ON THE DISTRIBUTION OF THE SECOND ELEMENTARY SYMMETRIC FUNCTION OF THE ROOTS OF A MATRIX

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1. Summary and introduction

Distribution problems in multivariate analysis are often related to the joint distribution of the characteristic roots of a matrix derived from sample observations taken from multivariate normal populations. This joint distribution (under certain null hypotheses) of s non-null characteristic roots given by Fisher [4], Girshick [6], Hsu [7], and Roy [20] can be expressed in the form

$$f(\theta_1,\cdots,\theta_s)=C(s,m,n)\prod_{i=1}^{s}\theta_i^m(1-\theta_i)^n\prod_{i>j}(\theta_i-\theta_j) \qquad 0<\theta_1\leqq\cdots\leqq\theta_s<1 \tag{1.1}$$

where

$$C(s,m,n)=\frac{\pi^{s/2}\prod_{i=1}^{s}\Gamma\left(\frac{2m+2n+s+i+2}{2}\right)}{\prod_{i=1}^{s}\left[\Gamma\left(\frac{2m+i+1}{2}\right)\Gamma\left(\frac{2n+i+1}{2}\right)\Gamma(i/2)\right]} \tag{1.2}$$

and m and n are defined differently for various situations described by Pillai [12], [14]. Nanda [10] has shown that if $\xi_i=n\theta_i$ $(i=1,\cdots,s)$, then the limiting distribution of $\xi_i'^s$ as n tends to infinity is given by

$$f_1(\xi_1,\xi_2,\cdots,\xi_s)=K(s,m)\prod_{i=1}^{s}\xi_i^m e^{-\xi_i}\prod_{i>j}(\xi_i-\xi_j) \qquad 0<\xi_1\leqq\cdots\leqq\xi_s<\infty \tag{1.3}$$

where

$$K(s,m)=\pi^{s/2}\Big/\left[\prod_{i=1}^{s}\Gamma\left(\frac{2m+i+1}{2}\right)\Gamma(i/2)\right]. \tag{1.4}$$

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The distribution (1.3) can also be arrived at as that of $\xi_i=\frac{1}{2}\gamma_i$ $(i=1, 2,\cdots, s)$ where $\gamma_i'^s$ are the roots of the equation $|S-\gamma\Sigma|=0$ where S is the variance-covariance matrix computed from a sample taken from an s-variate normal population with dispersion matrix Σ. In this paper, the first four moments of $W_2^{(s)}$, the second elementary symmetric function (esf) in s ξ'^s, have been obtained and approximations to its distribution suggested. In addition, the variances of the third and fourth esf's are also obtained. An example is given to illustrate the use of $W_2^{(s)}$ as a test criterion.

2. Formulae for the first four moments of $W_2^{(s)}$

The joint distribution (1.3) can be thrown into a determinantal form of the Vandermonde type and integrated over the range R: $0<\xi_1\leqq\cdots\leqq\xi_s<\infty$, giving

$$(2.1)\qquad \int_R f_1(\xi_1,\xi_2,\cdots,\xi_s)\prod_{i=1}^{s}d\xi_i=K(s,m)\begin{vmatrix} \int_0^\infty \xi_s^{m+s-1}e^{-\xi_s}d\xi_s\cdots\int_0^\infty \xi_s^{m}e^{-\xi_s}d\xi_s \\ \vdots \qquad \cdots \qquad \vdots \\ \int_0^{\xi_2} \xi_1^{m+s-1}e^{-\xi_1}d\xi_1\cdots\int_0^{\xi_2} \xi_1^{m}e^{-\xi_1}d\xi_1 \end{vmatrix}.$$

Now denote by $W(s-1, s-2,\cdots,1,0)$ the determinant on the right side of (2.1). Using lemma 1 in Pillai [15], the first four moments of $W_2^{(s)}$ can be obtained as follows: (denoting $\mathrm{E}(W_2^{(s)})^r$ by μ_r')

$$(2.2)\qquad \mu_1'=K(s,m)W(s,s-1,s-3,\cdots,1,0)$$

$$(2.3)\qquad \begin{aligned}\mu_2'=K(s,m)[&W(s+1,s,s-3,\cdots,1,0)\\ &+W(s+1,s-1,s-2,s-4,\cdots,1,0)\\ &+W(s,s-1,s-2,s-3,s-5,\cdots,1,0)]\end{aligned}$$

$$(2.4)\qquad \begin{aligned}\mu_3'=K(s,m)[&W(s+2,s+1,s-3,\cdots,1,0)\\ &+2W(s+2,s,s-2,s-4,\cdots,1,0)\\ &+3W(s+1,s,s-2,s-3,s-5,\cdots,1,0)\\ &+W(s+2,s-1,s-2,s-3,s-5,\cdots,1,0)\\ &+W(s+1,s,s-1,s-4,\cdots,1,0)\\ &+2W(s+1,s-1,s-2,s-3,s-4,s-6,\cdots,1,0)\\ &+W(s,s-1,s-2,s-3,s-4,s-5,s-7,\cdots,1,0)]\end{aligned}$$

and

$$(2.5)\qquad \mu_4'=K(s,m)[W(s+3,s+2,s-3,\cdots,1,0)$$

$$
\begin{aligned}
&+3W(s+3, s+1, s-2, s-4, \cdots, 1, 0)\\
&+6W(s+2, s+1, s-2, s-3, s-5, \cdots, 1, 0)\\
&+2W(s+3, s, s-1, s-4, \cdots, 1, 0)\\
&+3W(s+3, s, s-2, s-3, s-5, \cdots, 1, 0)\\
&+3W(s+2, s+1, s-1, s-4, \cdots, 1, 0)\\
&+7W(s+2, s, s-1, s-3, s-5, \cdots, 1, 0)\\
&+8W(s+2, s, s-2, s-3, s-4, s-6, \cdots, 1, 0)\\
&+3W(s+1, s, s-1, s-2, s-5, \cdots, 1, 0)\\
&+6W(s+1, s, s-1, s-3, s-4, s-6, \cdots, 1, 0)\\
&+6W(s+1, s, s-2, s-3, s-4, s-5, s-7, \cdots, 1, 0)\\
&+W(s+3, s-1, s-2, s-3, s-4, s-6, \cdots, 1, 0)\\
&+3W(s+2, s-1, s-2, s-3, s-4, s-5, s-7, \cdots, 1, 0)\\
&+3W(s+1, s-1, s-2, s-3, s-4, s-5, s-6, s-8, \cdots, 1, 0)\\
&+W(s, s-1, s-2, s-3, s-4, s-5, s-6, s-7, s-9, \cdots, 1, 0)].
\end{aligned}
$$

3. A method of evaluation of the W-determinants

Let us denote by $V(q_s, q_{s-1}, \cdots, q_1)$ the determinant which could be obtained from $W(q_s, q_{s-1}, \cdots, q_1)$ by replacing ξ_i by θ_i in (1.1), $e^{-\xi_i}$ by $(1-\theta_i)^n$ and the range of integration by that in (1.1). Pillai, [11], [13], has given a method of reducing the sth order determinant $V(q_s, q_{s-1}, \cdots, q_1)$ in terms of $(s-2)$th order determinants and an sth order determinant with q_s changed to q_s-1, the last one being zero if $q_s-1=q_{s-1}$. The method of reduction for $W(q_s, \cdots, q_1)$ can be deduced from that for $V(q_s, \cdots, q_1)$ in Pillai [13] and we obtain the following:

$$
\begin{aligned}
(3.1)\qquad & W(q_s, q_{s-1}, \cdots, q_1)\\
&=2\sum_{j=s-1}^{1}(-1)^{s-j-1}I(q_s+q_j; 2)W(q_{s-1}, \cdots, q_{j+1}, q_{j-1}, \cdots, q_1)\\
&\quad+(m+q_s)W(q_s-1, q_{s-1}, \cdots, q_1)
\end{aligned}
$$

where

$$
I(p; 2)=\int_0^\infty x^p e^{-2x}\,dx=\Gamma(p+1)/2^{p+1}.
$$

The values of the W-determinants involved in (2.2)-(2.5) are obtained using (3.1) and presented in the following section.

4. Values of the W-determinants

Let us set

$$(2m+a)(2m+b)\cdots=M(a, b, \cdots). \tag{4.1}$$

Then for the *first moment*

$$K(s, m)W(s, s-1, s-3, \cdots, 1, 0)=s(s-1)M(s, s+1)/2^3, \tag{4.2}$$

where, using (4.1),

$$M(s, s+1)=(2m+s)(2m+s+1).$$

In fact, in general

$$\begin{aligned} &K(s, m)W(s, s-1, s-2, \cdots, s-i+1, s-i-1, \cdots, 1, 0) \\ &=\binom{s}{i}M(s-i+2, \cdots, s+1)/2^i. \end{aligned} \tag{4.3}$$

For the *second raw moment*, we get

$$\begin{aligned} &K(s, m)W(s+1, s, s-3, \cdots, 1, 0) \\ &=\left[\binom{s}{2}M(s, s+1)/2^4 3!\right] \\ &\quad\times\left[4s(s+1)m^2+2s(2s^2+5s+9)m+s^4+4s^3+11s^2+8s+12\right]. \end{aligned} \tag{4.4}$$

$$\begin{aligned} &K(s, m)W(s+1, s-1, s-2, s-4, \cdots, 1, 0) \\ &=\left[\binom{s}{3}M(s-1, s, s+1)/2^6\right]\left[2(3s-1)m+3s^2+s+10\right] \\ &\quad+(m+s+1)K(s, m)W(s, s-1, s-2, s-4, \cdots, 1, 0). \end{aligned} \tag{4.5}$$

The last determinant on the right side of (4.5) is evaluated by putting $i=3$ in (4.3). In general

$$\begin{aligned} &K(s, m)W(s+1, s-1, s-2, \cdots, s-i+1, s-i-1, \cdots, 1, 0) \\ &=\left[i\binom{s}{i}M(s-i+2, \cdots, s+1)/2^{i+1}(i+1)\right] \\ &\quad\times\left[2(s+1)m+(s+1)(s+2)+i+1\right]. \end{aligned} \tag{4.6}$$

$K(s, m)W(s, s-1, s-2, s-3, s-5, \cdots, 1, 0)$ is obtained from (4.3) by putting $i=4$. Now using the results (4.2)-(4.6) we get μ_2, the *second central moment* of $W_2^{(s)}$, as

$$\mu_2=\left[\binom{s}{2}M(s, s+1)/2^3\right]\left[4(s-1)m+2s^2-2s+3\right]. \tag{4.7}$$

For the *third raw moment*, we get

$$\begin{aligned} &K(s, m)W(s+2, s+1, s-3, \cdots, 1, 0) \\ &=\left[\binom{s+2}{4}M(s, s+1, s+2, s+3)/2^6\cdot 3!\right] \\ &\quad\times\left[4s(s+1)m^2+2s(2s^2+5s+21)m+s^4+4s^3+23s^2+20s+72\right]. \end{aligned} \tag{4.8}$$

In fact, in general

$$
\begin{aligned}
(4.9)\quad & K(s, m)W(s+2, s+1, s-2, \cdots, s-i+1, s-i-1, \cdots, 1, 0) \\
&= \left[i(i-1)\binom{s+2}{i+2}M(s-i+2, \cdots, s+3)/2^{i+5}3!\right] \\
&\quad \times \left[4s(s+1)m^2+2s(2s^2+5s+4i+13)m \right. \\
&\quad \left. +s(s+1)(s^2+3s+4i+12)+6(i+1)(i+2)\right].
\end{aligned}
$$

$$
\begin{aligned}
(4.10)\quad & K(s, m)W(s+2, s, s-2, s-4, \cdots, 1, 0) \\
&= \left[\binom{s+1}{4}M(s-1, s, s+1, s+2)/2^5\cdot 15\right] \\
&\quad \times \left[2s(8s+1)m^2+s(16s^2+19s+109)m+4s^4+9s^3+59s^2+54s+180\right] \\
&\quad +(m+s+2)K(s, m)W(s+1, s, s-2, s-4, \cdots, 1, 0).
\end{aligned}
$$

The value of the determinant in the last term on the right side of (4.10) is obtained by putting $i=3$ in the following general result:

$$
\begin{aligned}
(4.11)\quad & K(s, m)W(s+1, s, s-2, \cdots, s-i+1, s-i-1, \cdots, 1, 0) \\
&= \left[(i-1)\binom{s}{i}M(s-i+2, \cdots, s+1)/2^{i+3}(i+1)\right] \\
&\quad \times \left[4s(s+1)m^2+2s(2s^2+5s+2i+5)m \right. \\
&\quad \left. +s^4+4s^3+(2i+7)s^2+(2i+4)s+2i(i+1)\right].
\end{aligned}
$$

Now $K(s, m)W(s+1, s, s-2, s-3, s-5, \cdots, 1, 0)$ is obtained from (4.11) by putting $i=4$.

$$
\begin{aligned}
(4.12)\quad & K(s, m)W(s+2, s-1, s-2, s-3, s-5, \cdots, 1, 0) \\
&= \left[\binom{s+1}{5}M(s-2, \cdots, s+2)/2^5 3!\right]\left[2(5s-2)m+5s^2+s+42\right] \\
&\quad +(m+s+2)K(s, m)W(s+1, s-1, s-2, s-3, s-5, \cdots, 1, 0).
\end{aligned}
$$

In fact, in general

$$
\begin{aligned}
(4.13)\quad & K(s, m)W(s+j, s-1, s-2, \cdots, s-i+1, s-i-1, \cdots, 1, 0) \\
&= \left[\binom{i+j-2}{j-1}\binom{s+j-1}{i+j-1}M(s-i+2, \cdots, s+j)/2^{i+j}j(i+j)\right] \\
&\quad \times \left[2\{(i+j-1)s-j\}m+(i+j-1)s^2+(i-j-1)s+j(i-1)(i+j+1)\right] \\
&\quad +(m+s+j)K(s, m) \\
&\quad \times W(s+j-1, s-1, s-2, \cdots, s-i+1, s-i-1, \cdots, 1, 0).
\end{aligned}
$$

The value of the last determinant on the right side of (4.12) is obtained easily from (4.6) by putting $i=4$.

$$
(4.14)\quad K(s, m)W(s+1, s, s-1, s-4, \cdots, 1, 0)
$$

$$=\left[\binom{s}{3}M(s-1,\, s,\, s+1)/2^6 4!\right]\left[8(s-1)s(s+1)m^3\right.$$
$$+12(s-1)s(s^2+2s+5)m^2+2(s-1)(3s^4+9s^3+32s^2+14s+72)m$$
$$\left.+s(s-1)(s^4+4s^3+17s^2+14s+72)+144\right].$$

Now, $K(s, m)W(s+1, s-1, s-2, s-3, s-4, s-6, \cdots, 1, 0)$ is obtained from (4.6) by putting $i=5$ and $K(s, m)W(s, s-1, s-2, s-3, s-4, s-5, s-7, \cdots, 1, 0)$ from (4.3) with $i=6$.

For the *fourth raw moment* we get

(4.15) $$K(s, m)W(s+3, s+2, s-3, \cdots, 1, 0)$$
$$=\left[\binom{s+2}{4}M(s, s+1, s+2, s+3)/2^8 5!\right]$$
$$\times\left[16s(s+1)(s+2)(s+3)m^4+8s(s+2)(s+3)(4s^2+14s+46)m^3\right.$$
$$+4(s+1)(s+2)(6s^4+48s^3+233s^2+609s+720)m^2$$
$$+2(s+2)(4s^6+46s^5+310s^4+1320s^3+3542s^2+5802s+6480)m$$
$$\left.+(s+2)(s+3)(s^6+11s^5+81s^4+373s^3+1118s^2+2256s+4320)+2880\right].$$

(4.16) $$K(s, m)W(s+3, s+1, s-2, s-4, \cdots, 1,0)$$
$$=\left[\binom{s+2}{5}M(s-1, \cdots, s+3)/2^8 4!\right]\left[8s(s+1)(5s+3)m^3\right.$$
$$+4s(15s^3+47s^2+189s+213)m^2+2(15s^5+70s^4+403s^3+966s^2$$
$$\left.+1842s+1440)m+5s^6+31s^5+217s^4+769s^3+2210s^2+4512s+5760\right]$$
$$+(m+s+3)K(s,m)W(s+2, s+1, s-2, s-4, \cdots, 1, 0).$$

The value of the determinant in the last term of the right side of (4.16) is obtained from (4.9) by putting $i=3$. $K(s, m)W(s+2, s+1, s-2, s-3, s-5, \cdots, 1, 0)$ is deduced from (4.9) with $i=4$.

(4.17) $$K(s, m)W(s+3, s, s-1, s-4, \cdots, 1, 0)$$
$$=\left[\binom{s+2}{5}M(s-1, \cdots, s+3)/2^8 3!\right]\left[8s^2(s-1)m^3\right.$$
$$+4s(s-1)(3s^2+2s+24)m^2+2(s-1)(3s^4+4s^3+49s^2+24s+180)m$$
$$\left.+s(s-1)(s^4+2s^3+25s^2+24s+180)+360\right]$$
$$+(m+s+3)K(s, m)W(s+2, s, s-1, s-4, \cdots, 1, 0).$$

The value of the determinant in the last term on the right side of (4.17) is obtained from the following result by putting $i=3$.

(4.18) $$K(s, m)W(s+2, s, s-1, s-3, \cdots, s-i+1, s-i-1, \cdots, 1, 0)$$
$$=\left[(i-2)\binom{s+1}{i+1}M(s-i+2, \cdots, s+2)/2^{i+4}(i+2)4!\right]$$
$$\times\left[8s(s-1)\{3(i+1)s+2(i-1)\}m^3+4s(s-1)\{9(i+1)s^2+2(7i-1)s\right.$$

$$
\begin{aligned}
&+2(6i^2+20i+4)\}m^2+2(s-1)\{9(i+1)s^4+2(11i+1)s^3\\
&+(24i^2+85i+17)s^2+2(6i^2+19i+5)s+24i(i+1)(i+2)\}m\\
&+s(s-1)\{3(i+1)s^4+2(5i+1)s^3+3(4i^2+15i+3)s^2+2(6i^2+19i+5)s\\
&+24i(i+1)(i+2)\}+24(i-1)i(i+1)(i+2)\big]+(m+s+2)\\
&\times K(s,m)W(s+1,s,s-1,s-3,\cdots,s-i+1,s-i-1,\cdots,1,0),
\end{aligned}
$$

where the value of the last determinant on the right side of (4.18) is obtained from the following:

$$
\begin{aligned}
(4.19)\quad & K(s,m)W(s+1,s,s-1,s-3,\cdots,s-i+1,s-i-1,\cdots,1,0)\\
&=\left[(i-2)\binom{s}{i}M(s-i+2,\cdots,s+1)/2^{i+1}(i+1)4!\right]\\
&\times\big[8(s+1)s(s-1)m^3+12s(s-1)(s^2+2s+i+2)m^2\\
&+2(s-1)\{3s^4+9s^3+(6i+14)s^2+(3i+5)s+6i(i+1)\}m\\
&+s(s-1)\{s^4+4s^3+(3i+8)s^2+(3i+5)s+6i(i+1)\}+6(i-1)i(i+1)\big].
\end{aligned}
$$

$K(s, m)W(s+3, s, s-2, s-3, s-5,\cdots, 1, 0)$ is deduced from the following result by putting $i=4$.

$$
\begin{aligned}
(4.20)\quad & K(s,m)W(s+3,s,s-2,\cdots,s-i+1,s-i-1,\cdots,1,0)\\
&=\left[(i^2-1)\binom{s+2}{i+2}M(s-i+2,\cdots,s+3)/2^{i+4}(i+3)4!\right]\\
&\times\big[4s\{3(i+2)s+(i-6)\}m^2+2s\{6(i+2)s^2+3(3i-2)s\\
&+12i^2+49i-6\}m+s(s+1)\{3(i+2)s^2+(5i-6)s+12i(i+4)\}\\
&+12i(i+2)(i+3)\big]+(m+s+3)K(s,m)\\
&\times W(s+2,s,s-2,\cdots,s-i+1,s-i-1,\cdots,1,0)
\end{aligned}
$$

where the value of the determinant in the last term on the right side of (4.20) is given by

$$
\begin{aligned}
(4.21)\quad & K(s,m)W(s+2,s,s-2,\cdots,s-i+1,s-i-1,\cdots,1,0)\\
&=\left[(i-1)\binom{s+1}{i+1}M(s-i+2,\cdots,s+2)/2^{i+3}(i+2)3!\right]\\
&\times\big[4s\{2(i+1)s+i-2\}m^2+2s\{4(i+1)s^2+(7i-2)s+(6i^2+19i-2)\}m\\
&+s(s+1)\{2(i+1)s^2+2(2i-1)s+6i(i+3)\}+6i(i+1)(i+2)\big]\\
&+(m+s+2)K(s,m)W(s+1,s,s-2,\cdots,s-i+1,s-i-1,\cdots,1,0),
\end{aligned}
$$

where the value of the determinant in the last term on the right side of (4.21) is obtained from (4.11).

$$
\begin{aligned}
(4.22)\quad & K(s,m)W(s+2,s+1,s-1,s-4,\cdots,1,0)\\
&=\left[\binom{s+2}{5}M(s-1,\cdots,s+3)/2^{11}\right]\big[8(s+1)s(s-1)m^3
\end{aligned}
$$

$$+4s(s-1)(3s^2+6s+31)m^2+2(s-1)(3s^4+9s^3+64s^2+30s+240)m$$
$$+s(s-1)(s^4+4s^3+33s^2+30s+240)+480\big].$$

$K(s, m)W(s+2, s, s-1, s-3, s-5,\cdots, 1, 0)$ is obtained from (4.18) by putting $i=4$ and $K(s, m)W(s+2, s, s-2, s-3, s-4, s-6,\cdots, 1, 0)$ from (4.21) with $i=5$.

$$(4.23)\quad K(s, m)W(s+1, s, s-1, s-2, s-5,\cdots, 1, 0)$$
$$=\left[\binom{s}{4}M(s-2,\cdots, s+1)/2^8 5!\right]$$
$$\times\big[16(s+1)s(s-1)(s-2)m^4+16s(s-1)(s-2)(2s^2+3s+11)m^3$$
$$+4(s-1)(s-2)(6s^4+12s^3+65s^2-s+240)m^2$$
$$+2(s-2)(4s^6+6s^5+54s^4-68s^3+462s^2-698s+1680)m$$
$$+s^8+14s^6-60s^5+269s^4-900s^3+2596s^2-4800s+5760\big].$$

$K(s,m)W(s+1, s, s-1, s-3, s-4, s-6,\cdots, 1, 0)$ is obtained from (4.19) by putting $i=5$; $K(s, m)W(s+1, s, s-2, s-3, s-4, s-5, s-7,\cdots, 1, 0)$ from (4.11) with $i=6$; $K(s, m)W(s+3, s-1, s-2, s-3, s-4, s-6,\cdots, 1, 0)$ from (4.13) by putting $j=3$ and $i=5$; $K(s, m)W(s+2, s-1, s-2, s-3, s-4, s-5, s-7,\cdots, 1, 0)$ from (4.13) by putting $j=2$ and $i=6$; $K(s, m)\times W(s+1, s-1, s-2, s-3, s-4, s-5, s-6, s-8,\cdots, 1, 0)$ from (4.6) with $i=7$; and $K(s, m)W(s, s-1, s-2, s-3, s-4, s-5, s-6, s-7, s-9,\cdots, 1, 0)$ from (4.3) by putting $i=8$.

5. Moments of second, third and fourth esf's

Using the values of the determinants evaluated in the preceding section, the third central moment of the second esf, $W_2^{(s)}$, is obtained in the following simple form:

$$(5.1)\quad \mu_3\{W_2^{(s)}\}=\left[\binom{s}{2}M(s, s+1)/2^3\right]\big[5(s-1)^2(2m)^2$$
$$+(10s^3-20s^2+30s-23)2m+5s^4-10s^3+25s^2-26s+21\big].$$

Similarly the fourth central moment of $W_2^{(s)}$ is given by

$$(5.2)\quad \mu_4\{W_2^{(s)}\}=\left[3\binom{s}{2}M(s, s+1)/2^7\right]$$
$$\times\big[4s(s-1)^3(2m)^4+4(s-1)^2(4s^3-3s^2+30s-28)(2m)^3$$
$$+(24s^6-60s^5+408s^4-1056s^3+1833s^2-2173s+1048)(2m)^2$$
$$+(16s^7-36s^6+384s^5-1036s^4+2634s^3-4209s^2$$
$$+4503s-2760)(2m)+4s^8-8s^7+124s^6-340s^5+1145s^4$$
$$-2148s^3+3479s^2-3360s+1944\big].$$

Further, the results of the preceding section can also be used to obtain the first two moments of $W_3^{(s)}$ and $W_4^{(s)}$, the third and fourth esf's respectively in the $s\xi$'s. It may be observed in general that

$$(5.3)\quad \mu_1'\{W_i^{(s)}\}=K(s,m)W(s,s-1,s-2,\cdots,s-i+1,s-i-1,\cdots,1,0),\quad i=1,\cdots,s$$

and the value of the right side of (5.3) is given in (4.3). Now using the methods in section 3, we get

$$(5.4)\quad \begin{aligned}\mu_2'\{W_3^{(s)}\}=K(s,m)\big[&W(s+1,s,s-1,s-4,\cdots,1,0)\\&+W(s+1,s,s-2,s-3,s-5,\cdots,1,0)\\&+W(s+1,s-1,s-2,s-3,s-4,s-6,\cdots,1,0)\\&+W(s,s-1,s-2,s-3,s-4,s-5,s-7,\cdots,1,0)\big]\end{aligned}$$

and

$$(5.5)\quad \begin{aligned}\mu_2'\{W_4^{(s)}\}=K(s,m)\big[&W(s+1,s,s-1,s-2,s-5,\cdots,1,0)\\&+W(s+1,s,s-1,s-3,s-4,s-6,\cdots,1,0)\\&+W(s+1,s,s-2,s-3,s-4,s-5,s-7,\cdots,1,0)\\&+W(s+1,s-1,s-2,s-3,s-4,s-5,s-6,s-8,\cdots,1,0)\\&+W(s,s-1,s-2,s-3,s-4,s-5,s-6,s-7,s-9,\cdots,1,0)\big].\end{aligned}$$

It may be pointed out that the values of the determinants on the right side of (5.4) and (5.5) are available in the preceding section and using these values and (5.3), the variances of $W_3^{(s)}$ and $W_4^{(s)}$ were obtained and are given below.

$$(5.6)\quad \begin{aligned}\mu_2\{W_3^{(s)}\}=\Big[3\binom{s}{3}M(s-1,s,s+1)/2^6\Big]\big[&(s-1)(s-2)(2m)^2\\&+(s-2)(2s^2-3s+7)2m+s^4-4s^3+11s^2-20s+20\big],\end{aligned}$$

and

$$(5.7)\quad \begin{aligned}\mu_2\{W_4^{(s)}\}=\Big[\binom{s}{4}&M(s-2,\cdots,s+1)/4!\,2^4\Big]\\&\times\big[2(s-1)(s-2)(s-3)(2m)^3+3(s-2)(s-3)(2s^2-4s+11)\\&\times(2m)^2+(s-3)(6s^4-30s^3+106s^2-225s+314)2m\\&+2s^6-18s^5+89s^4-318s^3+845s^2-1500s+1368\big].\end{aligned}$$

6. Approximations to the distribution of $W_2^{(s)}$

Using the results on moments of $W_2^{(s)}$ given in (4.3), (4.7), (5.1) and (5.2) the following approximation to the distribution of $W_2^{(s)}$ is suggested:

$$f(W_2^{(s)})=\frac{\alpha^{\nu}}{2\Gamma(\nu)}\exp\{-\alpha(W_2^{(s)})^{1/2}\}(W_2^{(s)})^{\nu/2-1} \qquad 0<W_2^{(s)}<\infty, \tag{6.1}$$

where

$$\nu=s(2m+s+1)/2 \tag{6.2}$$

and

$$\alpha^2=2[s(2m+s+1)+2]/(s-1)(2m+s). \tag{6.3}$$

It may be pointed out that the first moment is the same for the exact and approximate distributions. For further comparison, numerical values of the first four moments from the exact and approximate distributions and the ratios of the respective approximate and exact moments and the moment quotients were computed (not given here) for values of $s=3$, 4, 5, 7 and 10 and selected values of m. The tables showed that the ratio of the respective approximate to the exact moments tends to unity as m increases or s increases or both. On the basis of these ratios the approximate distribution might be recommended for $m=5$ and above when $s=3$, $m=3$ and above for $s=4$, $m=2$ and above for $s=5$ and $m=0$ and above for $s=7$ and all values of m and all values of s beyond 7. The values of the approximate and exact standard deviations, β_1's and β_2's practically agree in the first two places at the smallest values of m recommended for each value of s and this in turn almost guarantees sufficient accuracy for upper or lower percentage points from the approximate distribution. It may further be observed that an interesting feature of the distribution of $W_2^{(s)}$ is that it is asymptotically normal for large values of m or s.

An alternate approximation (which is exact for $s=2$) is obtained by replacing the value of ν in (6.2) by $s(2m+s)/2$ and α^2 in (6.3) by $2[s(2m+s)+2]/(s-1)(2m+s+1)$. But this second approximation is not as good as the one suggested in (6.1) even for $s=3$.

7. Some remarks

It may be pointed that $2\sum_{i=1}^{s}\xi_i$ is distributed [11] as a chi-square with $s(2m+s+1)$ degrees of freedom and hence the distribution problem in this case is very simple. The results of this paper show that we can also have a simple approximation to the distribution of the second esf in the s ξ's. While the former chi-square distribution can be interpreted as the limiting distribution of Pillai's $V^{(s)}$ criterion [11], [13], [14], [17], the same is also true in the present case that the distribution of

$W_2^{(s)}$ can also be considered as the limiting distribution of the second esf in the s θ's following the joint density (1.1). It might also be pointed out that the distribution problem studied in this paper has great use since it has been shown that several tests based on the esf's of the characteristic roots have been observed to have monotonicity of power and other optimum properties [1], [2], [3], [9].

8. An example

The criterion $W_2^{(s)}$ may now be used to test the equality of p-dimensional vectors of l p-variate normal populations having a common covariance matrix. The values of m and n in (1.1) appropriate for this test are given by

$$m=\frac{1}{2}|l-p-1|-\frac{1}{2}, \qquad n=\frac{1}{2}(N-l-p-1), \tag{8.1}$$

where N is the total of l sample sizes. The data studied by Rao [19, p. 263] may be used for the test, which consist of measurements on (1) head length (2) height and (3) weight of 140 school boys of almost the same age belonging to six different schools in an Indian city. The problem is to test the equality of the three mean characters from the six different schools. Let S^* and S be the sum of product matrices 'between' and 'within' schools for the three characters. These are available in Pillai and Samson [18]. Now

$$(S+S^*)^{-1}=\begin{pmatrix} .0^478984 & -.0^461351 & .0^511246 \\ & .001388366 & -.0^3269036 \\ & & .0^497857 \end{pmatrix} \tag{8.2}$$

and

$$S^*(S+S^*)^{-1}=\begin{pmatrix} .04684095 & .11100391 & -.00576916 \\ .00808723 & .08898122 & -.00120418 \\ .01837418 & .09115586 & .05046243 \end{pmatrix}. \tag{8.3}$$

Now, from (8.3), $V_2^{(3)}=\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3=0.010340$. Further from (8.1), $m=0.5$ and $n=65$. For, this value of m and $s=3$, the first four moments, β_1 and β_2 of $W_2^{(3)}$ were computed using the results of sections 4 and 5, and their values are as follows:

$$\mu_1=15, \qquad \mu_2=142.5, \qquad \mu_3=3600, \qquad \mu_4=221400,$$
$$\sqrt{\mu_2}=11.9373, \qquad \beta_1=4.4788, \qquad \sqrt{\beta_1}=2.1163, \qquad \beta_2=10.9030.$$

Now using the above values of $\sqrt{\beta_1}$ and β_2 and extrapolating from

"Tables of percentage points of Pearson curves, for given $\sqrt{\beta_1}$ and β_2, expressed in standard measure" (Johnson et al. [8]), the upper 5 per cent point of $W_2^{(3)}$ was determined as 38. Further, taking $\xi_i = n\theta_i$ $(i=1, \cdots, s)$,

$$W_2^{(3)} = n^2 V_2^{(3)} = 44,$$

which shows that the test rejects the null hypothesis of equality of the mean characters of boys from six different schools. However, the test does not reject the null hypothesis at the upper 1% level. This agrees with (a) the findings of Rao [18] who examined the data using the Λ criterion of Pearson and Wilks which is the product, $\prod_{i=1}^{s}(1-\theta_i)$; (b) the findings of Pillai and Samson [18] who tested the same hypothesis based on the criterion $U^{(s)} = \sum_{i=1}^{s}[\theta_i/(1-\theta_i)] = \sum_{i=1}^{s}\lambda_i$, and (c) the findings of Pillai [16] who further considered the test of this hypothesis using the criterion $U_{s-1}^{(s)}$, the $(s-1)$th elementary symmetric function in the s λ's. Foster [5], however, finds that the largest root is significant only at the upper 15% level.

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