ASSOCIATION SCHEMES OF NEW TYPES AND NECESSARY CONDITIONS FOR EXISTENCE OF REGULAR AND SYMMETRICAL PBIB DESIGNS WITH THOSE ASSOCIATION SCHEMES

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1. Introduction

The purpose of this paper is to present methods of constructing Kronecker-product association schemes, reduced association schemes and enlarged association schemes.

The Kronecker-product designs and the reduced designs were defined by M. N. Vartak for the first time [8], but the association schemes which were considered in relation to those designs were not defined. In this paper the Kronecker-product association schemes and the reduced association schemes are defined independently of the incidence matrices of the designs. Those associations including some association schemes of known types, i.e. the group divisible association schemes [2], the rectangular association schemes [9], the group divisible m-dimensional association schemes [4], the right angular association schemes [7], and the cubic association schemes [5] are derived from some association schemes of BIB designs.

In the next section the characteristic matrix which characterizes the association scheme is defined. In the section 3 the Kronecker-product association scheme is defined and the designs with the Kronecker-product association schemes are discussed. In the section 4 the conditions for an association scheme to be reduced are given, and then reduced association schemes and the designs with those association schemes are defined. In the section 5 the enlarged association schemes are defined and the uniqueness of such an association scheme is shown. In the sections 6 and 7 some association schemes of new types and necessary conditions for existence of regular and symmetrical PBIB designs having those schemes are given.

2. The characteristic matrix of an association scheme

Among v treatments an association of m-associate classes is defined and let the association matrices $A_0, A_1, A_2, \cdots, A_m$ be

$$A_k = \| a^i_{jk} \| \qquad i, j = 1, 2, \cdots, v \quad \text{and} \quad k = 1, 2, \cdots, m$$

where

$$a^i_{jk} = \begin{cases} 1, & \text{if } i\text{th and } j\text{th treatments are } k\text{th associates,} \\ 0, & \text{otherwise.} \end{cases}$$

The parameters of the association scheme may be defined by the following equations:

$$(1) \qquad A_i A_j = \textstyle\sum_{k=0}^{m} p^k_{ij} A_k \qquad \text{for } i, j = 0, 1, \cdots, m.$$

These relations are rewritten as

$$(2) \qquad \begin{Vmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{Vmatrix} A_k = (P_k \otimes I) \begin{Vmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{Vmatrix}$$

where

$$P_k = \begin{Vmatrix} p^0_{0k} & p^1_{0k} & \cdots & p^m_{0k} \\ p^0_{1k} & p^1_{1k} & \cdots & p^m_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ p^0_{mk} & p^1_{mk} & \cdots & p^m_{mk} \end{Vmatrix} \qquad \text{for } k = 0, 1, \cdots, m,$$

$A \otimes B$ denotes the Kronecker-product of A and B, and I is the unit matrix of degree $m+1$.

Following S. Yamamoto and Y. Fujii [10], if we denote the association algebra which is generated by the matrices $A_0, A_1, \cdots, A_m$ by $\mathfrak{A}$, then the relation (2) defines the regular representation of the association algebra $\mathfrak{A}$,

$$(\mathfrak{A}): \quad A_k \to P_k .$$

Consider a non-singular real matrix

$$C = \begin{Vmatrix} c_{00} & c_{01} & \cdots & c_{0m} \\ c_{10} & c_{11} & \cdots & c_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m0} & c_{m1} & \cdots & c_{mm} \end{Vmatrix} \qquad \text{with } c_{00} = c_{01} = \cdots = c_{0m} = 1$$

which makes all P_k diagonal simultaneously, in such a way that

$$(3)\qquad CP_kC^{-1}=\begin{Vmatrix} z_{0k} & & & & 0 \\ & z_{1k} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & z_{mk} \end{Vmatrix} \qquad \text{for } k=0,1,\cdots,m.$$

The matrices defined by

$$(4)\qquad P_j^{\natural}=\left(\textstyle\sum_{u=0}^{m} c_{ju}z_{ju}\right)^{-1}\textstyle\sum_{k=0}^{m} c_{jk}P_k \qquad \text{for } j=0,1,\cdots,m$$

are mutually orthogonal idempotents of the algebra generated by $P_0, P_1, \cdots, P_m$.

Let the matrix Z, whose $(k+1)$st row is the diagonal elements of the matrices (3), be defined by

$$(5)\qquad Z=\begin{Vmatrix} z_{00} & z_{10} & \cdots & z_{m0} \\ z_{01} & z_{11} & \cdots & z_{m1} \\ \cdots & \cdots & \cdots & \cdots \\ z_{0m} & z_{1m} & \cdots & z_{mm} \end{Vmatrix} \qquad \text{with } z_{00}=z_{10}=\cdots=z_{m0}=1.$$

Then it follows from (4) that

$$\begin{Vmatrix} P_0^{\natural} \\ P_1^{\natural} \\ \cdot \\ \cdot \\ \cdot \\ P_m^{\natural} \end{Vmatrix}=((CZ)^{-1}C\otimes I)\begin{Vmatrix} P_0 \\ P_1 \\ \cdot \\ \cdot \\ \cdot \\ P_m \end{Vmatrix}=(Z^{-1}\otimes I)\begin{Vmatrix} P_0 \\ P_1 \\ \cdot \\ \cdot \\ \cdot \\ P_m \end{Vmatrix}.$$

Therefore the mutually orthogonal idempotents $A_0^{\natural}=(1/v)G_v, A_1^{\natural},\cdots, A_m^{\natural}$ of the algebra $\mathfrak{A}$ corresponding to $P_0^{\natural}, P_1^{\natural},\cdots, P_m^{\natural}$ respectively are given by

$$(6)\qquad \begin{Vmatrix} A_0^{\natural} \\ A_1^{\natural} \\ \cdot \\ \cdot \\ \cdot \\ A_m^{\natural} \end{Vmatrix}=(Z^{-1}\otimes I)\begin{Vmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{Vmatrix}$$

where G_v is the matrix of degree v whose elements are all unity.

DEFINITION 1. The matrix Z in (5) is called the characteristic matrix of an association scheme, when the elements of Z satisfy the following relations

$$(7)\qquad z_{00}=z_{10}=\cdots=z_{m0}=1$$

$$\sum_{i=0}^{m} z_{ui}=0 \qquad \text{for } u=1, 2, \cdots, m$$
$$\sum_{i=0}^{m} z_{0i}=v$$
$$z_{ui}z_{uj}=\sum_{k=0}^{m} p_{ij}^{k} z_{uk} \qquad \text{for } u=0, 1, \cdots, m,$$

where p_{ij}^{k} $(i, j, k=0, 1, \cdots, m)$ are non-negative integers.

The following lemma 1 and the necessity of the word incidence in the lemma were shown by R. C. Bose and D. M. Mesner [1].

LEMMA 1. *If A_i $(i=0, 1, \cdots, m)$ are symmetric incidence matrices satisfying*

$$A_0=I_v$$
$$\sum_{i=0}^{m} A_i=G$$
$$A_jA_k=\sum_{i=0}^{m} p_{jk}^{i} A_i$$

for some set of constants p_{jk}^{i}, then A_i are the association matrices of an association scheme and p_{jk}^{i} $(i, j, k=0, 1, \cdots, m)$ are the parameters of the association scheme.

THEOREM 1. *Necessary and sufficient conditions that symmetric incidence matrices $A_0=I, A_1, \cdots, A_m$ of degree v be the association matrices of an association scheme are: (i) there exists a characteristic matrix Z of degree $m+1$, (ii) $m+1$ matrices $A_0^{\sharp}=(1/v)G_v, A_1^{\sharp}, \cdots, A_m^{\sharp}$ defined by*

$$\begin{Vmatrix} A_0^{\sharp} \\ A_1^{\sharp} \\ \cdot \\ \cdot \\ \cdot \\ A_m^{\sharp} \end{Vmatrix} = (Z^{-1} \otimes I) \begin{Vmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{Vmatrix} \tag{8}$$

are mutually orthogonal idempotents of an algebra.

PROOF. It can be shown without any difficulty that the resultant conditions of theorem 1 are necessary, so we omit the proof of the necessity. Let the matrix Z given by (5) be a characteristic matrix, that is to say, let the elements of the matrix Z satisfy the equations (7). And let $A_0^{\sharp}=(1/v)G_v, A_1^{\sharp}, \cdots, A_m^{\sharp}$ which are defined by (8) be mutually orthogonal idempotents of an algebra. Then it follows that

$$A_i=\sum_{u=0}^{m} z_{ui}A_u^{\sharp} \qquad \text{for } i=0, 1, \cdots, m.$$

Therefore

$$A_iA_j=\sum_{u=0}^{m} z_{ui}z_{uj}A_u^{\sharp}$$
$$=\sum_{u=0}^{m}\sum_{k=0}^{m} p_{ij}^{k} z_{uk}A_u^{\sharp}=\sum_{k=0}^{m} p_{ij}^{k} A_k$$

$$(j'\otimes I)\begin{Vmatrix} A_0 \\ A_1 \\ \vdots \\ A_m \end{Vmatrix} = (j'\otimes I)(Z\otimes I)\begin{Vmatrix} A_0^* \\ A_1^* \\ \vdots \\ A_m^* \end{Vmatrix} = vA_0^* = G_v$$

where $j'=(1, 1, \cdots, 1)$ is the vector of degree $m+1$. From lemma 1 it follows that $A_0=I, A_1, \cdots, A_m$ are association matrices of an association scheme. This theorem provides a useful algebraic method of verifying whether a given relation satisfies the conditions of an association.

THEOREM 2. *When the matrices Z_1 and Z_2 are characteristic matrices having the properties defined by* (7), *then $Z_{1\otimes 2}=Z_1\otimes Z_2$ is the characteristic matrix of an association scheme.*

PROOF. The elements of such a matrix as

$$Z_{1\otimes 2} = \begin{Vmatrix} z_{100}Z_2 & z_{110}Z_2 & \cdots & z_{1m_10}Z_2 \\ z_{101}Z_2 & z_{111}Z_2 & \cdots & z_{1m_11}Z_2 \\ \cdots & \cdots & \cdots & \cdots \\ z_{10m_1}Z_2 & z_{11m_1}Z_2 & \cdots & z_{1m_1m_1}Z_2 \end{Vmatrix} \tag{10}$$

have the following properties

(i) $\sum_{i=0}^{m_1}\sum_{k=0}^{m_2} z_{1ui}z_{2vk} = \sum_{i=0}^{m_1} z_{1ui}\sum_{k=0}^{m_2} z_{2vk}=0$ for $u=1, 2, \cdots, m_1$ and $v=0, 1, \cdots, m_2$

(ii) $\sum_{i=0}^{m_1}\sum_{k=0}^{m_2} z_{10i}z_{2vk} = \sum_{i=0}^{m_1} z_{10i}\sum_{k=0}^{m_2} z_{2vk}=0$ for $v=1, 2, \cdots, m_2$

(iii) $\sum_{i=0}^{m_1}\sum_{k=0}^{m_2} z_{10i}z_{20k} = \sum_{i=0}^{m_1} z_{10i}\sum_{k=0}^{m_2} z_{20k}=v_1v_2$

(iv) $z_{1ui}z_{2vk}z_{1uj}z_{2vh} = \sum_{x=0}^{m_1} p_{1ij}^x z_{1ux}\sum_{y=0}^{m_2} p_{2kh}^y z_{2vy} = \sum_{x=0}^{m_1}\sum_{y=0}^{m_2} p_{1ij}^x p_{2kh}^y z_{1ux}z_{2vy}$ for $u=0, 1, \cdots, m_1$ and $v=0, 1, \cdots, m_2$

where p_{1ij}^x and p_{2kh}^y are non-negative integers. Therefore $Z_{1\otimes 2}$ is the characteristic matrix.

DEFINITION 2. The association scheme corresponding to $Z_{1\otimes 2}$ is called the Kronecker-product association scheme and denoted by $\mathfrak{M}_1\otimes\mathfrak{M}_2$, where $\mathfrak{M}_i$ is the association corresponding to the characteristic matrix Z_i.

DEFINITION 3. An $(n\times m)$-matrix $L=\|l_{ij}\|$ is called incidental, if each element of the matrix is 0 or 1 and $l_{11}=1$, $l_{1j}=0$ $(j=2, 3, \cdots, m)$, and if $\sum_{i=2}^{n} l_{ij}=1$ $(j=2, 3, \cdots, m)$, $\sum_{j=1}^{m} l_{ij}\geqq 1$ $(i=1, 2, \cdots, n)$.

DEFINITION 4. The characteristic matrix Z of an association scheme with m-associate classes is called reducible, if there exist incidental $((n+1)\times(m+1))$-matrices L and L^* and a non-singular square matrix $\bar{Z}$ of order $(n+1)$ such that $LZ=\bar{Z}L^*$ for $n<m$. The matrix $\bar{Z}$ is said to be reduced from the characteristic matrix Z.

THEOREM 3. *The reduced matrix $\bar{Z}$ is the characteristic matrix of an association scheme, if there exists a set of non-negative integers $\bar{p}_{ij}^k$ such that*

$$\bar{z}_{ui}\bar{z}_{uj}=\sum_{k=0}^{m}\bar{p}_{ij}^k\bar{z}_{uk} \qquad \text{for} \quad i, j, u=0, 1, \cdots, m, \tag{11}$$

where $\bar{z}_{ui}$ are the elements of the reduced matrix $\bar{Z}$.

DEFINITION 5. The association scheme corresponding to the reducible matrix Z is called reducible. If the original association scheme is denoted by $\mathfrak{M}$, the association scheme $\overline{\mathfrak{M}}$ corresponding to the characteristic matrix $\bar{Z}$ is said to be the reduced association scheme from $\mathfrak{M}$.

PROOF OF THE THEOREM 3. Since $l_{00}=1$, $l_{0j}=0$ $(j=1, 2, \cdots, m)$, then we have $\bar{z}_{00}=\bar{z}_{10}=\cdots=\bar{z}_{n0}=1$. Multiplying $LZ=\bar{Z}L^*$ by $j'=(1, 1, \cdots, 1)$ from the left, we have

$$\underbrace{(1, 1, \cdots, 1)}_{n+1}LZ=\underbrace{(1, 1, \cdots, 1)}_{m+1}Z=\underbrace{(v, 0, \cdots, 0)}_{m+1}=\underbrace{(1, 1, \cdots, 1)}_{n+1}\bar{Z}L^*.$$

If we put $(1, 1, \cdots, 1)\bar{Z}=(\alpha_0, \alpha_1, \cdots, \alpha_n)$, then we have $\alpha_0=v$, $\alpha_1=\alpha_2=\cdots=\alpha_n=0$. Therefore $(1, 1, \cdots, 1)\bar{Z}=(v, 0, \cdots, 0)$. Moreover, we have

$$\bar{z}_{ui}\bar{z}_{uj}=\sum_{k=0}^{m}\bar{p}_{ij}^k\bar{z}_{uk} \qquad (i, j, u=0, 1, \cdots, m).$$

From definition 1, the proof is complete.

Here we list the necessary properties of the Kronecker-products for the reader's convenience.

LEMMA 2. $(AB)\otimes(CD)=(A\otimes C)(B\otimes D)$.

LEMMA 3. $(A+B)\otimes(C+D)=A\otimes C+B\otimes C+A\otimes D+B\otimes D$.

LEMMA 4.

$$(B_1, B_2, \cdots, B_s)\begin{Vmatrix} C_1 \\ C_2 \\ \vdots \\ C_s \end{Vmatrix}\otimes(D_1, D_2, \cdots, D_m)\begin{Vmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{Vmatrix}$$

$$=(B_1\otimes D_1, B_1\otimes D_2,\cdots, B_1\otimes D_m, B_2\otimes D_1,\cdots, B_s\otimes D_m)\left\|\begin{matrix} C_1\otimes F_1 \\ C_1\otimes F_2 \\ \vdots \\ C_1\otimes F_m \\ C_2\otimes F_1 \\ \vdots \\ C_2\otimes F_m \\ C_3\otimes F_1 \\ \vdots \\ C_s\otimes F_m \end{matrix}\right\|.$$

3. Kronecker-product association scheme

Let the matrices $A_{10}=I_{v_1}, A_{11},\cdots, A_{1m_1}$ of degree v_1 be the association matrices of the association scheme $\mathfrak{M}_1$ and also let the matrices $A_{20}=I_{v_2}$, $A_{21},\cdots, A_{2m_2}$ of degree v_2 be the association matrices of the association scheme $\mathfrak{M}_2$. Let the matrices Z_1, Z_2 and $Z_{1\otimes 2}$ be the characteristic matrices of $\mathfrak{M}_1$, $\mathfrak{M}_2$ and the Kronecker-product association scheme $\mathfrak{M}_1\otimes\mathfrak{M}_2$ respectively.

THEOREM 4. *The association matrices of* $\mathfrak{M}_1\otimes\mathfrak{M}_2$ *are*

$$A_{i\otimes j}=A_{1i}\otimes A_{2j} \qquad (i=0, 1,\cdots, m_1, \quad j=0, 1,\cdots, m_2). \tag{12}$$

PROOF. Since A_{1i} and A_{2j} are the symmetric and incidence matrices, $A_{i\otimes j}$ are symmetric and incidence matrices. From theorem 1, the matrices $A^\sharp_{10}, A^\sharp_{11},\cdots, A^\sharp_{1m_1}$ defined by

$$\left\|\begin{matrix} A^\sharp_{10} \\ A^\sharp_{11} \\ \vdots \\ A^\sharp_{1m_1} \end{matrix}\right\| = (Z_1^{-1}\otimes I_{v_1})\left\|\begin{matrix} A_{10} \\ A_{11} \\ \vdots \\ A_{1m_1} \end{matrix}\right\| \tag{13}$$

are mutually orthogonal idempotents of the algebra. And also the matrices $A^\sharp_{20}, A^\sharp_{21},\cdots, A^\sharp_{2m_2}$ defined by

$$\left\|\begin{matrix} A^\sharp_{20} \\ A^\sharp_{21} \\ \vdots \\ A^\sharp_{2m_2} \end{matrix}\right\| = (Z_2^{-1}\otimes I_{v_2})\left\|\begin{matrix} A_{20} \\ A_{21} \\ \vdots \\ A_{2m_2} \end{matrix}\right\| \tag{14}$$

are mutually orthogonal idempotents of the algebra.

Therefore the matrices $A^{\sharp}_{i\otimes j}=A^{\sharp}_{1i}\otimes A^{\sharp}_{2j}$ $(i=0, 1, \cdots, m_1,\ j=0, 1, \cdots, m_2)$ are mutually orthogonal idempotents of an algebra. Since

$$A_{u\otimes v}=A_{1u}\otimes A_{2v}=\sum_{\alpha=0}^{m_1} z_{1\alpha u}\sum_{\beta=0}^{m_2} z_{2\beta v}A^{\sharp}_{1\alpha}\otimes A^{\sharp}_{2\beta} =\sum_{\alpha=0}^{m_1}\sum_{\beta=0}^{m_2} z_{1\alpha u}z_{2\beta v}A^{\sharp}_{\alpha\otimes\beta} \tag{15}$$

we have

$$\left\| \begin{matrix} A_{0\otimes 0} \\ A_{0\otimes 1} \\ \vdots \\ A_{m_1\otimes m_2} \end{matrix} \right\| =(Z_1\otimes Z_2)\otimes(I_{v_1}\otimes I_{v_2}) \left\| \begin{matrix} A^{\sharp}_{0\otimes 0} \\ A^{\sharp}_{0\otimes 1} \\ \vdots \\ A^{\sharp}_{m_1\otimes m_2} \end{matrix} \right\| . \tag{16}$$

Therefore, from theorem 1 and theorem 2 the proof is complete.

When we have the association scheme $\mathfrak{M}$, then we get a partially balanced design with r replications and blocks if we can arrange the v treatments into b blocks such that

(i) each block contains k treatments (all different)

(ii) each treatment is contained in r blocks

(iii) if two treatments α and β are ith associate, then they occur together in λ_i blocks, the number λ_i being independent of the particular pair of ith associates α and β.

The numbers v, b, r, k, λ_i $(i=1, 2, \cdots, m)$ are the parameters of the design with association scheme $\mathfrak{M}$.

Let the parameters of the design with $\mathfrak{M}_1\otimes\mathfrak{M}_2$ be $v, b, r, k, \lambda_{i\otimes j}$ $(i=0, 1, \cdots, \mathrm{m}_1,\ j=0, 1, \cdots, m_2)$, where $r=\lambda_{0\otimes 0}$. And let the parameters of the design with $\mathfrak{M}_p$ $(p=1, 2)$ be $v_p, b_p, r_p, k_p, \lambda_{pj}$ $(j=1, 2, \cdots, m_p)$.

The incidence matrix $N=||n_{ij}||$ of a design is defined by

$$n_{ij}=\begin{cases} 1, & \text{if treatment } i \text{ occurs in block } j, \\ 0, & \text{otherwise.} \end{cases}$$

Let their incidence matrices be denoted by N_1, N_2 and N respectively, then

$$\begin{aligned} N_1N_1'&=r_1A_{10}+\lambda_{11}A_{11}+\cdots+\lambda_{1m_1}A_{1m_1} \\ N_2N_2'&=r_2A_{20}+\lambda_{21}A_{21}+\cdots+\lambda_{2m_2}A_{2m_2} \\ N_{\otimes}N_{\otimes}'&=rA_{0\otimes 0}+\lambda_{0\otimes 1}A_{0\otimes 1}+\cdots+\lambda_{m_1\otimes m_2}A_{m_1\otimes m_2}, \end{aligned} \tag{17}$$

also

$$\begin{aligned} N_1N_1'&=\rho_{10}A^{\sharp}_{10}+\rho_{11}A^{\sharp}_{11}+\cdots+\rho_{1m_1}A^{\sharp}_{1m_1} \\ N_2N_2'&=\rho_{20}A^{\sharp}_{20}+\rho_{21}A^{\sharp}_{21}+\cdots+\rho_{2m_2}A^{\sharp}_{2m_2} \\ N_{\otimes}N_{\otimes}'&=\rho_{0\otimes 0}A^{\sharp}_{0\otimes 0}+\rho_{0\otimes 1}A^{\sharp}_{0\otimes 1}+\cdots+\rho_{m_1\otimes m_2}A^{\sharp}_{m_1\otimes m_2} \end{aligned} \tag{18}$$

where ρ_{1i}, ρ_{2j} and $\rho_{i\otimes j}$ are characteristic roots of the matrices N_1N_1', N_2N_2' and $N_\otimes N_\otimes'$ respectively. From (13), (14) and (16) it follows that

$$(19)\qquad \begin{aligned}(\rho_{10}, \rho_{11}, \cdots, \rho_{1m_1}) &= (r_1, \lambda_{11}, \cdots, \lambda_{1m_1})Z_1\\ (\rho_{20}, \rho_{21}, \cdots, \rho_{2m_2}) &= (r_2, \lambda_{21}, \cdots, \lambda_{2m_2})Z_2\\ (\rho_{0\otimes 0}, \rho_{0\otimes 1}, \cdots, \rho_{m_1\otimes m_2}) &= (r, \lambda_{0\otimes 1}, \cdots, \lambda_{m_1\otimes m_2})Z_{1\otimes 2}.\end{aligned}$$

In general, $N_\otimes \neq N_1 \otimes N_2$, so it can be said that the design with the Kronecker-product association scheme is not the Kronecker-product design defined by M. N. Vartak.

If $N_\otimes = N_1 \otimes N_2$, then from lemmas 2 and 3 we have for any i and j

$$(20)\qquad \rho_{i\otimes j} = \rho_{1i}\rho_{2j}$$

and also

$$(21)\qquad r = r_1r_2,\quad \lambda_{0\otimes j} = r_1\lambda_{2j},\quad \lambda_{i\otimes 0} = \lambda_{1i}r_2,\quad \lambda_{i\otimes j} = \lambda_{1i}\lambda_{2j}$$
$$(i = 1, 2, \cdots, m_1 \text{ and } j = 1, 2, \cdots, m_2).$$

The design with the Kronecker-product association scheme is said to be the Kronecker-product design, if and only if $N_\otimes = N_1 \otimes N_2$, and we are to be concerned with the Kronecker-product design hereafter.

The latent vectors corresponding to the characteristic roots ρ_{p0} of N_pN_p' are denoted by the column vectors of G_{v_p} for $p = 1, 2$. Since latent vectors corresponding to the characteristic root ρ_{1i} of N_1N_1' are denoted by column vectors of $A_{1i}^\sharp$, and since trace$(A_{1i}^\sharp) = \alpha_i$, without loss of generality, the α_i linear independent vectors among the column vectors of $A_{1i}^\sharp$ can be represented by $\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \cdots, \boldsymbol{x}_{i\alpha_i}$. By the same reason, the linear independent vectors among latent vectors corresponding to the characteristic root ρ_{2i} of N_2N_2' can be represented by $\boldsymbol{y}_{i1}, \boldsymbol{y}_{i2}, \cdots, \boldsymbol{y}_{i\beta_i}$, and they are column vectors of $A_{2i}^\sharp$ with $\beta_i = $ trace$(A_{2i}^\sharp)$. Linear independent latent vectors corresponding to the characteristic root $\rho_{i\otimes j}$ of $N_\otimes N_\otimes'$ are denoted by the linear independent column vectors of the idempotent $A_{i\otimes j}^\sharp$, and without loss of generality they can be represented by

$$\boldsymbol{x}_{i1} \otimes \boldsymbol{y}_{j1}, \boldsymbol{x}_{i1} \otimes \boldsymbol{y}_{j2}, \cdots, \boldsymbol{x}_{i1} \otimes \boldsymbol{y}_{j\beta_j}, \cdots, \boldsymbol{x}_{i\alpha_i} \otimes \boldsymbol{y}_{j\beta_j}.$$

Therefore, we have the following theorem.

THEOREM 5. *Let the Gramian matrices of systems of vectors corresponding to the characteristic root ρ_{1i} of N_1N_1', to the characteristic root ρ_{2j} of N_2N_2' and to the characteristic root $\rho_{i\otimes j}$ of $N_\otimes N_\otimes'$ be Q_{1i}, Q_{2j} and $Q_{i\otimes j}$ respectively. If the above-mentioned vectors are rational, then $Q_{i\otimes j}$ is rationally congruent to $Q_{1i} \otimes Q_{2j}$.*

4. Reduced association scheme

Let us consider a reducible association scheme $\mathfrak{M}$ with m-associate classes. Let $A_0=I, A_1,\cdots, A_m$ of degree v be the association matrices of $\mathfrak{M}$.

If Z is the characteristic matrix of $\mathfrak{M}$, then there exists a reduced matrix $\bar{Z}$ of degree $n+1$ $(n<m)$ such that $LZ=\bar{Z}L^*$ where L and L^* are incidental $((n+1)\times(m+1))$-matrices.

THEOREM 6. *If there exists a set of non-negative integers* $\bar{p}_{ij}^k$ *such as those in* (11), *the matrices* $\bar{A}_0, \bar{A}_1,\cdots, \bar{A}_n$ *which are denoted by*

$$\begin{Vmatrix} \bar{A}_0 \\ \bar{A}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}_n \end{Vmatrix} = (L\otimes I) \begin{Vmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{Vmatrix} \tag{22}$$

are the association matrices of the reduced association scheme from $\mathfrak{M}$. *The reduced association scheme from* $\mathfrak{M}$ *is denoted by* $\overline{\mathfrak{M}}$.

PROOF. Let the idempotents of the association algebra generated by the association matrices $A_0, A_1,\cdots, A_m$ be $A_0^\sharp, A_1^\sharp,\cdots, A_m^\sharp$. Then

$$\begin{Vmatrix} \bar{A}_0 \\ \bar{A}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}_n \end{Vmatrix} = (LZ\otimes I) \begin{Vmatrix} A_0^\sharp \\ A_1^\sharp \\ \cdot \\ \cdot \\ \cdot \\ A_m^\sharp \end{Vmatrix}. \tag{23}$$

When the matrices $\bar{A}_0^\sharp, \bar{A}_1^\sharp,\cdots, \bar{A}_n^\sharp$ are denoted by

$$\begin{Vmatrix} \bar{A}_0^\sharp \\ \bar{A}_1^\sharp \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}_n^\sharp \end{Vmatrix} = (L^*\otimes I) \begin{Vmatrix} A_0^\sharp \\ A_1^\sharp \\ \cdot \\ \cdot \\ \cdot \\ A_m^\sharp \end{Vmatrix}, \tag{24}$$

they are mutually orthogonal idempotents of an algebra. Since $LZ=\bar{Z}L^*$, then

$$\begin{Vmatrix} \bar{A}_0 \\ \bar{A}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}_n \end{Vmatrix} = (\bar{Z}L^* \otimes I) \begin{Vmatrix} A_0^\sharp \\ A_1^\sharp \\ \cdot \\ \cdot \\ \cdot \\ A_m^\sharp \end{Vmatrix} = (\bar{Z} \otimes I) \begin{Vmatrix} \bar{A}_0^\sharp \\ \bar{A}_1^\sharp \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}_n^\sharp \end{Vmatrix}. \tag{25}$$

Hence $\bar{A}_0 = A_0 = I$. Therefore, from theorem 1 and theorem 3 the result of theorem 6 follows.

The design with the reduced association scheme is called the reduced design.

Let the parameters of the design with $\bar{\mathfrak{M}}$ be denoted by $\bar{v}, \bar{b}, \bar{r}, \bar{k}, \bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_n$, and also the parameters of the design with $\mathfrak{M}$ be denoted by $v, b, r, k, \lambda_1, \lambda_2, \cdots, \lambda_m$.

COROLLARY. *There is a PBIB design with n associate classes with the reduced association $\bar{\mathfrak{M}}$ reduced from $\mathfrak{M}$ with m associate classes, if and only if*

$$(r, \lambda_1, \cdots, \lambda_m) = (\bar{r}, \bar{\lambda}_1, \cdots, \bar{\lambda}_n) L. \tag{26}$$

PROOF. If (26) holds, then

$$\textstyle\sum_{i=0}^m \lambda_i A_i = ((\bar{\lambda}_0, \bar{\lambda}_1, \cdots, \bar{\lambda}_n) L \otimes I) \begin{Vmatrix} A_0 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{Vmatrix}, \tag{27}$$

where $\lambda_0 = r$, $\bar{\lambda}_0 = \bar{r}$. Hence,

$$\textstyle\sum_{j=0}^n \bar{\lambda}_j \bar{A}_j = ((\bar{\lambda}_0, \bar{\lambda}_1, \cdots, \bar{\lambda}_n) \bar{Z} \otimes I) \begin{Vmatrix} \bar{A}_0^\sharp \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}_n^\sharp \end{Vmatrix}$$

$$= ((\bar{\lambda}_0, \bar{\lambda}_1, \cdots, \bar{\lambda}_n) \bar{Z} L \otimes I) \begin{Vmatrix} A_0^\sharp \\ \cdot \\ \cdot \\ \cdot \\ A_m^\sharp \end{Vmatrix}$$

$$= ((\bar{\lambda}_0, \bar{\lambda}_1, \cdots, \bar{\lambda}_n) L \otimes I) \begin{Vmatrix} A_0 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{Vmatrix}.$$

Therefore $\sum_{i=0}^{m}\lambda_i A_i=\sum_{j=0}^{n}\bar{\lambda}_j\bar{A}_j$. Vice versa, if $\sum_{i=0}^{m}\lambda_i A_i=\sum_{j=0}^{n}\bar{\lambda}_j\bar{A}_j$ then (27) holds and (26) follows. It may be said that the design with $\overline{\mathfrak{M}}$ is a reduced design, when (26) holds.

From (24), it is easy to see that the latent vectors corresponding to the characteristic root $\bar{\rho}_i$ of $\bar{N}\bar{N}'$ are column vectors of $\bar{A}_i^{\sharp}=\sum_{j=0}^{m}l_{ij}^{*}A_j^{\sharp}$, where l_{ij}^{*} is the element of L^{*}. For fixed i, if $l_{ij_1}^{*}, l_{ij_2}^{*},\cdots, l_{ij_{\beta_i}}^{*}$ are unity and the other l_{ij}^{*} are zero, the linear independent vectors among column vectors of $\bar{A}_i^{\sharp}$ are represented by

$$
\begin{array}{l}
x_{j_1 1},\ x_{j_1 2},\ \cdots,\ x_{j_1 \alpha_{j_1}},\\
x_{j_2 1},\ x_{j_2 2},\ \cdots,\ x_{j_2 \alpha_{j_2}}\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{j_{\beta_i} 1},\ x_{j_{\beta_i} 2},\cdots,\ x_{j_{\beta_i} \alpha_{j_{\beta_i}}},
\end{array}
\tag{28}
$$

where $x_{j_s u}$ $(u=1, 2,\cdots, \alpha_{j_s})$ are the linear independent vectors among column vectors of $A_{j_s}^{\sharp}$.

Since $A_i^{\sharp}A_j^{\sharp}=0$ for $i\neq j$, then each vector in a row of (28) is orthogonal to all vectors in the other rows. Therefore, we have the following theorem.

THEOREM 7. *Let the gramian matrix of the system of the latent vectors corresponding to the characteristic root ρ_j of NN' be Q_j. So long as we restrict ourselves to rational elements, the gramian matrix $\bar{Q}_i$ of the system of the vectors corresponding to the characteristic root $\bar{\rho}_i$ of $\bar{N}\bar{N}$ is rationally congruent to*

$$
\left\| \begin{array}{cccc}
Q_{j_1} & & & 0\\
& Q_{j_2} & & \\
& & \ddots & \\
0 & & & Q_{j_{\beta_i}}
\end{array} \right\| .
\tag{29}
$$

5. Enlarged association scheme

Consider the association scheme with m-associate classes. Let the characteristic matrix of the association scheme be Z.

Let the elements of the matrix

$$
Z^{+}=\left\| \begin{array}{cccc}
z_{00}^{+} & z_{10}^{+} & \cdots & z_{m+1\,0}^{+}\\
z_{01}^{+} & z_{11}^{+} & \cdots & z_{m+1\,1}^{+}\\
\cdots & \cdots & \cdots & \cdots\\
z_{0\,m+1}^{+} & z_{1\,m+1}^{+} & \cdots & z_{m+1\,m+1}^{+}
\end{array} \right\|
\tag{30}
$$

be defined by

$$
\begin{aligned}
(31)\qquad & z_{00}^{+}=z_{10}^{+}=\cdots=z_{m0}^{+}=z_{m+1\,0}^{+}=1\,,\\
& z_{m+1\,1}^{+}=z_{m+1\,2}^{+}=\cdots=z_{m+1\,m}^{+}=0\,,\\
& z_{m+1\,m+1}^{+}=-1,\quad z_{0\,m+1}^{+}=z_{1\,m+1}^{+}=\cdots=z_{m\,m+1}^{+}=s-1\,,\\
& z_{ij}^{+}=sz_{ij}\qquad \text{for } i=0,1,\cdots,m \quad\text{and}\quad j=1,2,\cdots,m\,,
\end{aligned}
$$

where z_{ij} is the element of the characteristic matrix Z. Since

$$
(32)\qquad Z^{+}\begin{Vmatrix}
1 & 0\cdots0 & 0\cdots0\\
0 & 1 \quad 0 & 0\cdots0\\
\cdot & \cdot & \\
\cdot & \quad\cdot & \\
\cdot & \qquad\cdot & \cdots\cdots\\
0 & 0 \quad 1 & 0\cdots0\\
0 & 0\cdots0 & 1\cdots1
\end{Vmatrix}
=\begin{Vmatrix}
1 & 0\cdots0 & 0 & \cdots\cdot 0\\
0 & 1 \quad 0 & 0 & 1 \quad 0\\
\cdot & \cdot & \cdot & \cdot\\
\cdot & \quad\cdot & \cdot & \quad\cdot\\
\cdot & \qquad\cdot & \cdot & \qquad\cdot\\
0 & 0 \quad 1 & 0 & 0 \quad 1\\
0 & 0\cdots0 & 1 & 0\cdots0
\end{Vmatrix}
\left(\begin{Vmatrix} 1 & 1\\ s-1 & -1\end{Vmatrix}\otimes Z\right),
$$

Z^{+} is the reduced matrix from the characteristic matrix $\begin{Vmatrix} 1 & 1\\ s-1 & -1\end{Vmatrix}\otimes Z$ of the Kronecker-product association scheme, which is Kronecker-product of a BIB association scheme and the given association scheme.

Let the parameters p_{jk}^{+i} $(i, j, k=0, 1, 2,\cdots, m+1)$ be

$$
\begin{aligned}
(33)\qquad & p_{jk}^{+i}=sp_{jk}^{i},\quad p_{jk}^{+0}=p_{jk}^{+m+1}=sp_{jk}^{0},\\
& p_{0k}^{+i}=p_{0k}^{i},\qquad p_{m+1\,k}^{+i}=(s-1)p_{0k}^{i},\\
& p_{00}^{+i}=p_{0\,m+1}^{+i}=p_{m+1\,m+1}^{+i}=p_{0k}^{+0}=p_{m+1\,k}^{+0}=p_{m+1\,k}^{+m+1}=p_{0\,k}^{+m+1}\\
& \quad =p_{m+1\,0}^{+0}=p_{0\,0}^{+m+1}=0,\\
& p_{00}^{+0}=p_{m+1\,0}^{+m+1}=1,\quad p_{m+1\,m+1}^{+0}=(s-1),\quad p_{m+1\,m+1}^{+m+1}=s-2,
\end{aligned}
$$

where p_{jk}^{i} $(i, j, k=0, 1,\cdots, m)$ are the parameters of the given association scheme. So, p_{jk}^{+i} $(i, j, k=0, 1,\cdots, m+1)$ are non-negative integers. Hence

$$
(34)\qquad z_{ui}^{+}z_{uj}^{+}=\textstyle\sum_{k=0}^{m+1} p_{ij}^{+k}z_{uk}^{+}\,.
$$

Therefore the matrix Z^{+} is the characteristic matrix of an association scheme.

DEFINITION 6. When the association scheme corresponding to the characteristic matrix Z is denoted by $\mathfrak{M}$, the association scheme corresponding to the characteristic matrix Z^{+} is called the enlarged association scheme of $\mathfrak{M}$.

For example, group divisible association scheme (*GD* association scheme) is enlarged from the association scheme of a BIB design which may be called *B* association scheme, and the association scheme of the *GD* m-associate schemes is enlarged from the association scheme of the *GD*$(m-1)$-associate scheme.

Let the matrices $A_0^+, A_1^+, \cdots, A_{m+1}^+$ be given by

$$(35)\qquad \left\| \begin{array}{c} A_0^+ \\ A_1^+ \\ \vdots \\ A_{m+1}^+ \end{array} \right\| = \left(\left\| \begin{array}{cccccc} 1 & 0\cdots 0 & 0 & 0\cdots 0 \\ 0 & 1 \quad 0 & 0 & 1 \quad 0 \\ & \ddots & & \ddots \\ 0 & 0 \quad 1 & 0 & 0 \quad 1 \\ 0 & 0\cdots 0 & 1 & 0\cdots 0 \end{array} \right\| \otimes I_{vs} \right) \left\| \begin{array}{l} A_0 \otimes I_s \\ A_1 \otimes I_s \\ \vdots \\ A_m \otimes I_s \\ A_0 \otimes (G_s - I_s) \\ A_1 \otimes (G_s - I_s) \\ \vdots \\ A_m \otimes (G_s - I_s) \end{array} \right\| ,$$

where $A_0 = I_v, A_1, \cdots, A_m$ are the association matrices of $\mathfrak{M}$. Since

$$\left\| \begin{array}{l} A_0 \otimes I_s \\ A_1 \otimes I_s \\ \vdots \\ A_m \otimes I_s \\ A_0 \otimes (G_s - I_s) \\ \vdots \\ A_m \otimes (G_s - I_s) \end{array} \right\| = \left(\left(\left\| \begin{array}{cc} 1 & 1 \\ s-1 & -1 \end{array} \right\| \otimes Z \right) \otimes I_{sv} \right) \left\| \begin{array}{l} A_0^{\sharp} \otimes (1/s) G_s \\ A_1^{\sharp} \otimes (1/s) G_s \\ \vdots \\ A_m^{\sharp} \otimes (1/s) G_s \\ A_0^{\sharp} \otimes (I_s - (1/s) G_s) \\ \vdots \\ A_m^{\sharp} \otimes (I_s - (1/s) G_s) \end{array} \right\| ,$$

where $A_0^{\sharp}, A_1^{\sharp}, \cdots, A_m^{\sharp}$ are the idempotents of the association algebra generated by the matrices $A_0, A_1, \cdots, A_m$, then it follows from (32) that

$$(36)\qquad \left\| \begin{array}{c} A_0^+ \\ A_1^+ \\ \vdots \\ A_{m+1}^+ \end{array} \right\| = \left(\left(Z^+ \left\| \begin{array}{cccc} 1 & 0\cdots 0 & 0\cdots 0 \\ 0 & 1 \quad 0 & 0\cdots 0 \\ \vdots & \ddots & \cdots\cdots \\ 0 & 0 \quad 1 & 0\cdots 0 \\ 0 & 0\cdots 0 & 1\cdots 1 \end{array} \right\| \right) \otimes I_{sv} \right) \left\| \begin{array}{l} A_0^{\sharp} \otimes (1/s) G_s \\ A_1^{\sharp} \otimes (1/s) G_s \\ \vdots \\ A_m^{\sharp} \otimes (1/s) G_s \\ A_0^{\sharp} \otimes (I_s - (1/s) G_s) \\ \vdots \\ A_m^{\sharp} \otimes (I_s - (1/s) G_s) \end{array} \right\| .$$

Let the matrices $A_0^{+\sharp}, A_1^{+\sharp}, \cdots, A_{m+1}^{+\sharp}$ be given by

$$(37)\qquad \left\| \begin{array}{c} A_0^{+\sharp} \\ A_1^{+\sharp} \\ \vdots \\ A_{m+1}^{+\sharp} \end{array} \right\| = \left(\left\| \begin{array}{cccc} 1 & 0\cdots 0 & 0\cdots 0 \\ 0 & 1 \quad 0 & 0\cdots 0 \\ \vdots & \ddots & \\ 0 & 0 \quad 1 & 0\cdots 0 \\ 0 & 0\cdots 0 & 1\cdots 1 \end{array} \right\| \otimes I_{vs} \right) \left\| \begin{array}{l} A_0^{\sharp} \otimes (1/s) G_s \\ \vdots \\ A_m^{\sharp} \otimes (1/s) G_s \\ A_0^{\sharp} \otimes (I_s - (1/s) G_s) \\ \vdots \\ A_m^{\sharp} \otimes (I_s - (1/s) G_s) \end{array} \right\| ,$$

then $A_0^{+\sharp}, A_1^{+\sharp}, \cdots, A_{m+1}^{+\sharp}$ are mutually orthogonal idempotents of an algebra. Then it follows from (36) that

$$\begin{Vmatrix} A_0^+ \\ A_1^+ \\ \cdot \\ \cdot \\ \cdot \\ A_{m+1}^+ \end{Vmatrix} = (Z^+ \otimes I_{sv}) \begin{Vmatrix} A_0^{+*} \\ A_1^{+*} \\ \cdot \\ \cdot \\ \cdot \\ A_{m+1}^{+*} \end{Vmatrix} .$$

Therefore $A_0^+, A_1^+, \cdots, A_{m+1}^+$ are the association matrices of the enlarged association scheme. Moreover it is straightforward to see that p_{jk}^{+i} $(i, j, k = 0, 1, \cdots, m+1)$ are the association parameters of the enlarged association scheme.

THEOREM 8. *If the association scheme corresponding to the characteristic matrix Z is unique, then the enlarged association scheme corresponding to the characteristic matrix Z^+ is unique.*

PROOF. The parameters p_{jk}^{+i} $(i, j, k=0, 1, \cdots, m+1)$ are uniquely determined from (34) by the elements of Z^+ and are represented as in (33). If two treatments α and β are the $(m+1)$st associates, then from $p_{m+1\,k}^{+m+1}=\delta_{0k}$ $(k=0, 1, \cdots, m)$ and $p_{m+1\,m+1}^{+m+1}=s-2$, every treatment which is the $(m+1)$st associate of α is the $(m+1)$st associate of β, where $\delta_{ij}=0$ or 1 if $i\neq j$ or $i=j$ respectively. If two treatments α and β are the ith associates, from $p_{j\,m+1}^{+i}=(s-1)\delta_{ij}$, every treatment which is the $(m+1)$st associate of β is the ith associate of α. Therefore we have v sets each of which contains s treatments and in which all pairs of treatments are the $(m+1)$st associates. Moreover, each treatment is contained in one and only one set. Consequently, from $p_{jk}^{+i}=sp_{jk}^{i}$ and $p_{jk}^{+m+1}=sp_{jk}^{0}$ $(i=0, 1, \cdots, m,\ j, k=1, 2, \cdots, m)$, the association scheme which has the parameters p_{jk}^{+i} $(i, j, k=0, 1, \cdots, m+1)$ is uniquely derived from the association scheme which has the parameters p_{jk}^{i} $(i, j, k=0, 1, \cdots, m)$.

From theorem 5, theorem 7, (32) and (37), we have the following theorem.

THEOREM 9. *Let the Gramian matrices of the systems of the linear independent column vectors of A_i^+ and A_j be Q_i^+ and Q_j $(i=0, 1, \cdots, m+1,$ $j=0, 1, \cdots, m)$ respectively. So long as we restrict ourselves to rational elements, Q_i^+ $(i=0, 1, \cdots, m)$ is rationally congruent to sQ_i and $|Q_{m+1}^+|$ is rationally congruent to s^v.*

If there exists a design with parameters

$$v, b, r, k, \lambda_i \ (i=1, 2, \cdots, m),$$

that is to say, if the incidence matrix N of such a design is given, then the incidence matrix N^+ of the design with parameters

$$v^+=vs,\quad b^+=bk\binom{s}{u},\quad r^+=r\left\{(k-1)\binom{s}{u}+\binom{s-1}{u-1}\right\},$$
$$k^+=u+(k-1)s,\quad \lambda_i^+=\lambda_i\left\{(k-2)\binom{s}{u}+2\binom{s-1}{u-1}\right\}\ (i=1,2,\cdots,m),$$
$$\lambda_{m+1}^+=r\left\{(k-1)\binom{s}{u}+\binom{s-2}{u-2}\right\}\qquad \text{for all } u=1,2,\cdots,s$$

can be given as follows:

Let the jth treatment in the αth set, which corresponds to the αth treatment of the given design, be $\{(\alpha-1)s+j\}$th treatment of the design with the enlarged association scheme for $j=1,2,\cdots,s$ and $\alpha=1,2,\cdots,v$. If the αth treatment of the given design is the ith $(i\neq 0)$ associate of the βth treatment, then the $\{(\alpha-1)s+j\}$th treatment is the ith associate of the $\{(\beta-1)s+j'\}$th treatment for all $j, j'=1,2,\cdots,s$.

Let us consider the incidence matrix N^* of a PBIB design with $\binom{s-1}{u-1}$ replications and $\binom{s}{u}$ blocks such that s treatments can be arranged into $\binom{s}{u}$ blocks each of which contains u treatments. Such a design has the blocks each of which contains a selection of u of s treatments. And also let us consider the $\left(s\times\binom{s}{u}\right)$-matrix K whose elements are all unity. Let the $\left(s\times\binom{s}{u}k\right)$-matrices N_j^* $(j=1,2,\cdots,k)$ be denoted by

$$N_j^*=\|\underbrace{K\cdots K}_{j-1}\,N^*\,\underbrace{K\cdots K}_{k-j}\|.$$

If the non-zero elements in the jth column of N are $n_{j_1 j}, n_{j_2 j},\cdots,n_{j_k j}$, $n_{j_i j}$ are replaced by N_i^*, and zero elements of N are replaced by the zero $\left(s\times\binom{s}{u}k\right)$-matrix. Then we get the incidence matrix N^+ of the design with the enlarged association scheme in place of N.

For illustration: If the following incidence matrix

$$N=\begin{Vmatrix}1&0&1&0\\1&0&0&1\\0&1&1&0\\0&1&0&1\end{Vmatrix}$$

of the design exists with parameters

$$v=b=4,\quad r=2,\quad k=2,\quad \lambda_1=1,\quad \lambda_2=0,$$

then the incidence matrix of the design with parameters

$$v^+=16,\quad b^+=48,\quad r^+=18,\quad k^+=6,\quad \lambda_1^+=6,\quad \lambda_2^+=0,\quad \lambda_3^+=14,$$

is given as follows:

$$N^{+}=\left\|\begin{matrix} N^* & K & 0 & 0 & N^* & K & 0 & 0 \\ K & N^* & 0 & 0 & 0 & 0 & N^* & K \\ 0 & 0 & N^* & K & K & N^* & 0 & 0 \\ 0 & 0 & K & N^* & 0 & 0 & K & N^* \end{matrix}\right\|,$$

where

$$N^*=\left\|\begin{matrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{matrix}\right\|, \qquad K=\left\|\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}\right\|.$$

6. The design with Kronecker-product $B\otimes GD$ association scheme, and the design with Kronecker-product $B\otimes L_i$ association scheme

GD association scheme has $v=ms$ treatments which are each divided into m groups of s treatments, such that any two treatments in the same group are first associates and any two treatments in different groups are second associates. Let GD association algebra be generated by the association matrices $A_{20}=I_v$, A_{21} and A_{22} of GD association scheme. If $A^{\#}_{20}=(1/v)G_v$, $A^{\#}_{21}$ and $A^{\#}_{22}$ are mutually orthogonal idempotents of GD association algebra, then

$$\begin{aligned} N_2N_2' &= r_2A_{20}+\lambda_{21}A_{21}+\lambda_{22}A_{22} \\ &= \rho_{20}A^{\#}_{20}+\rho_{21}A^{\#}_{21}+\rho_{22}A^{\#}_{22} \end{aligned}$$

where N_2 is the incidence matrix of GD design with parameters v_2, b_2, r_2, k_2, λ_{21}, λ_{22}, and ρ_{20}, ρ_{21}, ρ_{22} are the distinct characteristic roots of N_2N_2' with multiplicities trace$(A^{\#}_{20})$, trace$(A^{\#}_{21})$ and trace$(A^{\#}_{22})$ respectively. If $|Q_{20}|$, $|Q_{21}|$ and $|Q_{22}|$ are Gramians corresponding to the rational characteristic roots ρ_{20}, ρ_{21} and ρ_{22} respectively, that is, Gramians of the sets of linearly independent column vectors of $A^{\#}_{20}$, $A^{\#}_{21}$ and $A^{\#}_{22}$ respectively, we have

$$|Q_{20}|\sim sm, \quad |Q_{21}|\sim s^{m-1}m, \quad |Q_{22}|\sim s^m.$$

When the square-free parts of two rational number a and b are equal, we denote the fact by $a\sim b$. We have

$$\text{trace}(A^{\#}_{20})=1, \quad \text{trace}(A^{\#}_{21})=m-1, \quad \text{trace}(A^{\#}_{22})=m(s-1).$$

If there exists a BIB design with parameters $v_1=l, b_1, r_1, k_1$, then we have

$$N_1N_1'=rA_{10}+\lambda A_{11}=\rho_{10}A^{\#}_{10}+\rho_{11}A^{\#}_{11},$$

where ρ_{10} and ρ_{11} are the distinct characteristic roots of N_1N_1', and N_1 is the incidence matrix of BIB design. If Gramians of the sets of linearly independent column vectors of $A_{10}^\sharp$ and $A_{11}^\sharp$ are $|Q_{10}|$ and $|Q_{11}|$, then we have

$$|Q_{10}|\sim l \qquad |Q_{11}|\sim l.$$

And yet, we have

$$\text{trace}(A_{10}^\sharp)=1, \qquad \text{trace}(A_{11})=l-1.$$

Now, the characteristic matrix of the Kronecker-product $B\otimes GD$ association scheme is defined by the following matrix :

$$(38) \quad Z=\begin{Vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ s-1 & s-1 & -1 & s-1 & s-1 & -1 \\ s(m-1) & -s & 0 & s(m-1) & -s & 0 \\ l-1 & l-1 & l-1 & -1 & -1 & -1 \\ (l-1)(s-1) & (l-1)(s-1) & -(l-1) & -(s-1) & -s+1 & 1 \\ s(m-1)(l-1) & -s(l-1) & 0 & -s(m-1) & s & 0 \end{Vmatrix}.$$

From theorem 4, it is known that the association matrices $A_0, A_1, \cdots, A_5$ of such an association scheme are represented as follows :

$$A_0=A_{10}\otimes A_{20}, \quad A_1=A_{10}\otimes A_{21}, \quad A_3=A_{11}\otimes A_{20},$$
$$A_2=A_{10}\otimes A_{22}, \quad A_4=A_{11}\otimes A_{21}, \quad A_5=A_{11}\otimes A_{22}.$$

If N is the incidence matrix of the design with the Kronecker-product $B\otimes GD$ association scheme and with parameters b, r, k, λ_1, λ_2, λ_3, λ_4, λ_5, we have the following spectral resolution of NN'

$$NN'=\rho_0A_0^\sharp+\rho_1A_1^\sharp+\rho_2A_2^\sharp+\rho_3A_3^\sharp+\rho_4A_4^\sharp+\rho_5A_5^\sharp,$$

where $\rho_0, \rho_1, \cdots, \rho_5$ are characteristic roots of NN', and $A_0^\sharp, A_1^\sharp, \cdots, A_5^\sharp$ are idempotents of the association algebra which is generated by $A_0=I_v$, $A_1, \cdots, A_5$. If $|Q_0|, |Q_1|, \cdots, |Q_5|$ are Gramians corresponding to the rational characteristic roots $\rho_0, \rho_1, \cdots, \rho_5$ of NN', it follows from theorem 5 that

$$|Q_0|\sim lms, \qquad |Q_1|\sim l^{m-1}s^{m-1}m, \qquad |Q_2|\sim l^{(s-1)m}s^m,$$
$$|Q_3|\sim l(ms)^{l-1}, \qquad |Q_4|\sim m^{l-1}s^{(m-1)(l-1)}l^{m-1},$$
$$|Q_5|\sim s^{m(l-1)}l^{(s-1)m}.$$

And further

$$\alpha_0=\text{trace}(A_0^\sharp)=1, \qquad \alpha_1=\text{trace}(A_1^\sharp)=m-1,$$
$$\alpha_2=\text{trace}(A_2^\sharp)=m(s-1), \quad \alpha_3=\text{trace}(A_3^\sharp)=l-1,$$

$$\alpha_4=\mathrm{trace}(A_4^2)=(l-1)(m-1),$$
$$\alpha_5=\mathrm{trace}(A_5^2)=m(l-1)(s-1).$$

From (19) one gets

$$\rho_0=r+(s-1)\lambda_1+s(m-1)\lambda_2+(l-1)\lambda_3+(l-1)(s-1)\lambda_4+s(m-1)(l-1)\lambda_5$$
$$\rho_1=r+(s-1)\lambda_1-s\lambda_2+(l-1)\lambda_3+(l-1)(s-1)\lambda_4-s(l-1)\lambda_5$$
$$\rho_2=r-\lambda_1+(l-1)\lambda_3-(l-1)\lambda_4$$
$$\rho_3=r+(s-1)\lambda_1+s(m-1)\lambda_2-\lambda_3-(s-1)\lambda_4-s(m-1)\lambda_5$$
$$\rho_4=r+(s-1)\lambda_1-s\lambda_2-\lambda_3-(s-1)\lambda_4-s\lambda_5$$
$$\rho_5=r-\lambda_1-\lambda_3+\lambda_4.$$

If the regular and symmetrical PBIB design of the type above mentioned does exist, it is known [11] that Hasse-Minkowski p-invariant of NN' should be equal to $(-1,-1)_p$ for all primes p, i.e.,

$$C_p(NN')=(-1,-1)_p\prod_{i=0}^{5}(-1,\rho_i)_p^{\alpha_i(\alpha_i+1)/2}(\rho_i,|Q_i|)_p^{\alpha_i+1}\cdot\prod_{i>j}(\rho_i,|Q_j|)_p^{\alpha_i}(\rho_j,|Q_i|)_p^{\alpha_j}(\rho_i,\rho_j)_p^{\alpha_i\alpha_j}=(-1,-1)_p \quad \text{for all primes } p, \tag{39}$$

where $\rho_i>0$ $(i=0,1,\cdots,5)$, moreover

$$\textstyle\prod_{i=0}^{5}\rho_i^{\alpha_i}\sim 1. \tag{40}$$

We now distinguish eight cases: (i) v odd, (ii) l and m both odd and s even, (iii) l and s both odd and m even, (iv) m and s both odd and l even, (v) s and m both even and l odd, (vi) l and m both even and s odd, (vii) l and s both even and m odd, and (viii) l, m and s all even.

In case (i), $|Q_1|\sim m$, $|Q_2|\sim s$, $|Q_3|\sim l$, $|Q_4|\sim|Q_5|\sim 1$,

$$\alpha_1\equiv\alpha_2\equiv\alpha_3\equiv 0 \pmod 2, \quad \text{and } \alpha_4\equiv\alpha_5\equiv 0 \pmod 4.$$

Hence

$$(m-1)m\equiv(m-1)^2+(m-1)\equiv(m-1) \pmod 4$$
$$\{m(s-1)+1\}m(s-1)\equiv m(s-1)\equiv(s-1) \pmod 4.$$

Then (39) implies that

$$(-1,\rho_1)_p^{(m-1)/2}(-1,\rho_2)_p^{(s-1)/2}(-1,\rho_3)_p^{(l-1)/2}(\rho_1,m)_p(\rho_2,s)_p(\rho_3,l)_p=1. \tag{41}$$

In case (ii), (39) and (40) imply that

$$\rho_2\sim 1, \quad (-1,\rho_1)_p^{(m-1)/2}(-1,\rho_3\rho_5)_p^{(l-1)/2}(\rho_1,m)_p(\rho_3\rho_5,l)_p=1. \tag{42}$$

Similarly, in case (iii)

$$(43)\qquad \rho_1\sim 1,\quad (-1,\rho_3\rho_4)_p^{(l-1)/2}(\rho_3\rho_4, l)_p=1,$$

in case (iv)

$$(44)\qquad \rho_3\sim 1,\quad (-1,\rho_1\rho_4)_p^{(m-1)/2}(-1,\rho_2\rho_5)_p^{(s-1)/2}(\rho_1\rho_4, m)_p(\rho_2\rho_5, s)_p=1,$$

and in case (v)

$$(45)\qquad \rho_1\sim 1,\quad (-1,\rho_2)_p^{m/2}(-1,\rho_3\rho_4)_p^{(l-1)/2}(\rho_3\rho_4, l)_p=1.$$

In case (vi), $|Q_1|\sim|Q_3|\sim|Q_4|\sim lsm$, $|Q_2|\sim|Q_5|\sim 1$,

$$\alpha_1\equiv\alpha_3\equiv\alpha_4\equiv 1 \pmod 2,\quad \text{and } \alpha_2\equiv\alpha_5\equiv 0 \pmod 4.$$

Then $\rho_1\rho_3\rho_4$ is a perfect square and furthermore

$$(-1,\rho_1)_p^{m/2}(-1,\rho_3)_p^{l/2}(-1,\rho_4)_p^{((l-1)(m-1)+1)/2}(\rho_1,\rho_3)_p(\rho_1,\rho_4)_p(\rho_3,\rho_4)_p=1.$$

Hence

$$(\rho_1,\rho_4)_p(\rho_3,\rho_4)_p=(\rho_4,\rho_4)_p=(-1,\rho_4)_p=(-1,\rho_1)_p(-1,\rho_3)_p,$$

$$\{(l-1)(m-1)+1\}/2+1=(l+m)/2 \pmod 2.$$

Then (39) and (40) imply that

$$(46)\qquad \rho_1\rho_3\rho_4\sim 1,\quad (-1,\rho_1)_p^{l/2}(-1,\rho_3)_p^{m/2}(\rho_1,\rho_3)_p=1.$$

Similarly, in case (vii)

$$(47)\qquad \begin{aligned}&\rho_2\rho_3\rho_5\sim 1,\\ &(-1,\rho_1)_p^{(m-1)/2}(-1,\rho_2)_p^{l/2}(-1,\rho_3)_p^{(m+s-1)/2}(-1,\rho_4)_p^{(m-1)/2}\\ &\qquad\qquad\cdot(\rho_1\rho_3\rho_4, m)_p(\rho_2,\rho_3)_p=1,\end{aligned}$$

and in case (viii)

$$(48)\qquad \rho_1\rho_3\rho_4\sim 1,\quad (-1,\rho_1)_p^{l/2}(-1,\rho_2\rho_3\rho_5)_p^{m/2}(\rho_1,\rho_3)_p=1.$$

Thus we can state the above results as the following theorem.

THEOREM 10. *Necessary conditions for the existence of a regular and symmetrical PBIB design with Kronecker-product $B\otimes GD$ association scheme are:*

$$\rho_1,\ \rho_2,\ \rho_3,\ \rho_4,\ \rho_5>0,$$

and in the above-mentioned cases (i), (ii), $\cdots$, (viii), *relations* (41), (42), $\cdots$, (48) *must be satisfied respectively.*

The designs with the following parameters violate the conditions of the above statement and hence are non-existent.

	l	m	s	r	λ_1	λ_2	λ_3	λ_4	λ_5	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
(1)	43	5	3	60	0	15	20	0	5	225	900	160	10	40
(2)	7	5	3	13	3	5	1	1	1	4	10	64	4	10
(3)	5	7	3	16	10	2	4	3	2	46	10	26	26	5
(4)	5	7	3	17	11	3	5	4	2	56	10	44	23	5

L_i design as $v=s^2$ treatments which are arranged in a square L and $i-2$ mutually orthogonal Latin square are superimposed on L. For each treatment θ in L, treatments occurring in the same row or column of L as θ or treatments corresponding to the same letters of the superimposed orthogonal Latin squares are first associates and others are second associates.

Let $A^{\#}_{30}=(1/v)G_v$, $A^{\#}_{31}$ and $A^{\#}_{32}$ be mutually orthogonal idempotents of L_i association algebra being generated by the association matrices of L_i association scheme. If $|Q_{30}|$, $|Q_{31}|$ and $|Q_{32}|$ are Gramians corresponding to $A^{\#}_{30}$, $A^{\#}_{31}$ and $A^{\#}_{32}$ respectively, we have

(49) $$|Q_{30}|\sim 1,\quad |Q_{31}|\sim s^{is},\quad |Q_{32}|\sim s^{is}.$$

And yet, we have

(50) $$\alpha_0=\text{trace}(A^{\#}_{30})=1,\quad \alpha_1=\text{trace}(A^{\#}_{31})=i(s-1),$$
$$\alpha_2=\text{trace}(A^{\#}_{32})=(s-i+1)(s-1).$$

Now, the characteristic matrix of the Kronecker-product $B\otimes L_i$ association scheme is defined by the following matrix:

(51) $$Z=\begin{Vmatrix} 1 & 1 \\ l-1 & -1 \end{Vmatrix}\otimes\begin{Vmatrix} 1 & 1 & 1 \\ i(s-1) & s-i & -i \\ (s-1)(s-i+1) & -s+i-1 & i-1 \end{Vmatrix}.$$

From theorem 4 it is known that the association matrices $A_0, A_1, \cdots, A_5$ of such an association scheme are represented as follows:

(52) $$A_0=A_{10}\otimes A_{30},\quad A_1=A_{10}\otimes A_{31},\quad A_2=A_{10}\otimes A_{32},$$
$$A_3=A_{11}\otimes A_{30},\quad A_4=A_{11}\otimes A_{31},\quad A_5=A_{11}\otimes A_{32}.$$

If $|Q_0|, |Q_1|, \cdots, |Q_5|$ are Gramians corresponding to the idempotents $A^{\#}_0, A^{\#}_1, \cdots, A^{\#}_5$ respectively, where $A^{\#}_0=A^{\#}_{10}\otimes A^{\#}_{30}$, $A^{\#}_1=A^{\#}_{10}\otimes A^{\#}_{31}$, $A^{\#}_2=A^{\#}_{10}\otimes A^{\#}_{32}$, $A^{\#}_3=A^{\#}_{11}\otimes A^{\#}_{30}$, $A^{\#}_4=A^{\#}_{11}\otimes A^{\#}_{31}$ and $A^{\#}_5=A^{\#}_{11}\otimes A^{\#}_{32}$, we have

(53) $$|Q_0|\sim l,\quad |Q_1|\sim s^{is}l^{i(s-1)},\quad |Q_2|\sim s^{is}l^{(s-i+1)(s-1)},$$
$$|Q_3|\sim l,\quad |Q_4|\sim s^{is(l-1)}l^{i(s-1)},$$
$$|Q_5|\sim s^{is(l-1)}l^{(s-i+1)(s-1)}.$$

And yet, we have

(54) $$\alpha_1=\mathrm{tr}(A_1^*)=i(s-1),\quad \alpha_2=\mathrm{tr}(A_2^*)=(s-i+1)(s-1),$$
$$\alpha_3=\mathrm{tr}(A_3^*)=l-1,\qquad \alpha_4=\mathrm{tr}(A_4^*)=(l-1)(s-1)i,$$
$$\alpha_5=\mathrm{tr}(A_5^*)=(l-1)(s-1)(s-i+1).$$

This association scheme is reducible to the enlarged association scheme being enlarged from L_i association scheme. If $i=2$ and $l=s$, this association scheme is reducible to the cubic association scheme.

If N is the incidence matrix of the design with parameters $v, b, r, k, \lambda_1, \cdots, \lambda_5$ and with Kronecker-product $B\otimes L_i$ association scheme, the characteristic roots $\rho_0, \rho_1, \cdots, \rho_5$ of NN' can be represented by the characteristic matrix Z in (51) as follows:

$$(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)=(r, \lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5)Z.$$

From (39), (40), (53) and (54) we have the following theorem.

THEOREM 11. *Necessary conditions for the existence of a regular and symmetrical PBIB design with the Kronecker-product $B\otimes L_i$ association scheme are:*

$$\rho_1,\ \rho_2,\ \rho_3,\ \rho_4,\ \rho_5>0$$

(i) *if i is even and if l and s are odd, then*

$$(-1, \rho_3)_p^{(l-1)/2}(\rho_3, l)_p=1,$$

(ii) *if i is even and if s is odd and l is even, then*

$$\rho_3\sim 1,$$

(iii) *if i is even and if l is odd and s is even, then*

$$\rho_2\sim 1,\quad (-1, \rho_1)_p^{i/2}(-1, \rho_3\rho_5)_p^{(l-1)/2}(\rho_3\rho_5, l)_p=1,$$

(iv) *if i is even and if l and s are even, then*

$$\rho_2\rho_3\rho_5\sim 1,\quad (-1, \rho_1\rho_3\rho_4)_p^{i/2}(-1, \rho_2)_p^{l/2}(\rho_2, \rho_3)_p=1,$$

(v) *if i is odd and if l and s are odd, then*

$$(-1, \rho_1\rho_2)_p^{(s-1)/2}(-1, \rho_3)_p^{(l-1)/2}(\rho_1\rho_2, s)_p(\rho_3, l)_p=1,$$

(vi) *if i is odd and if s is odd and l is even, then*

$$\rho_3\sim 1,\quad (-1, \rho_1\rho_2\rho_4\rho_5)_p^{(s-1)/2}(\rho_1\rho_2\rho_4\rho_5, s)_p=1,$$

(vii) *if i is odd and if l is odd and s is even, then*

$$\rho_1\sim 1,\quad (-1, \rho_2)_p^{(s-i+1)/2}(-1, \rho_3\rho_4)_p^{(l-1)/2}(\rho_3\rho_4, l)_p=1,$$

(viii) *if i is odd and if l and s are even, then*

$$\rho_1\rho_3\rho_4 \sim 1, \quad (-1, \rho_1)_p^{l/2}(-1, \rho_2\rho_3\rho_5)_p^{(s-t+1)/2}(\rho_1, \rho_3)_p = 1.$$

Furthermore we can obtain the Kronecker-product $B \otimes T_2$ association scheme from B association scheme and triangular association scheme, and also the necessary conditions of a regular and symmetrical design with such an association scheme.

7. Extended right angular design and the designs with the enlarged association scheme being enlarged from L_i association scheme

The rectangular association scheme has $v = ml$ treatments which are arranged in a rectangular form R having m rows and l columns. For each treatment θ in R, treatments occurring in the same row as θ are first associates, treatments occurring in the same column as θ are second associates and others are third associates.

Since the characteristic matrix of the rectangular association scheme is

$$\left\| \begin{matrix} 1 & 1 & 1 & 1 \\ (m-1) & -1 & (m-1) & -1 \\ (l-1) & (l-1) & -1 & -1 \\ (m-1)(l-1) & -(l-1) & -(m-1) & 1 \end{matrix} \right\|,$$

then from (30) and (31) it follows that the characteristic matrix of the enlarged association scheme of R association scheme is

$$(55) \qquad Z = \left\| \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ s(m-1) & -s & s(m-1) & -s & 0 \\ s(l-1) & s(l-1) & -s & -s & 0 \\ s(m-1)(l-1) & -s(l-1) & -s(m-1) & s & 0 \\ s-1 & s-1 & s-1 & s-1 & -1 \end{matrix} \right\|.$$

DEFINITION 8. The association scheme corresponding to the characteristic matrix Z in (55) is called the extended right angular association scheme.

As a simple example of the argument given in section 3, we see that the rectangular association scheme R is formed as Kronecker-product association scheme of two B association schemes having m and l treatments respectively.

Since the matrix (38) is reducible to the matrix (55), the extended right angular association scheme is reduced from $B \otimes GD$ association scheme.

From the fact that rectangular association scheme is determined uniquely [9] by its parameters, and from the theorem 8, we have the following theorem.

THEOREM 12. *The extended right angular association scheme is determined uniquely by the characteristic matrix* Z *in* (55). *Especially, if either* $l=2$ *or* $m=2$, *the extended right angular association scheme is the same as the right angular association scheme* [7].

Now, some designs with the extended right angular association scheme may be constructed as follows:

(i) If the *BIB* design with parameters $v_1=l, b_1, r_1, k_1, \lambda$ and *GD* design with parameters $v_2=sm, b_2, r_2, k_2, \lambda_{21}, \lambda_{22}$ are given, and if $r_2=\lambda_{21}$, then we have the extended right angular design with parameters $v=lsm$, $b=b_1b_2$, $r=r_1r_2$, $k=k_1k_2$, $\lambda_1=r_1\lambda_{22}$, $\lambda_2=\lambda r_2$, $\lambda_3=\lambda\lambda_{22}$, $\lambda_4=r_1r_2$.

This design is the reduced design of Kronecker-product design which is the Kronecker product of *BIB* design and *GD* design.

(ii) If there exist *BIB* designs with parameters $v_1, b_1, r_1, k_1, \lambda$ and with parameters $v_2, b_2, r_2, k_2, \lambda^*$ respectively, we get the extended right angular design with parameters

$$v=v_1v_2s,\quad b=b_1b_2k_1k_2\binom{s}{u},\quad r=r_1r_2\left\{(k_1k_2-1)\binom{s}{u}+\binom{s-1}{u-1}\right\},$$

$$k=u+(k_1k_2-1)s,\quad \lambda_1=r_1\lambda^*\left\{(k_1k_2-2)\binom{s}{u}+2\binom{s-1}{u-1}\right\},$$

$$\lambda_2=r_2\lambda\left\{(k_1k_2-2)\binom{s}{u}+2\binom{s-1}{u-1}\right\},$$

$$\lambda_3=\lambda\lambda^*\left\{(k_1k_2-2)\binom{s}{u}+2\binom{s-1}{u-1}\right\},$$

$$\lambda_4=r_1r_2\left\{(k_1k_2-2)\binom{s}{u}+\binom{s-2}{u-2}\right\},\qquad \text{for all } u=1,2\cdots,s.$$

Let the design with the extended right angular association scheme be called the extended right angular design.

If $A_0=I_v, A_1,\cdots,A_4$ are the association matrices of the extended right angular association scheme, we have

$$A_0=I_{lms},\quad A_1=(G_m-I_m)\otimes I_l\otimes G_s,\quad A_2=I_m\otimes(G_l-I_l)\otimes G_s,$$
$$A_3=(G_m-I_m)\otimes(G_l-I_l)\otimes G_s,\qquad A_4=I_m\otimes I_l\otimes(G_s-I_s).$$

If N is the incidence matrix of the extended right angular design with parameters $v, b, r, k, \lambda_1,\cdots,\lambda_4$, we have the following spectral resolution

$$NN'=\textstyle\sum_{i=0}^4\rho_iA_i^\sharp$$

where $\rho_0, \rho_1, \cdots, \rho_4$ are characteristic roots of NN', and $A_0^\sharp, A_1^\sharp, \cdots, A_4^\sharp$ are the idempotents of the association algebra which is generated by $A_0, A_1, \cdots, A_4$. If $|Q_0|, |Q_1|, \cdots, |Q_4|$ are Gramians corresponding to the rational roots $\rho_0, \rho_1, \cdots, \rho_4$ of NN' respectively, it follows from theorem 4 and theorem 2 that

$$(56) \qquad |Q_0| \sim lms, \quad |Q_1| \sim ml^{m-1}s^{m-1}, \quad |Q_2| \sim lm^{l-1}s^{l-1}, \\ |Q_3| \sim s^{(l-1)(m-1)}l^{(m-1)}m^{(l-1)}, \qquad |Q_4| \sim s^{lm}.$$

And yet, we have

$$(57) \qquad \alpha_0 = 1, \quad \alpha_1 = \mathrm{tr}(A_1^\sharp) = m-1, \quad \alpha_2 = \mathrm{tr}(A_2^\sharp) = l-1, \\ \alpha_3 = \mathrm{tr}(A_3^\sharp) = (l-1)(m-1), \quad \alpha_4 = \mathrm{tr}(A_4^\sharp) = ml(s-1).$$

From (19) it follows that

$$(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4) = (r, \lambda_1, \lambda_2, \lambda_3, \lambda_4)Z,$$

where Z is the characteristic matrix (55).

From (39), (40), (56) and (57), we have the following theorem.

THEOREM 13. *Necessary conditions for existence of a regular and symmetrical extended right angular design are:*

$$\rho_1, \ \rho_2, \ \rho_3, \ \rho_4 > 0,$$

(i) *if v is odd, then*

$$(-1, \rho_1)_p^{(m-1)/2}(-1, \rho_2)_p^{(l-1)/2}(-1, \rho_4)_p^{(s-1)/2}(\rho_1, m)_p(\rho_2, l)_p(\rho_4, s)_p = 1,$$

(ii) *if l and m are odd and if s is even, then*

$$\rho_4 \sim 1, \quad (-1, \rho_1)_p^{(m-1)/2}(-1, \rho_2)_p^{(l-1)/2}(\rho_1, m)_p(\rho_2, l)_p = 1,$$

(iii) *if l and s are odd and if m is even, then*

$$\rho_1 \sim 1, \quad (-1, \rho_2\rho_3)_p^{(l-1)/2}(\rho_2\rho_3, l)_p = 1,$$

(iv) *if m and s are odd and if l is even, then*

$$\rho_2 \sim 1, \quad (-1, \rho_1\rho_3)_p^{(m-1)/2}(\rho_1\rho_3, m)_p = 1,$$

(v) *if l and m are even, then*

$$\rho_1\rho_2\rho_3 \sim 1, \quad (-1, \rho_1)_p^{l/2}(-1, \rho_2)_p^{m/2}(\rho_1, \rho_2)_p = 1,$$

(vi) *if s and m are even and if l is odd, then*

$$\rho_1 \sim 1, \quad (-1, \rho_2\rho_3)_p^{(l-1)/2}(-1, \rho_4)_p^{m/2}(\rho_2\rho_3, l)_p = 1,$$

(vii) *if l and s are even and if m is odd, then*

$$\rho_2 \sim 1, \quad (-1, \rho_1\rho_3)_p^{(m-1)/2}(-1, \rho_4)_p^{l/2}(\rho_1\rho_3, m)_p = 1.$$

The designs with the following parameters violate the conditions of the above statement and hence are non-existent.

	l	m	s	r	λ_1	λ_2	λ_3	λ_4	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4
(1)	3	3	5	10	3	2	1	5	100	25	40	10	5
(2)	3	3	5	11	1	2	3	5	121	16	1	31	6
(3)	5	3	5	8	2	1	0	4	64	34	39	9	4
(4)	5	3	5	9	2	2	0	3	81	51	31	1	6
(5)	5	3	5	12	2	3	1	3	144	54	19	4	9

Since the characteristic matrix of L_i association scheme having m^2 treatments is

$$\begin{Vmatrix} 1 & 1 & 1 \\ i(m-1) & m-i & -i \\ (m-1)(m-i+1) & -m+i-1 & i-1 \end{Vmatrix},$$

then from (30) and (31) it follows that the characteristic matrix of the enlarged association scheme which is enlarged from L_i association scheme is

$$Z = \begin{Vmatrix} 1 & 1 & 1 & 1 \\ is(m-1) & s(m-i) & -is & 0 \\ s(m-1)(m-i+1) & -s(m-i+1) & s(i-1) & 0 \\ s-1 & s-1 & s-1 & -1 \end{Vmatrix}. \tag{58}$$

If $A_0 = I_v$, A_1, A_2 and A_3 are the association matrices of the enlarged association scheme, we have

$$A_0 = I_{sm^2}, \quad A_1 = A_{11} \otimes G_s, \quad A_2 = A_{12} \otimes G_s, \quad A_3 = I_{m^2} \otimes (G_s - I_s),$$

where A_{11} and A_{12} are the association matrices of L_i association scheme. If $A_0^\sharp$, $A_1^\sharp$, $A_2^\sharp$ and $A_3^\sharp$ are the idempotents of the association algebra which is generated by A_0, A_1, A_2 and A_3, and if $|Q_0|$, $|Q_1|$, $|Q_2|$ and $|Q_3|$ are Gramians corresponding to $A_0^\sharp$, $A_1^\sharp$, $A_2^\sharp$ and $A_3^\sharp$ respectively, then we have

$$\begin{aligned} &|Q_0| \sim s, \quad |Q_1| \sim s^{i(m-1)} m^{im}, \\ &|Q_2| \sim s^{(m-1)(m-i+1)} m^{im}, \quad |Q_3| \sim s^{m^2}. \end{aligned} \tag{59}$$

And yet, we have

(60) $$\alpha_0=\mathrm{tr}(A_0^\sharp)=1,\quad \alpha_1=\mathrm{tr}(A_1^\sharp)=i(m-1),$$
$$\alpha_2=\mathrm{tr}(A_2^\sharp)=(m-1)(m-i+1),\quad \alpha_3=\mathrm{tr}(A_3^\sharp)=m^2(s-1).$$

If N is the incidence matrix of the design with parameters v, b, r, k, λ_1, λ_2, λ_3 and with the enlarged association scheme from L_i, the characteristic roots ρ_0, ρ_1, ρ_2 and ρ_3 of NN' can be represented by the characteristic matrix Z in (58) as follows:

$$(\rho_0, \rho_1, \rho_2, \rho_3)=(r, \lambda_1, \lambda_2, \lambda_3)Z.$$

From (39), (40), (59) and (60), we have the following theorem.

THEOREM 14. *Necessary conditions for the existence of a regular and symmetrical design with the enlarged association scheme are*

$$\rho_1, \rho_2, \rho_3, \rho_4>0$$

(i) *if i is even and if m is even, then*

$$\rho_2\sim 1,\quad (-1, \rho_1)_p^{i/2}=1,$$

(ii) *if i is even and if m and s are odd, then*

$$(-1, \rho_3)_p^{(s-1)/2}(\rho_3, s)_p=1,$$

(iii) *if i is even and if m is odd and s is even, then*

$$\rho_3\sim 1,$$

(iv) *if i is odd and if m and s are odd, then*

$$(-1, \rho_1\rho_2)_p^{(m-1)/2}(-1, \rho_3)_p^{(s-1)/2}(\rho_1\rho_2, m)_p(\rho_3, s)_p=1,$$

(v) *if i is odd and if m is odd and s is even, then*

$$\rho_3\sim 1,\quad (-1, \rho_1\rho_2)_p^{(m-1)/2}(\rho_1\rho_2, m)_p=1,$$

(vi) *if i is odd and if m is even, then*

$$\rho_1\sim 1,\quad (-1, \rho_2)_p^{(m-i+1)/2}=1.$$

By the same method, we will get designs with the other enlarged association schemes.

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