lambda_{pq}^{(6)ij}=r\sum_\alpha\lambda_\alpha p_{ij}^{\alpha}n_\alpha v-\delta_{i0}(1-\delta_{0j})\lambda_j n_j v$$
$$-\delta_{0j}(1-\delta_{0i})\lambda_i n_i v-\lambda_{ij}^{(1)},\qquad i,j=0,1,\cdots,m,$$

where, as before, δ_{ij} stands for the Kronecker delta and

$$\text{(4.7)}\qquad \lambda_{ij}^{(1)}=\sum_p\lambda_{pp}^{(2)ij},\qquad \lambda_{ij}^{(2)}=\sum_p\lambda_{pp}^{(3)ij}.$$

Hence, from (1.2), (2.4) and (3.3), it follows that

$$\text{(4.8)}\qquad \mathcal{E}(\bar{\theta})=\frac{\bar{\alpha}}{b(k-1)}\quad\text{and}\quad \mathcal{E}(\bar{\bar{\theta}})=\frac{\bar{\bar{\alpha}}}{b(k-1)}.$$

From (4.4) and (4.6), it follows that

$$\mathrm{Var}(\bar{\theta})=\left[\frac{2(2k-3)}{k(k-1)}-\frac{2\Gamma_0}{\Delta_0^2}\right]\left[\frac{v}{b^2k(k-1)}\sum_{i=1}^{m}\bar{\mu}_i^2\lambda_i n_i\right.$$

$$-\frac{4}{b^2k(k-1)(k-2)}\sum_{i,j=1}^{m}\mu_i\mu_j\lambda_{ij}^{(1)}$$

$$+\frac{3}{b^2k(k-1)(k-2)(k-3)}\sum_{i,j=1}^{m}\bar{\mu}_i\bar{\mu}_j\lambda_{ij}^{(2)}\Bigg]-\frac{2v}{b^2k(k-1)^2}\sum_{i=1}^{m}(\bar{\mu}_0-\bar{\mu}_i)^2\lambda_i n_i$$

$$+\frac{2v}{b^2k^2(k-1)^2}\Bigg[r^2(k-1)^2\sum_{i=0}^{m}\bar{\mu}_i^2 n_i$$

$$+\sum_{i,j,l=0}^{m}\sum_{\alpha,\beta=1}^{m}\bar{\mu}_i\bar{\mu}_j\lambda_\alpha\lambda_\beta p_{\alpha\beta}^{l}p_{ij}^{l}n_l-2r(k-1)\sum_{i,j=1}^{m}\sum_{\alpha=1}^{m}\bar{\mu}_i\bar{\mu}_j p_{ij}^{\alpha}n_\alpha\Bigg].$$

Since

$$\sum_{i=1}^{m}(\bar{\mu}_0-\bar{\mu}_i)n_i\lambda_i=k\bar{\alpha}/v,$$

the second term of the right-hand side of the above expression is

$$-\frac{2v}{b^2k(k-1)^2}\sum_{i=1}^{m}(\bar{\mu}_0-\bar{\mu}_i)^2\lambda_i n_i$$

$$=-\frac{2v}{b^2k(k-1)^2}\left(\frac{k\bar{\alpha}}{v}\right)^2\frac{1}{r(k-1)}\frac{\left(\sum_{i=1}^{m}n_i\lambda_i\right)\left(\sum_{i=1}^{m}(\bar{\mu}_0-\bar{\mu}_i)^2 n_i\lambda_i\right)}{\left[\sum_{i=1}^{m}(\bar{\mu}_0-\bar{\mu}_i)n_i\lambda_i\right]^2}$$

$$=-\frac{2\bar{\alpha}^2}{b^3(k-1)^3}\left[1-\frac{\sum_{i\neq j}(\bar{\mu}_0-\bar{\mu}_i)(\bar{\mu}_0-\bar{\mu}_j)n_i n_j\lambda_i\lambda_j}{k^2\bar{\alpha}^2/v^2}\right]$$

$$=-\frac{2\bar{\alpha}^2}{b^3(k-1)^3}+\frac{2v^2}{b^3k^2(k-1)^3}\sum_{i\neq j}^{m}(\bar{\mu}_0-\bar{\mu}_i)(\bar{\mu}_0-\bar{\mu}_j)n_i n_j\lambda_i\lambda_j,$$

and the third term becomes

$$\frac{2v}{b^2k^2(k-1)^2}\Bigg[r^2(k-1)^2\sum_{i=0}^{m}\bar{\mu}_i^2 n_i+\sum_{i,j,l=0}^{m}\sum_{\alpha,\beta=1}^{m}\bar{\mu}_i\bar{\mu}_j\lambda_\alpha\lambda_\beta p_{\alpha\beta}^{l}p_{ij}^{l}n_l$$

$$-2r(k-1)\sum_{i,j=0}^{m}\sum_{\alpha=1}^{m}\bar{\mu}_i\bar{\mu}_j\lambda_\alpha p_{ij}^{\alpha}n_\alpha\Bigg]$$

$$=\frac{2v}{b^2k^2(k-1)^2}\sum_{s,u=1}^{h}\Bigg[r^2(k-1)^2\sum_{i=0}^{m}\mu_{si}\mu_{ui}n_i+\sum_{i,j,l=0}^{m}\sum_{\alpha,\beta=1}^{m}\mu_{si}\mu_{ui}p_{\alpha\beta}^{l}p_{ij}^{l}n_l\lambda_\alpha\lambda_\beta$$

$$-2r(k-1)\sum_{i,j=0}^{m}\sum_{\alpha=1}^{m}\mu_{si}\mu_{ui}\lambda_\alpha p_{ij}^{\alpha}n_\alpha\Bigg]$$

$$=\frac{2v}{b^2k^2(k-1)^2}\sum_{u=1}^{h}\frac{k^2}{v^2}\left(\frac{\alpha_u}{rk-\rho_u}\right)^2\left(\frac{v}{\alpha_u}\right)(rk-\rho_u)^2$$

$$=\frac{2}{b^2(k-1)^2}\sum_{u=1}^{h}\alpha_u=\frac{2\bar{\alpha}}{b^2(k-1)^2},$$

we have

$$(4.9)\qquad \mathrm{Var}(\bar{\theta})=\Big[\frac{2\bar{\alpha}}{b^2(k-1)^2}-\frac{2\bar{\alpha}^2}{b^3(k-1)^3}\Big]$$
$$+\frac{2v^2}{b^3k^2(k-1)^3}\sum_{i\neq j}(\bar{\mu}_0-\bar{\mu}_i)(\bar{\mu}_0-\bar{\mu}_j)n_in_j\lambda_i\lambda_j$$
$$+\Big[\frac{2(2k-3)}{k(k-1)}-\frac{2\Gamma_0}{\Delta_0^2}\Big]\Big[\frac{v}{b^2k(k-1)}\sum_{i=1}^{m}\bar{\mu}_i^2n_i\lambda_i$$
$$-\frac{4}{b^2k(k-1)(k-2)}\sum_{i,j=1}^{m}\bar{\mu}_i\bar{\mu}_j\lambda_{ij}^{(1)}$$
$$+\frac{3}{b^2k(k-1)(k-2)(k-3)}\sum_{i,j=1}^{m}\bar{\mu}_i\bar{\mu}_j\lambda_{ij}^{(2)}\Big].$$

Likewise

(4.10) $\mathrm{Var}(\bar{\bar{\theta}})$ may be obtained from (4.9) by replacing $\bar{\mu}$'s by $\bar{\bar{\mu}}$'s throughout.

Finally, by using the relation (3.11), one gets

$$(4.11)\qquad \mathrm{Cov}(\bar{\theta},\bar{\bar{\theta}})=-\frac{2\bar{\alpha}\bar{\bar{\alpha}}}{b^3(k-1)^3}+\frac{v^2}{b^3k^2(k-1)^3}\sum_{i\neq j}\Big[(\bar{\mu}_0-\bar{\mu}_i)(\bar{\bar{\mu}}_i-\bar{\bar{\mu}}_j)$$
$$+(\bar{\bar{\mu}}_0-\bar{\bar{\mu}}_i)(\bar{\mu}_i-\bar{\mu}_j)\Big]n_in_j\lambda_i\lambda_j+\Big[\frac{2(2k-3)}{k(k-1)}-\frac{2\Gamma_0}{\Delta_0^2}\Big]$$
$$\cdot\Big[\frac{v}{b^2k(k-1)}\sum_{i,j=1}^{m}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_in_i-\frac{4}{b^2k(k-1)(k-2)}$$
$$\cdot\sum_{i,j=1}^{m}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_{ij}^{(1)}+\frac{3}{b^2k(k-1)(k-2)(k-3)}\sum_{i,j=1}^{m}\bar{\mu}_i\bar{\bar{\mu}}_j\lambda_{ij}^{(2)}\Big].$$

When $k=2$ or 3, the last term of the expressions $\mathrm{Var}(\bar{\theta})$, $\mathrm{Var}(\bar{\bar{\theta}})$ and $\mathrm{Cov}(\bar{\theta},\bar{\bar{\theta}})$ vanishes.

Now, let us consider the limiting process such that

$$b\to\infty$$

whereas $v, k, n_1,\cdots,n_m$ and p^i_{jk} are fixed. Then, since $vr=bk$ and $\sum_{i=1}^{m}n_i\lambda_i=r(k-1)$, r and at least one λ_i are of the same order as b, and

$$\bar{\mu}_i=O\Big(\frac{1}{b}\Big)\quad\text{and}\quad\bar{\bar{\mu}}_i=O\Big(\frac{1}{b}\Big),\qquad i=0,1,\cdots,m.$$

It is also seen that

$$\lambda_{ij}^{(1)}\leqq bk(k-1)(k-2),\qquad \lambda_{ij}^{(2)}\leqq bk(k-1)(k-2)(k-3)$$

and

$$\Gamma_0/\Delta_0^2 \leqq 1.$$

Thus, we get

$$\text{(4.12)}\qquad \begin{aligned} \operatorname{Var}(\bar\theta) &= \frac{2\bar\alpha}{b^2(k-1)^2}\left(1+O\left(\frac{1}{b}\right)\right), \\ \operatorname{Var}(\bar{\bar\theta}) &= \frac{2\bar{\bar\alpha}}{b^2(k-1)^2}\left(1+O\left(\frac{1}{b}\right)\right), \\ \operatorname{Cov}(\bar\theta, \bar{\bar\theta}) &= O\left(\frac{1}{b^3}\right). \end{aligned}$$

Hence we can treat that for sufficiently large values of b

$$\operatorname{Var}(\bar\theta) = \frac{2\bar\alpha}{b^2(k-1)^2}, \qquad \operatorname{Var}(\bar{\bar\theta}) = \frac{2\bar{\bar\alpha}}{b^2(k-1)^2}, \qquad \operatorname{Cov}(\bar\theta, \bar{\bar\theta}) = 0.$$

In such a situation, we take the following two-dimensional continuous distribution:

$$\text{(4.13)}\qquad \frac{\Gamma\left(\frac{\nu_1+\nu_2+\nu_3}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)\Gamma\left(\frac{\nu_3}{2}\right)} \bar\theta^{\nu_1/2-1}\bar{\bar\theta}^{\nu_2/2-1}(1-\bar\theta-\bar{\bar\theta})^{\nu_3/2-1} d\bar\theta d\bar{\bar\theta}$$

$$\text{for } \bar\theta \geqq 0,\ \bar{\bar\theta} \geqq 0,\ \bar\theta+\bar{\bar\theta} \leqq 1,$$

as an approximation to the permutation distribution of $(\bar\theta, \bar{\bar\theta})$ due to the randomization.

Then, equating up to the second order moments, we have

$$\text{(4.14)}\qquad \begin{aligned} &\frac{\nu_1}{\nu_1+\nu_2+\nu_3} = \mathcal{E}(\bar\theta), \qquad \frac{\nu_2}{\nu_1+\nu_2+\nu_3} = \mathcal{E}(\bar{\bar\theta}), \\ &\frac{2\nu_1(\nu_2+\nu_3)}{(\nu_1+\nu_2+\nu_3)^2(\nu_1+\nu_2+\nu_3+2)} = \operatorname{Var}(\bar\theta), \\ &\frac{2\nu_2(\nu_1+\nu_3)}{(\nu_1+\nu_2+\nu_3)^2(\nu_1+\nu_2+\nu_3+2)} = \operatorname{Var}(\bar{\bar\theta}), \\ &\frac{-2\nu_1\nu_2}{(\nu_1+\nu_2+\nu_3)^2(\nu_1+\nu_2+\nu_3+2)} = \operatorname{Cov}(\bar\theta, \bar{\bar\theta}). \end{aligned}$$

From the first three equations of (4.14), one obtains

$$\text{(4.15)}\qquad \begin{aligned} \nu_1 &= \frac{2}{\operatorname{Var}(\bar\theta)}[\mathcal{E}(\bar\theta)(1-\mathcal{E}(\bar\theta)) - \operatorname{Var}(\bar\theta)]\mathcal{E}(\bar\theta), \\ \nu_2 &= \frac{2}{\operatorname{Var}(\bar\theta)}[\mathcal{E}(\bar\theta)(1-\mathcal{E}(\bar\theta)) - \operatorname{Var}(\bar\theta)]\mathcal{E}(\bar{\bar\theta}), \end{aligned}$$

$$\nu_3=\frac{2}{\mathrm{Var}(\bar{\theta})}[\mathcal{E}(\bar{\theta})(1-\mathcal{E}(\bar{\theta}))-\mathrm{Var}(\bar{\theta})][1-\mathcal{E}(\bar{\theta})-\mathcal{E}(\bar{\bar{\theta}})].$$

If we put

(4.16) $$\nu_1=\phi\bar{\alpha},\quad \nu_2=\phi\bar{\bar{\alpha}}\quad \text{and}\quad \nu_3=\phi(n-b-v+1),$$

then, from (4.8) and (4.12), we can see that

(4.17) $$\phi=1+O\left(\frac{1}{b}\right).$$

It can easily be checked that the solutions (4.15) satisfy the remaining two equations of (4.14) under the limiting situation described above.

Thus, for sufficiently large values of b, the permutation distribution of $(\bar{\theta},\bar{\bar{\theta}})$ due to the randomization may be approximated by

(4.18) $$\frac{\Gamma\left(\frac{n-b}{2}\right)}{\Gamma\left(\frac{\bar{\alpha}}{2}\right)\Gamma\left(\frac{\bar{\bar{\alpha}}}{2}\right)\Gamma\left(\frac{n-b-v+1}{2}\right)}\bar{\theta}^{\bar{\alpha}/2-1}\bar{\bar{\theta}}^{\bar{\bar{\alpha}}/2-1}(1-\bar{\theta}-\bar{\bar{\theta}})^{(n-b-v+1)/2-1}d\bar{\theta}d\bar{\bar{\theta}}.$$

5. The approximate null-distribution of the F-statistic after the randomization

Now, we can calculate the expectation

$$\mathcal{E}(\bar{\theta}^{\mu}\bar{\bar{\theta}}^{\gamma}(1-\bar{\theta}-\bar{\bar{\theta}})^{\nu})$$

with respect to the permutation distribution due to the randomization in the sense of the approximation described in the preceding section. Thus we have

(5.1) $$\mathcal{E}(\bar{\theta}^{\mu}\bar{\bar{\theta}}^{\gamma}(1-\bar{\theta}-\bar{\bar{\theta}})^{\nu})$$
$$=\frac{\Gamma\left(\frac{n-b}{2}\right)\Gamma\left(\frac{\bar{\alpha}}{2}+\mu\right)\Gamma\left(\frac{\bar{\bar{\alpha}}}{2}+\gamma\right)\Gamma\left(\frac{n-b-v+1}{2}+\nu\right)}{\Gamma\left(\frac{\bar{\alpha}}{2}\right)\left(\frac{\bar{\bar{\alpha}}}{2}\right)\Gamma\left(\frac{n-b-v+1}{2}\right)\Gamma\left(\frac{n-b}{2}+\mu+\gamma+\nu\right)}.$$

Inserting (5.1) into the right-hand side of (2.15), we get an approximate null-distribution of the F-statistic after the randomization as follows:

(5.2) $$\frac{\Gamma\left(\frac{n-b-\bar{\bar{\alpha}}}{2}\right)}{\Gamma\left(\frac{\bar{\alpha}}{2}\right)\Gamma\left(\frac{n-b-v+1}{2}\right)}\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2-1}$$

$$\cdot\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-(n-b-\bar{\bar{\alpha}})/2} d\left(\frac{\bar{\alpha}}{n-b-v+1}F\right).$$

When $h=m$, this turns out to be the central F-distribution with degrees of freedom $(v-1,\ n-b-v+1)$.

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REFERENCES

[1] J. Ogawa, S. Ikeda and M. Ogasawara, "On the null-distribution of the F-statistic in a randomized partially balanced incomplete block design with two associate classes under the Neyman model," Inst. of Statist., University of North Carolina. *Mimeo Series* 414, 1964.

[2] R. C. Bose, "Least square aspects of analysis of variance," Inst. Statist., University of North Carolina, *Mimeo Series* 6, 1948.

[3] J. Ogawa and G. Ishii, "The relationship algebra and the analysis of variance of a partially balanced incomplete block design," *Ann. Math. Statist.*, 36 (1965), 1815-1828.