ON THE RESOLVENT OF A BROWNIAN MOTION WITH DRIFT

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0. Introduction

Let us consider a Brownian motion on a bounded open set G and a Brownian motion with drift $a(x)$ on G which is obtained by the transformation of probability measure connected with multiplicative functional

$$M_t=\exp\left(\int_0^t a(x_u)dx_u-\frac{1}{2}\int_0^t a^2(x_u)du\right) \text{ [2].}$$

Let $G^\lambda f$ and $\tilde{G}^\lambda f$ be the resolvents of these two processes. Operating $\tilde{G}^\lambda$ to both sides of

(0.1) $$\left(\lambda-\frac{1}{2}\Delta\right)G^\lambda f=f$$

and noticing

(0.2) $$\left(\lambda-\frac{1}{2}\Delta-a(\cdot)\,\mathrm{grad}\right)\tilde{G}^\lambda f=f,$$

we get

(0.3) $$\tilde{G}^\lambda f-G^\lambda f=\tilde{G}^\lambda[a(\cdot)\,\mathrm{grad}\,G^\lambda f],$$

and similarly

(0.4) $$\tilde{G}^\lambda f-G^\lambda f=G^\lambda[a(\cdot)\,\mathrm{grad}\,\tilde{G}^\lambda f].$$

Rigorous proofs of (0.3) and (0.4) will be given in theorem 2 and theorem 3, respectively. Simpler but less suggestive proofs of (0.3) and (0.4) will be given in the appendix. As a corollary of theorem 2, we get

(0.5) $$h(x_t)-h(x_0)=\int_0^t \mathrm{grad}\,h(x_u)dx_u+\int_0^t \mathfrak{G}h(x_u)du \qquad \text{for } h\in D(\mathfrak{G}),$$

where $\mathfrak{G}$ is the generator of a Brownian motion. (0.5) is an extension of the well-known formula of stochastic integral

$$(0.6)\qquad h(x_t)-h(x_0)=\int_0^t \operatorname{grad} h(x_u)dx_u+\int_0^t \frac{1}{2}\Delta h(x_u)du \qquad \text{for } h\in C^2.$$

A similar form will be obtained for the Brownian motion with drift. (See the corollary of lemma 5.) In section 1 we will consider a Hunt process $\boldsymbol{M}$. Let $\tilde{\boldsymbol{M}}$ be obtained from $\boldsymbol{M}$ by the transformation connected with multiplicative functional with mean 1. We shall find the relation between the resolvent of $\boldsymbol{M}$ and that of $\tilde{\boldsymbol{M}}$, and prove that $\mathfrak{N}=\{f; G^\lambda f=0\}$ coincides with $\tilde{\mathfrak{N}}=\{f;\tilde{G}^\lambda f=0\}$. In sections 2, 3, 4 and 5, we confine ourselves to a Brownian motion with absorbing barrier. Sections 4 and 5 are devoted to the proofs of (0.3), (0.4) and (0.5). These proofs also contain an interesting formula

$$(0.7)\qquad e^{-\lambda t}\tilde{G}^\lambda f(\tilde{x}_t)=\int_0^t e^{-\lambda u}\operatorname{grad}\tilde{G}^\lambda f(\tilde{x}_u)dZ_u+\tilde{G}^\lambda f(\tilde{x}_0)-\int_0^t e^{-\lambda u}f(\tilde{x}_u)du,$$

where $(Z_u, \mathcal{F}_u)$ is an appropriate martingale. (0.7) is nothing but the decomposed form of supermartingale $e^{-\lambda t}\tilde{G}^\lambda f(\tilde{x}_t)$ [5]. It is well-known that any continuous additive functional in $\mathfrak{M}$* of a Brownian motion is essentially written in the form $\int_0^t \operatorname{grad} G^\lambda f(x_u)dx_u$ [4]. (0.7) shows that a similar statement is valid for the Brownian motion with drift.

1. Resolvents of M and $\tilde{M}$

Let $\boldsymbol{M}=(x_t, P_x, \zeta, x\in S\cup\{\Delta\})$ be a standard Markov process. (S is separable locally compact Hausdorff space.) The Green operator G^λ of $\boldsymbol{M}$ is defined as

$$(1.1)\qquad U(x)=G^\lambda f(x)=E_x\int_0^\zeta e^{-\lambda t}f(x_t)dt$$

for any bounded Borel measurable function f on S, and $\lambda>0$. In this paper the notations and definitions of additive functional, excessive function, etc. are the same as those of [2], [4].

Let M_t be a uniformly integrable continuous multiplicative functional of $\boldsymbol{M}$, such as $P_x(M_0=1)=1$, $E_xM_t=1$ for ${}^\forall x\in S$ and $0\leqq{}^\forall t<\infty$. ($M_t=M_\zeta$ for $t\geqq\zeta$). Then M_t is a martingale of class (D) [5].

Let us consider

$$(1.2)\qquad \begin{aligned} X_t[U]&=e^{-\lambda t}U(x_t)-U(x_0)+\int_0^t e^{-\lambda s}f(x_s)ds \qquad (t<\zeta)\\ &=X_{\zeta-}[U] \qquad (t\geqq\zeta). \end{aligned}$$

* We denote by $\mathfrak{M}$ the set of all λ-additive functionals, A_t, having the properties $E_xA_t=0$, $E_x(A_t)^2<\infty$ $(0\leqq t<\infty)$.

This is a λ-additive functional of $\boldsymbol{M}$ such that $E_x X_t[U]=0$, $E_x(X_t[U])^2<\infty$ $(0\leqq t\leqq\infty)$. Define

$$\tilde{U}(x)=\tilde{G}^\lambda f(x)=E_x M_\zeta\cdot\int_0^\zeta e^{-\lambda s}f(x_s)ds=E_x\int_0^\zeta e^{-\lambda s}M_s f(x_s)ds \qquad (\lambda>0). \tag{1.3}$$

PROPOSITION 1.

$$\tilde{U}(x)-U(x)=E_x(M_\zeta\cdot X_\zeta[U]). \tag{1.4}$$

PROOF. First we notice

$$X_\zeta[U]=-U(x_0)+\int_0^\zeta e^{-\lambda s}f(x_s)ds$$

and

$$|E_x M_\zeta\cdot X_\zeta[U]|\leqq||U||+||\tilde{U}||<\infty.$$

Integrating both sides of

$$M_\zeta\cdot X_\zeta[U]=-M_\zeta\cdot U(x_0)+M_\zeta\cdot\int_0^\infty e^{-\lambda s}f(x_s)ds,$$

we get the proposition.

We can also prove

$$E_x\left(M_t\int_0^\zeta e^{-\lambda s}f(x_s)ds\right)-U(x)=E_x(M_t\cdot X_t[U]). \tag{1.5}$$

Now consider

$$Y_t[\tilde{U}]=e^{-\lambda t}\tilde{U}(x_t)-\tilde{U}(x_0)+\int_0^t e^{-\lambda s}f(x_s)ds \qquad (\lambda>0). \tag{1.6}$$

It can be verified that the following properties hold:

$$E_x(M_t\cdot Y_t[\tilde{U}])=0 \qquad (0\leqq t\leqq\zeta)$$

$$E_x M_t\cdot(Y_t[\tilde{U}])^2\leqq 2||\tilde{U}||\cdot\left|E_x\left(M_\zeta\cdot\int_0^\zeta e^{-2\lambda s}f(x_s)ds\right)\right|<\infty$$

and

$$Y_s[\tilde{U}](w)+e^{-\lambda s}\cdot Y_t[\tilde{U}](w_s^+)=Y_{t+s}[\tilde{U}](w). \tag{1.7}$$

PROPOSITION 2.

$$\tilde{U}(x)-U(x)=-E_x Y_\zeta[\tilde{U}]. \tag{1.8}$$

PROOF. Since $|E_x Y_t[\tilde{U}]|\leqq 2||\tilde{U}||+||U||$ $(0\leqq t\leqq\zeta)$, integrating both sides of (1.6), and noticing that

$$E_x e^{-\lambda t}\tilde{U}(x_t)=E_x\Big(M_\zeta(w_t^+)\cdot\int_t^\zeta e^{-\lambda s}f(x_s)ds;\ \zeta>t\Big)\to 0\quad(t\to\infty),$$

we get the proposition for $\lambda>0$.

LEMMA 1. *Let*

$$U_i(x)=E_x\int_0^\zeta e^{-\lambda u}f_i(x_u)du \qquad f_i\geqq 0, \qquad i=1,2.$$

Then

$$E_x(X_\zeta[U_1]-X_\zeta[U_2])^2\leqq 2\|U_1-U_2\|(\|U_1\|+\|U_2\|). \tag{1.9}$$

PROOF. Set $U=U_1-U_2$, $f=f_1-f_2$. Since the left-hand side is equal to

$$E_x(X_\zeta[U])^2=E_x\Big(\int_0^\zeta e^{-\lambda s}f(x_s)ds\Big)^2-U^2(x),$$

we have

$$E_x(X_\zeta[U])^2\leqq 2E_x\Big(\int_0^\zeta e^{-2\lambda s}U(x_s)f(x_s)ds\Big)\leqq 2\|U\|\,E_x\Big(\int_0^\zeta e^{-2\lambda s}|f(x_s)|ds\Big)$$
$$\leqq 2\|U\|(\|U_1\|+\|U_2\|).$$

LEMMA 2.

$$E_xM_\zeta(Y_\zeta[\tilde{U}_1]-Y_\zeta[\tilde{U}_2])^2\leqq 2\|\tilde{U}_1-\tilde{U}_2\|(\|\tilde{U}_1\|+\|U_2\|), \tag{1.10}$$

where

$$\tilde{U}_i(x)=E_xM_\zeta\cdot\int_0^\zeta e^{-\lambda u}f_i(x_u)du.$$

The proof of the lemma is similar to that of the previous one, and will be omitted.

Now we shall investigate the Markov process $\tilde{M}=(\tilde{x}_t, \tilde{P}_x, \zeta, x\in S\cup\{\Delta\})$ obtained from M by a transformation of probability measure such that the elements of $\tilde{M}$ are given as follows:

$$\tilde{x}_t=x_t\quad(t<\zeta),\quad =\Delta\quad(t\geqq\zeta) \tag{1.11}$$
$$\tilde{P}_x(A)=E_xM_\zeta\chi_A.$$

It is well-known that $\tilde{M}$ is also a standard process. The Green operator of $\tilde{M}$ is given by (1.3). We will assume $P_x(M_t>0, 0\leqq t<\zeta)=1$.

THEOREM 1.

$$\mathfrak{N}=\{f: G^\lambda f=0\}=\tilde{\mathfrak{N}}=\{f: \tilde{G}^\lambda f=0\}.$$

PROOF. Suppose $G^\lambda f=0$. From lemma 1 it follows that $E_x(X_\zeta[G^\lambda f])^2 =0$ and $X_\zeta[G^\lambda f]=0$ (a.s. P_x). This and proposition 1 yield that $\tilde{G}^\lambda f=0$. Conversely, if $\tilde{G}^\lambda f=0$, then $Y_\zeta[\tilde{G}^\lambda f]=0$ (a.s. P_x) from lemma 2 and the assumption $M_t>0$ $[0\leqq t<\zeta]$. This together with proposition 2 yields $G^\lambda f=0$.

2. Brownian motion with drift

Let $X=(x_t, \tau, P_x)$* be a part of a Brownian motion on a bounded open set G whose boundary is of class $A^{(1,\mu)}$, that is, any local coordinate function of ∂G is differentiable and its derivatives satisfy the Hölder condition, and $a(x)=(a_1(x_1,\cdots,x_n),\cdots,a_n(x_1,\cdots,x_n))$** be a bounded Borel measurable function on G and $\tau=\inf\{t>0, x_t\notin G\}$. We shall treat the Brownian motion with drift, $\tilde{X}=(\tilde{x}_t, \tau, \tilde{P}_x)$, the elements of which are given by

$$\tilde{x}_t=x_t \quad \text{for } t<\tau, \quad =\Delta \quad \text{for } t\geqq\tau,$$

$$\tilde{P}_x(A)=\int_A \exp\left[\int_0^\tau a(x_u)dx_u-\frac{1}{2}\int_0^\tau a^2(x_u)du\right]P_x(dw). \tag{2.1}$$

$\tilde{X}$ is a continuous strong Feller process and $\lim_{x\to a}\tilde{T}_t f(x)=0$ $(x\in G, a\in\partial G)$. If $a(x)$ is Hölder continuous on $\bar{G}$, then $\tilde{X}$ is a diffusion process with absorbing barrier, and its Green operator $\tilde{G}^\lambda f$ is differentiable on $\bar{G}$, and twice continuously differentiable on $\bar{G}$ if f is Hölder continuous on $\bar{G}$. Let $\mathcal{D}(G)$ denote the set of all twice continuously differentiable functions which vanish outside some compact set in G. If $f\in\mathcal{D}(G)$ and

$$Df=\frac{1}{2}\Delta f+a(\cdot)\,\mathrm{grad}\,f \tag{2.2}$$

is continuous, then f belongs to the domain of generator $\tilde{\mathfrak{G}}$ of $\tilde{X}$ and $\tilde{\mathfrak{G}}f =Df$. The above assertions are proved in theorems 14.17 and 14.18 of [2]. The proofs of the following formulas are also found in [2].

$$\sup_{x\in G} E_x\tau<\infty \tag{2.3}$$

$$M_t-1=\int_0^t a(x_u)\chi_{\tau>u}M_u dx_u \tag{2.4}$$

where

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* P_x is the probability measure of a Brownian motion in R^n.

** We extend $a(x)$ to the whole space by defining $a(x)=0$ for $x\notin G$.

$$M_t=\exp\left(\int_0^t a(x_s)\chi_{\tau>s}dx_s-\frac{1}{2}\int_0^t a^2(x_s)\chi_{\tau>s}ds\right)\qquad(0\leqq t\leqq\infty)\tag{2.5}$$

$$E_xM_t=1\qquad(0\leqq t\leqq\infty)\tag{2.6}$$

$$E_x(M_t)^2\leqq\exp(\|a\|^2t)\qquad(0\leqq t<\infty)$$

$$\begin{aligned}\tilde{H}_t^\lambda f(x)\equiv\tilde{E}_xe^{-\lambda t}f(x_t)&=E_xM_\tau e^{-\lambda t}f(x_t)\chi_{\tau>t}\\&=E_xM_te^{-\lambda t}f(x_t)\chi_{\tau>t}.\end{aligned}\tag{2.7}$$

3. $X_t[U]$ and $Y_t[\tilde{U}]$ for $U, \tilde{U}\in C^2(\bar{G})$

In this section we shall represent $X_t[G^\lambda f]$ and $Y_t[\tilde{G}^\lambda f]$ by stochastic integral when $G^\lambda f$ and $\tilde{G}^\lambda f$ are twice continuously differentiable. The formula of stochastic integral shows that for $h\in C^2$,

$$\int_0^t\operatorname{grad}h(x_u)dx_u{}^*=h(x_t)-h(x_0)-\int_0^t\frac{1}{2}\Delta h(x_u)du,\tag{3.1}$$

and its slight modification,

$$\int_0^t\operatorname{grad}h(x_u)dx_u=h(x_t)-h(x_0)-\int_0^tDh(x_u)du+\int_0^ta(x_u)\operatorname{grad}h(x_u)du\tag{3.1'}$$

hold. By using the relation,

$$\begin{aligned}&\int_0^te^{-\lambda u}\operatorname{grad}h(x_u)dx_u\\&=e^{-\lambda t}\int_0^t\operatorname{grad}h(x_u)dx_u+\lambda\int_0^te^{-\lambda u}\left(\int_0^u\operatorname{grad}h(x_s)dx_s\right)du,\end{aligned}$$

we get (3.2) and (3.3) from (3.1) and (3.1′) respectively.

$$\int_0^te^{-\lambda u}\operatorname{grad}h(x_u)dx_u=e^{-\lambda t}h(x_t)-h(x_0)+\int_0^te^{-\lambda u}\left(\lambda-\frac{1}{2}\Delta\right)h(x_u)du\tag{3.2}$$

$$\begin{aligned}&\int_0^te^{-\lambda u}\operatorname{grad}h(x_u)dx_u-\int_0^te^{-\lambda u}a(x_u)\operatorname{grad}h(x_u)du\\&=e^{-\lambda t}h(x_t)-h(x_0)+\int_0^te^{-\lambda s}(\lambda-D)h(x_s)ds\qquad(\lambda>0).\end{aligned}\tag{3.3}$$

Then, if both $U=G^\lambda f$ and $\tilde{U}=\tilde{G}^\lambda f$ are twice continuously differentiable on $\bar{G}$, we get

$$X_t[U]=e^{-\lambda t}U(x_t)-U(x_0)+\int_0^te^{-\lambda s}f(x_s)ds\tag{3.4}$$

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* $\int_0^t\operatorname{grad}h(x_u)dx_u=\sum_{j=1}^n\int_0^t\frac{\partial}{\partial x^{(j)}}h(x_u)dx_u^{(j)}$.

$$=\int_0^t e^{-\lambda s}\,\mathrm{grad}\, U(x_s)dx_s$$

and

$$\text{(3.5)}\qquad Y_t[\tilde{U}]=e^{-\lambda t}\tilde{U}(\tilde{x}_t)-\tilde{U}(\tilde{x}_0)+\int_0^t e^{-\lambda s}f(\tilde{x}_s)ds$$
$$=\int_0^t e^{-\lambda s}\,\mathrm{grad}\,\tilde{U}(\tilde{x}_s)dx_s-\int_0^t e^{-\lambda s}a(\tilde{x}_s)\,\mathrm{grad}\,\tilde{U}(\tilde{x}_s)ds \qquad (t\leqq\tau).$$

These facts will be extended in the following sections.

4. Proofs of (0.3) and (0.5)

First we raise the next lemma which can be proved if we notice that

$$\left|\frac{\partial\bar{P}(t,x,y)}{\partial x_j}\right|\leqq Mt^{-(n+1)/2}\exp\{-(|y-x|^2)/t\}.$$

(See [8] and the appendix of [2].)

LEMMA 3. *Let $\bar{G}^\lambda f$ and $\bar{P}(t,x,y)$ denote the Green operator and the transition density of a diffusion process on G with an absorbing barrier. Then*

$$\mathrm{grad}\,[\bar{G}^\lambda f](x)=\int_{\bar{G}}\bar{h}(x,y)f(y)dy^{*} \qquad (x\in\bar{G})$$

$$\int_{\bar{G}}\left(\int_{\bar{G}}\bar{h}(x,y)dy\right)^2dx<\infty$$

where

$$\bar{h}(x,y)=\int_0^\infty e^{-\lambda u}\,\mathrm{grad}_x\,\bar{P}(t,x,y)dt \qquad (x,y)\in\bar{G}\times\bar{G}.$$

Quantities analogous to $\bar{G}$, $\bar{h}$, etc. in lemma 3, but defined for X and $\tilde{X}$ of section 2, will be denoted by G, $\tilde{G}$, h, $\tilde{h}$, etc.

* $\bar{h}(x,y)=(h_1(x,y),h_2(x,y),\cdots,h_n(x,y))$,
$\frac{\partial}{\partial x_j}\bar{G}^\lambda f(x)=\int_{\bar{G}}\bar{h}_j(x,y)f(y)dy$,
$\bar{h}_j(x,y)=\int_0^\infty e^{-\lambda s}\frac{\partial\bar{P}(s,x,y)}{\partial x_j}ds$,
$\left(\int_{\bar{G}}\bar{h}(x,y)dy\right)^2=\sum_{j=1}^n\left(\int_{\bar{G}}\bar{h}_j(x,y)dy\right)^2$,
$\bar{G}^\lambda f(x)=\int_{\bar{G}}\bar{g}_\lambda(x,y)f(y)dy$.

LEMMA 4. *For any bounded Borel measurable function* $f(x)$ *on* $\bar{G}$, *we get*

$$(4.1)\qquad \int_{\bar{G}} E_x\left(X_\infty[G^\lambda f]-\int_0^\infty e^{-\lambda s}\operatorname{grad} G^\lambda f(x_s)dx_s\right)^2 dx=0$$

where $G^\lambda f$ *and* $X_t[G^\lambda f]$ *have the same meaning as in section* 3. *Moreover, the fact that the integrand of the left-hand side of* (4.1) *is* 2λ*-excessive function yields that it equals* 0 *for any* $x\in\bar{G}$ *and then*

$$(4.2)\qquad X_t[G^\lambda f]=\int_0^t e^{-\lambda u}\operatorname{grad} G^\lambda f(x_u)dx_u \qquad (a.s.\ P_x,\ t\leqq\tau)$$

holds.

PROOF. First we shall prove the lemma when $f(x)$ is a limit of bounded pointwise convergence of a sequence $f_n(x)$ in $C^{0,k}(\bar{G})$. Note that $f_n\in C^{0,k}(\bar{G})$ implies

$$G^\lambda f_n\in C^2(\bar{G})$$

and

$$X_\infty[G^\lambda f_n]=\int_0^\tau e^{-\lambda s}\operatorname{grad} G^\lambda f_n(x_s)dx_s.$$

By lemma 3 we have

$$\begin{aligned}(4.3)\qquad & E_x\left(\int_0^\tau e^{-\lambda u}\operatorname{grad} G^\lambda f(x_u)dx_u-\int_0^\tau e^{-\lambda u}\operatorname{grad} G^\lambda f_n(x_u)dx_u\right)^2\\ &=E_x\int_0^\tau e^{-2\lambda u}(\operatorname{grad} G^\lambda f(x_u)-\operatorname{grad} G^\lambda f_n(x_u))^2du\\ &=\int_{\bar{G}} g_{2\lambda}(x,y)(\operatorname{grad} G^\lambda f(y)-\operatorname{grad} G^\lambda f_n(y))^2dy\\ &=\int_{\bar{G}} g_{2\lambda}(x,y)\left(\int_{\bar{G}} h(y,z)f(z)dz-\int_{\bar{G}} h(y,z)f_n(z)dz\right)^2dy^*.\end{aligned}$$

Integrating both sides, we have

$$\begin{aligned}(4.4)\qquad & \int_{\bar{G}} E_x\left(\int_0^\tau e^{-\lambda u}\operatorname{grad} G^\lambda f(x_u)dx_u-\int_0^\tau e^{-\lambda u}\operatorname{grad} G^\lambda f(x_u)dx_u\right)^2dx\\ &=\int_{\bar{G}}\int_{\bar{G}} g_{2\lambda}(x,y)\left(\int_{\bar{G}} h(y,z)f(z)dz-\int_{\bar{G}} h(y,z)f_n(z)dz\right)^2dy\,dx\\ &\leqq K\int_{\bar{G}}\left(\int_{\bar{G}} h(y,z)f(z)dz-\int_{\bar{G}} h(y,z)f_n(z)dz\right)^2dy \quad\to 0 \text{ as } n\to\infty,\end{aligned}$$

by Lebesgue's bounded convergence theorem. On the other hand

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* $(\operatorname{grad} G^\lambda f(y))^2=\sum_{j=1}^n\left(\frac{\partial}{\partial y_j}G^\lambda f(y)\right)^2=\sum_{j=1}^n\left(\int_{\bar{G}} h_j(y,z)f(z)dz\right)^2.$

$$(4.5)\qquad \int_{\bar G} E_x(X_\tau[G^\lambda f]-X_\tau[G^\lambda f_n])^2 dx$$
$$=2\int_{\bar G} E_x\left(\int_0^\tau e^{-2\lambda s}(G^\lambda f-G^\lambda f_n)(x_s)(f-f_n)(x_s)ds\right)dx$$
$$-\int_{\bar G}(G^\lambda f(x)-G^\lambda f_n(x))^2 dx$$
$$=2\int_{\bar G}\int_{\bar G} g_{2\lambda}(x,y)(f-f_n)(y)(G^\lambda f-G^\lambda f_n)(y)dy\,dx$$
$$-\int_{\bar G}(G^\lambda f_n(x)-G^\lambda f(x))^2 dx\,.$$

Both terms in the right hand sides of (4.5) tend to 0 as $n\to\infty$, since $G^\lambda f_n\to G^\lambda f$ boundedly. Thus we conclude

$$\int_{\bar G} E_x\left(X_\tau[G^\lambda f]-\int_0^\tau e^{-\lambda u}\,\mathrm{grad}\,G^\lambda f(x_u)dx_u\right)^2 dx$$
$$\leqq 2\int_{\bar G} E_x(X_\tau[G^\lambda f]-X_\tau[G^\lambda f_n])^2 dx+2\int_{\bar G} E_x\left(\int_0^\tau e^{-\lambda s}\,\mathrm{grad}\,G^\lambda f_n(x_s)dx_s\right.$$
$$\left.-\int_0^\tau e^{-\lambda s}\,\mathrm{grad}\,G^\lambda f(x_s)dx_s\right)^2 dx\to 0\,,$$

which proves (4.1). In particular for any bounded continuous function (4.1) is true. We can also prove in the same way as above that the class of functions for which (4.1) is valid is a monotone class; thus, for any bounded Borel measurable function (4.1) is proved. The rest is proved by the fact that

$$X_t[G^\lambda f]-\int_0^t e^{-\lambda s}\,\mathrm{grad}\,G^\lambda f(x_s)dx_s$$

belongs to the class $\mathfrak{M}_c$. (See footnote of section 1.)

COROLLARY.

$$(4.6)\qquad e^{-\lambda t}G^\lambda f(x_t)-G^\lambda f(x_0)$$
$$=\int_0^t e^{-\lambda s}\,\mathrm{grad}\,G^\lambda f(x_s)dx_s-\int_0^t e^{-\lambda s}f(x_s)ds \qquad (a.s.\ t\leqq\tau)$$

$$(4.7)\qquad U(x_t)-U(x_0)=\int_0^t \mathrm{grad}\,U(x_s)dx_s+\int_0^t \mathfrak{G}U(x_s)ds \qquad (a.s.\ t\leqq\tau),$$

where

$$\mathfrak{G}U=(\lambda-(G^\lambda)^{-1})U,\qquad for\ U\in\mathfrak{D}(\mathfrak{G}),$$

is the generator of X in the sense of K. Ito [3]. *This is an extended formula of stochastic integral.*

From (4.2) it follows that

$$
\begin{aligned}
E_x M_\infty X_\infty[G^\lambda f] = E_x M_\tau X_\tau[G^\lambda f] &= E_x M_\tau \cdot \int_0^\tau e^{-\lambda u} \operatorname{grad} G^\lambda f(x_u) dx_u \\
&= E_x (M_\tau - 1) \cdot \int_0^\tau e^{-\lambda u} \operatorname{grad} G^\lambda f(x_u) dx_u \\
&= E_x \left(\int_0^\tau a(x_u) M_u dx_u \right) \left(\int_0^\tau e^{-\lambda u} \operatorname{grad} G^\lambda f(x_u) dx_u \right) \\
&= E_x \int_0^\tau M_u e^{-\lambda u} a(x_u) \operatorname{grad} G^\lambda f(x_u) du \\
&= \tilde{E}_x \int_0^\tau e^{-\lambda u} a(x_u) \operatorname{grad} G^\lambda f(x_u) du^*.
\end{aligned}
$$

By this and proposition 1 we get the following:

THEOREM 2.

$$
\tilde{G}^\lambda f(x) - G^\lambda f(x) = \tilde{G}^\lambda [a(\cdot) \operatorname{grad} G^\lambda f](x),
$$

$$
\textit{for } {}^\forall f \in B(\overline{G}) \textit{ and } {}^\forall x \in \overline{G}.
$$

5. Representing $Y_t[\tilde{U}]$ by martingale Z_t, and proof of (0.4)

From now on we assume that $a(x)$ is Hölder continuous, then as was mentioned in section 2, $\tilde{X}$ is a diffusion process with an absorbing barrier corresponding to differential operator $\frac{1}{2}\Delta + \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}$. The Green operator $\tilde{G}^\lambda f$ is in $C^1(\overline{G})$ and in $C^2(\overline{G})$ if $f \in C^{0,k}(\overline{G})$. From lemma 3 we have already known that

$$
\operatorname{grad} \tilde{G}^\lambda f(x) = \int_{\overline{G}} \tilde{h}(x, y) f(y) dy,
$$

$$
\tilde{h}(x, y) = \int_0^\infty e^{-\lambda t} \operatorname{grad}_x \tilde{p}(t, x, y) dt,
$$

and

$$
\int_{\overline{G}} \left(\int_{\overline{G}} \tilde{h}(x, y) dy \right)^2 dx < \infty.
$$

$\int_{\overline{G}} \int_0^\infty e^{-\lambda t} \tilde{p}(t, x, y) dt\, dx$ is also bounded.

Let $(x_{t\wedge\tau}, E_x, x \in \overline{G})$ be a stopped Brownian motion on $\overline{G}$. Obviously $x_{t\wedge\tau} = \tilde{x}_t = x_t (t < \tau)$**. Let us consider

$$
Z_t = (Z_t^{(1)}, Z_t^{(2)}, \cdots, Z_t^{(n)})
$$

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* $a(x) \operatorname{grad} G^\lambda f(x) = \sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j} G^\lambda f(x)$.

** $x_t(w)$ is the part of a Brownian motion. As for $\tilde{x}_t(w)$, see the definition (2.1).

(5.1) $$Z_t^{(j)}=x_{t\wedge\tau}^{(j)}-x_0^{(j)}-\int_0^{t\wedge\tau} a_j(x_u)du \qquad (0\leqq t<\infty).$$

Applying the formula of stochastic integral to Z_t and $M_t-1=\int_0^t a(x_u)\cdot M_u dx_u$ and $F(x_1, x_2)=x_1x_2$, we have

$$\begin{aligned} Z_t^{(j)}\cdot M_t &= \int_0^{t\wedge\tau} M_s dx_s^{(j)} - \int_0^{t\wedge\tau} M_s\cdot a_j(x_s)ds \\ &\quad + \sum_{k=1}^n \int_0^{t\wedge\tau} Z_s^{(j)}\cdot M_s\cdot a_k(x_s)\cdot dx_s^{(k)} + \int_0^{t\wedge\tau} M_s\cdot a_j(x_s)ds \\ &= \int_0^{t\wedge\tau} M_s dx_s^{(j)} + \sum_{k=1}^n \int_0^{t\wedge\tau} Z_s^{(j)}\cdot a_k(x_s) M_s dx_s^{(k)} \qquad (j=1, 2, \cdots, n). \end{aligned}$$

(See [3] and [7].)

Again applying the formula of stochastic integral to $Z_t^{(j)}, M_t$ and $Z_t^{(j)}$ and $F(x_1, x_2)=x_1\cdot x_2$, we have

$$(Z_t^{(j)})^2 M_t = 2\int_0^{t\wedge\tau} M_s\cdot Z_s^{(j)} dx_s^{(j)} + \sum_{k=1}^n \int_0^{t\wedge\tau} M_s\cdot (Z_s^{(j)})^2\cdot a_k(x_s) dx_s^{(k)} + \int_0^{t\wedge\tau} M_u du.$$

Since $|Z_t^{(j)}|\leqq |x_{t\wedge\tau}^{(j)}-x_0^{(j)}|+||a||t\leqq d+||a||t$, $(z_t^{(j)})^2\leqq 2(d^2+||a||^2t)$ and $E_x(M_t^2)\leqq \exp\{||a||^2t\}$ $(0\leqq t<\infty)$, where d is the diameter of G, we have

$$E_x\int_0^{t\wedge\tau} M_s^2 ds \leqq C_0(\exp\{||a||^2t\}-1),$$

$$E_x\int_0^{t\wedge\tau} (Z_s^{(j)} a_k(x_s) M_s)^2 ds \leqq C_1(d^2+||a||^2t)\cdot(\exp\{||a||^2t\}-1)<\infty,$$

and

$$E_x(Z_t^{(j)}\cdot M_t)^2 \leqq (J_1+J_2t)(\exp\{||a||^2t\}-1)<\infty.$$

Hence

$$E_x Z_t^{(j)}\cdot M_t = 0 \qquad (0\leqq t<\infty).$$

Similarly we have

$$E_x\int_0^{t\wedge\tau} ((Z_s^{(j)})^2 a_k(x_s) M_s)^2 ds \leqq (K_1+K_2t+K_3t^2)(\exp\{||a||^2t\}-1).$$

Hence

$$E_x M_t (Z_t^{(j)})^2 = E_x\int_0^{t\wedge\tau} M_u du \qquad (0\leqq t<\infty).$$

Noticing that $M_t=M_\tau$ $(t\geqq\tau)$, we have

(5.2) $$\begin{aligned} \tilde{E}_x Z_t &= E_x M_\tau Z_t = E_x(M_t\cdot Z_t\cdot E_{x_t} M_\tau;\, t<\tau) + E_x(M_t Z_t;\, t\geqq\tau) \\ &= E_x M_t Z_t = 0, \end{aligned}$$

$$\tilde{E}_x(Z_t)^2 = \tilde{E}_x t \wedge \tau . \tag{5.3}$$

$(Z_t, \mathcal{F}_{t\wedge\tau}, \tilde{P}_x)$ is a square integrable martingale, and we can define the stochastic integral for Z_t following to Courrège [6]

$$\tilde{E}_x\left(\int_0^t F_u dZ_u\right)^2 = \tilde{E}_x\int_0^{t\wedge\tau} |F_u|^2 du \qquad (0 \leqq t < \infty) \tag{5.4}$$

$$\tilde{E}_x((Z_t - Z_s)^2 | \mathcal{F}_{s\wedge\tau}) = \tilde{E}_x(t\wedge\tau - s\wedge\tau | \mathcal{F}_{s\wedge\tau}) \qquad (0 \leqq s \leqq t < \infty). \tag{5.5}$$

LEMMA 5.

$$\int_{\bar{G}} \tilde{E}_x\left(Y_s[\tilde{G}^\lambda f] - \int_0^s e^{-\lambda u} \operatorname{grad} \tilde{G}^\lambda f(x_u) dZ_u\right)^2 dx = 0 \qquad (0 \leqq s \leqq \tau) \tag{5.6}$$

$$Y_t[\tilde{G}^\lambda f] = \int_0^t e^{-\lambda u} \operatorname{grad} \tilde{G}^\lambda f(x_u) dZ_u \qquad (a.s.\ t \leqq \tau \cdot \tilde{P}_x) \tag{5.7}$$

for $^\forall f \in B(\bar{G})$ *and* $^\forall x \in \bar{G}$, *where* $\tilde{G}^\lambda f$ *and* $Y_t[\tilde{G}^\lambda f]$ *are the same as in section* 3.

COROLLARY.

$$e^{-\lambda t}\tilde{G}^\lambda f(x_t) - \tilde{G}^\lambda f(x_0) = \int_0^t e^{-\lambda u} \operatorname{grad} \tilde{G}^\lambda f(x_u) dZ_u - \int_0^t e^{-\lambda u} f(x_u) du \qquad (a.s.\ t \leqq \tau \cdot \tilde{P}_x) \tag{5.8}$$

$$\tilde{U}(x_t) - \tilde{U}(x_0) = \int_0^t \operatorname{grad} \tilde{U}(x_s) dZ_s + \int_0^t \tilde{\mathfrak{G}} \tilde{U}(x_s) ds \qquad (a.s.\ t \leqq \tau \cdot \tilde{P}_x), \tag{5.9}$$

where

$$\tilde{\mathfrak{G}}\tilde{U} = (\lambda - (\tilde{G}^\lambda)^{-1})\tilde{U} \qquad \text{for } \tilde{U} \in \mathcal{D}(\tilde{\mathfrak{G}}).$$

The proof of lemma 5 is carried out in the same way as that of lemma 4, if we remark the following:

$$\begin{aligned}
&\int_{\bar{G}} \tilde{E}_x\left(\int_0^\infty e^{-\lambda u} \operatorname{grad} \tilde{G}^\lambda f(x_u) dZ_u\right)^2 dx \\
&= \int_{\bar{G}} \tilde{E}_x \int_0^\infty e^{-2\lambda u} (\operatorname{grad} \tilde{G}^\lambda f(x_u))^2 du\, dx \\
&= \int_{\bar{G}}\int_{\bar{G}} \tilde{g}_{2\lambda}(x, y) (\operatorname{grad} \tilde{G}^\lambda f(y))^2 dy\, dx \\
&= \int_{\bar{G}}\int_{\bar{G}} \tilde{g}_{2\lambda}(x, y) \left(\int_{\bar{G}} \tilde{h}(y, z) f(z) dz\right)^2 dy\, dx \\
&\leqq K \int_{\bar{G}} \left(\int_{\bar{G}} \tilde{h}(y, z) f(z) dz\right)^2 dy .
\end{aligned}$$

(5.9) is the extended formula of a stochastic integral represented by the generator $\tilde{\mathfrak{G}}$ of $\tilde{X}$.

Now we shall prove a lemma which shows that the set of $\tilde{P}_x$-measure 0 is also P_x-measure 0, and that (5.8) and (5.9) are valid if we replace $\tilde{P}_x$ by P_x.

LEMMA 6.

$$P_x(A)=\tilde{E}_x N_\tau \cdot \chi_A(w), \qquad (A \in \mathcal{F}_{\tau\wedge t},\ {}^\forall t \geqq 0) \tag{5.10}$$

where

$$N_t=\exp\left[-\int_0^t a(x_u)dZ_u-\frac{1}{2}\int_0^{t\wedge\tau} a^2(x_u)du\right] \qquad \textit{for } 0\leqq t\leqq\infty.$$

PROOF. Notice that

$$\left(-\int_0^t a(x_u)dZ_u-\frac{1}{2}\int_0^t a^2(x_u)du\right)+\left(\int_0^t a(x_u)dx_u-\frac{1}{2}\int_0^t a^2(x_u)du\right)$$
$$=-\int_0^t a(x_u)dx_u+\int_0^t a^2(x_u)du-\frac{1}{2}\int_0^t a^2(x_u)du+\int_0^t a(x_u)dx_u$$
$$-\frac{1}{2}\int_0^t a^2(x_u)du=0,$$

by the definition of Z_t. This yields $M_tN_t=1$ $(0\leqq t\leqq\infty)$ and $\tilde{E}_xN_\infty\chi_A(w)$ $=E_xM_\tau N_\tau\chi_A(w)=P_x(A)$.

Remark. N_t has the same properties for $\tilde{X}$ as those of M_t for X, for example,

$$\tilde{E}_xN_t=1 \qquad (0\leqq t\leqq\infty), \tag{5.11}$$

$$N_t-1=-\int_0^t N_u\cdot a(\tilde{x}_u)dZ_u \qquad (0\leqq t\leqq\infty). \tag{5.12}$$

Finally we get the following:

THEOREM 3. *If $a(x)$ is Hölder continuous, then*

$$\tilde{G}^\lambda f(x)-G^\lambda f(x)=G^\lambda[a(\cdot)\operatorname{grad}\tilde{G}^\lambda f](x), \tag{5.13}$$

for any bounded Borel measurable function on $\overline{G}$ and any x in $\overline{G}$.

PROOF. By proposition 2 and the relation (5.8) for P_x-measure,

$$\tilde{G}^\lambda f(x)-G^\lambda f(x)=-E_xY_\infty[\tilde{G}^\lambda f]$$
$$=-E_x\int_0^\tau e^{-\lambda u}\operatorname{grad}\tilde{G}^\lambda f(x_u)dZ_u$$

$$=-E_x\int_0^\tau e^{-\lambda u}\operatorname{grad}\tilde{G}^\lambda f(x_u)dx_u+E_x\int_0^\tau e^{-\lambda u}\operatorname{grad}\tilde{G}^\lambda f(x_u)\cdot a(x_u)du$$
$$=E_x\int_0^\tau e^{-\lambda u}a(x_u)\operatorname{grad}\tilde{G}^\lambda f(x_u)du$$
$$=G^\lambda[a(\cdot)\operatorname{grad}\tilde{G}^\lambda f](x).$$

From theorems 1, 2 and 3, we can see that

$$\mathcal{D}(\mathfrak{G})=\mathfrak{R}(G^\lambda)/\mathfrak{N}=\mathfrak{R}(\tilde{G}^\lambda)/\tilde{\mathfrak{N}}=\mathcal{D}(\tilde{\mathfrak{G}})$$

and

$$\tilde{\mathfrak{G}}=\mathfrak{G}+a(\cdot)\operatorname{grad},$$

where $\mathfrak{R}(G^\lambda)$ and $\mathfrak{R}(\tilde{G}^\lambda)$ denote the ranges of G^λ ane $\tilde{G}^\lambda$ of bounded Borel measurable functions on $\bar{G}$, respectively, and they do not depend on λ [3].

Appendix

We shall give simpler proofs of (0.3) and (0.4). The assumption for $a(\cdot)$ and the boundary of G are the same as those of section 5. (0.3) and (0.4) are true for $f\in C^{0,k}(\bar{G})$ from the results of sections 1 and 2. For any function which is a limit of bounded pointwise convergence of a sequence $f_n(x)$ in $C^{0,k}(\bar{G})$, Lebesgue's bounded convergence theorem yields that $G^\lambda f_n(x)\to G^\lambda f(x)$, $\tilde{G}^\lambda f_n(x)\to\tilde{G}^\lambda f(x)$ and $\int_{\bar{G}}h(y,z)f_n(z)dz\to\int_{\bar{G}}h(y,z)\cdot f(z)dz$, boundedly. Then

$$\int_{\bar{G}}(\tilde{G}^\lambda f(x)-G^\lambda f(x))dx$$
$$=\lim_{n\to\infty}\int_{\bar{G}}(\tilde{G}^\lambda f_n(x)-G^\lambda f_n(x))dx$$
$$=\lim_{n\to\infty}\int_{\bar{G}}\int_{\bar{G}}\tilde{g}_\lambda(x,y)a(y)\int_{\bar{G}}h(y,z)f_n(z)dz\,dy\,dx$$
$$=\int_{\bar{G}}\int_{\bar{G}}\tilde{g}_\lambda(x,y)a(y)\int_{\bar{G}}h(y,z)f(z)dz\,dy\,dx.$$

Since the integrands of both sides of this equality are continuous ($\tilde{X}$ is a strong Feller process), we obtain (0.3). We can also prove that the class of functions for which (0.3) is valid is a monotone class, and we have (0.3) for any bounded Borel measurable function. In a similar way we can prove (0.4).

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