DIRICHLET'S TRANSFORMATION AND DISTRIBUTIONS OF LINEAR FUNCTIONS OF ORDERED GAMMA VARIATES[1)]

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The Dirichlet's transformation for evaluating the Dirichlet's multiple integral is well known, and is found in almost all text books on advanced calculus, see, e.g., Gibson ([5], p. 492). This transformation may be used to simplify certain multiple integrals which occur in the distribution theory of linear functions of ordered gamma variates. In fact, Dirichlet's transformation reduces certain unknown integrals, occurring in the distribution theory of ordered gamma variates, to integrals of known types. The transformation may be slightly modified and used in the distribution theory of linear functions of ordered normal variates. The Dirichlet's transformation does not appear to have been utilized in connection with ordered gamma or normal variates, and it might, perhaps, be useful to point out its applications to ordered gamma variate theory. Some other applications of Dirichlet's integral are given by Wilks [13].

1. Introduction

A considerable amount of research work has been recently done in connection with the distribution theory of linear functions of ordered exponential variates, see, e.g., [2], [3], [4], [9], [10], and [12]. Comparatively the ordered gamma variate theory is less developed, although recently it is drawing attention of many research workers, see, e.g., [6], [7], and [8]. One of the foremost obstruction encountered with the ordered gamma variate theory is the difficulty of integrating the ordered gamma variates, say, $x_1<x_2<\cdots<x_N$, over the range, $0<x_i<\infty$, $i=1, \cdots, N$, and $x_1<x_2<\cdots<x_N$. Often this difficulty is completely obviated by the Dirichlet's transformation, which transforms the ordered x variates to the unordered θ variates with the range, $0<\theta_N<\infty$, $0<\theta_i<1$, $i=1, \cdots, N-1$. We formally develop this general theory in the next

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section and in section 3 we give a few illustrations of our general theory.

2. General theory of Dirichlet's transformation

The Dirichlet's (or Dirichlet-Liouville's) multiple integral, which all of us know, may once again be stated as follows, Gibson ([5], p. 492). Integrate the following multiple integral

$$\int y_1^{\alpha_1-1} y_2^{\alpha_2-1} \cdots y_N^{\alpha_N-1} f(y_1+y_2+\cdots+y_N)\, dy_1 \cdots dy_N \,, \tag{2.1}$$

over the range, $0<y_1+y_2+\cdots+y_N<\theta_N$, $0<y_i<\infty$, $i=1, \cdots, N$. We assume $\alpha_i>0$, $i=1, \cdots, N$, and f a suitable function so that the resulting single integral, Gibson ([5], p. 492, last line), with respect to θ_N exists. The following transformation ([5], p. 147, example 2) may be used to evaluate (2.1).

$$\begin{aligned} y_1 &= \theta_1\theta_2 \cdots \theta_N \,, \\ y_1+y_2 &= \theta_2\theta_3 \cdots \theta_N \,, \\ \cdots\cdots &= \cdots\cdots \,, \\ y_1+y_2+\cdots+y_j &= \theta_j\theta_{j+1} \cdots \theta_N \,, \\ \cdots\cdots &= \cdots\cdots \,, \\ y_1+y_2+\cdots+\cdots+y_N &= \theta_N \,. \end{aligned} \tag{2.2}$$

The Jacobian J of the transformation from the y's to θ's is known to be ([5], p. 147, example 2)

$$J=\theta_2\theta_3^2\theta_4^3 \cdots \theta_{N-1}^{N-2}\theta_N^{N-1} \,. \tag{2.3}$$

From the left hand side of equation (2.2) we note that $\sum_{j=1}^{i} y_j > \sum_{j=1}^{i-1} y_j$, $i=2, \cdots, N$. This suggests us the following transformation of our ordered gamma variates. Set $x_i=\sum_{j=1}^{i} y_j$, $i=1, \cdots, N$. This transformation with Jacobian unity assures us that the range, $0<x_i<\infty$, $i=1, \cdots, N$; $x_1<x_2<\cdots<x_N$, is now transformed to the range $0<y_i<\infty$, $i=1, \cdots, N$. Since y's are now unordered we transform y's to θ's by the transformation (2.2). Thus our transformation from x's to θ's has the Jacobian J_1, where

$$J_1=1\times J=\theta_2\theta_3^2 \cdots \theta_{N-1}^{N-2}\theta_N^{N-1} \,. \tag{2.4}$$

Since $x_i>x_{i-1}$, $i=2, \cdots, N$, we note that the range of integration with respect to θ's is determined by the condition, $0<\theta_N<\infty$, $0<\theta_i<1$, $i=1, \cdots, N-1$. Further, we note that the process of integration must be carried on as follows. Integrate with respect to θ_N first, and then with respect to θ_{N-1} and so on, and finally integrate with respect to θ_1.

Now we proceed with the theory of ordered gamma variates. We suppose that an ordered sample $x_1 < x_2 < \cdots < x_N$ is drawn from a gamma population, say,

$$g(x) = (\Gamma(p))^{-1} x^{p-1} \exp\{-x\} , \qquad 0 < x < \infty . \tag{2.5}$$

Obviously the characteristic function (c.f.) $\phi(t)$ of the linear function $u_1 x_1 + \cdots + u_N x_N$, where u's are constants, is

$$\phi(t) = N!(\Gamma(p))^{-N} \int x_1^{p-1} \cdots x_N^{p-1} \exp\{-\sum_{j=1}^{N} (1 - u_j it) x_j\} \, dx_1 \cdots dx_N , \tag{2.6}$$

the range of the integration in (2.6) being, $0 < x_i < \infty$, $i = 1, \cdots, N$, and $x_1 < x_2 < \cdots < x_N$. Transforming x's to θ's by using (2.2) we find that

$$\phi(t) = N!(\Gamma(p))^{-N} \int \exp\{-\theta_N [D]\} \theta_N^{Np-1} \theta_{N-1}^{(N-1)p-1} \cdots \theta_2^{2p-1} \theta_1^{p-1} \, d\theta_N \cdots d\theta_1 , \tag{2.7}$$

where

$$\begin{aligned} [D] = [(1 - u_N it) + \theta_{N-1}((1 - u_{N-1} it) + (1 - u_{N-2} it)\theta_{N-2} + \cdots \\ + (1 - u_1 it)\theta_1 \theta_2 \cdots \theta_{N-2})] . \end{aligned} \tag{2.8}$$

On integrating (2.7) with respect to θ_N from 0 to ∞, we find that

$$\phi(t) = N!\Gamma(Np)(\Gamma(p))^{-N} \int [D]^{-Np} \theta_{N-1}^{(N-1)p-1} \cdots \theta_1^{p-1} \, d\theta_{N-1} \cdots d\theta_1 . \tag{2.9}$$

Our next problem is to integrate (2.9) with respect to θ_{N-1}. However, we note that the required integral is a known type of beta integral ([5], p. 490)

$$\begin{aligned} &\int_0^1 z^{m-1}(1-z)^{n-1}(a+bz)^{-(m+n+p)} \, dz \\ &\qquad = a^{-(n+p)}(a+b)^{-(m+p)} \int_0^1 t^{m-1}(1-t)^{n-1}(a+b(1-t))^p \, dt \\ &\qquad = a^{-n}(a+b)^{-m} B(m, n) , \qquad \text{if } \ p = 0 . \end{aligned} \tag{2.10}$$

If $p \neq 0$, then the right hand side of (2.10) is a series of beta integrals. Hence the integral (2.9) with respect to θ_{N-1} can at least be formally integrated. Suppose this integration with respect to θ_{N-1} has been performed. Then from the right hand side of (2.10) we note that the integral with respect to θ_{N-2} is also a beta integral of the type (2.10). Similarly we see that all the successive integrals with respect to θ_{N-3}, θ_{N-4}, $\cdots$, θ_1 are beta integrals of the type (2.10). Thus we have shown a formal solution to our ordered gamma variate problem, i.e., the c.f. (2.7) can be explicitly found as a multiple series of c.f.'s of linear functions of unordered gamma variates.

Now we proceed with the theory of ordered normal variates. Here the range of integration is $-\infty<x_1<x_2<\cdots<x_N<\infty$, and $-\infty<x_i<\infty$, $i=1,\cdots,N$. We transform the x variates to the v variates by the transformation $x_i=\sum_{j=1}^{i} v_j$, $i=1,\cdots,N$. The Jacobian of this transformation is unity. We note that this transformation assures us that v's are unordered and that $-\infty<v_1<\infty$, $0<v_i<\infty$, $i=2,\cdots,N$. We transform all v's except v_1 to new variates ξ's by the transformation

$$\sum_{j=2}^{i} v_j=\xi_{i-1}\xi_i\cdots\xi_{N-1}\,,\qquad i=2,\cdots,N\,. \tag{2.11}$$

The Jacobian J_2 of the transformation from $v_2, v_3,\cdots,v_N$ to $\xi_1,\xi_2,\cdots,\xi_{N-1}$ is known to be ([5], p. 147, example 2)

$$J_2=\xi_2\xi_3^2\cdots\xi_{N-2}^{N-3}\xi_{N-1}^{N-2}\,. \tag{2.12}$$

Thus the transformation from the x's to $v_1,\xi_1,\cdots,\xi_{N-1}$ is

$$\begin{aligned}
x_1&=v_1\,,\\
x_2&=(\xi_1\xi_2\cdots\xi_{N-1}+v_1)\,,\\
&\;\;\vdots\\
x_i&=(\xi_{i-1}\xi_i\cdots\xi_{N-1}+v_1)\,,\\
&\;\;\vdots\\
x_N&=(\xi_{N-1}+v_1)\,.
\end{aligned} \tag{2.13}$$

The Jacobian of the transformation (2.13) is given by (2.12). We note that, $-\infty<v_1<\infty$, $0<\xi_{N-1}<\infty$, $0<\xi_i<1$, $i=1,\cdots,N-2$. The order of integration in the multiple integral with respect to $v_1,\xi_1,\cdots,\xi_{N-1}$ is as follows. Integrate first with respect to v_1, then with respect to ξ_{N-1}, and then with respect to ξ_{N-2}, and so on, and finally with respect to ξ_1. Unfortunately, the transformation from x's to $v_1,\xi_1,\cdots,\xi_{N-1}$ does not reduce the original multiple integral to any of the known types, except for $N=2$, and $N=3$. However, for $N=2$, and $N=3$ we hope our method is at least of classroom interest.

We now proceed to consider a few illustrations of our general transformations developed in this section.

3. Some examples

We take our first example from Govindarajulu [6]. Let $\alpha_1^2 < \alpha_2^2$ be an ordered sample of size 2, from the α^2 distribution

$$g(\alpha^2) = (2\pi)^{1/2} \exp\left\{-\frac{1}{2}\alpha^2\right\}(\alpha^2)^{1/2-1}, \qquad 0 < \alpha^2 < \infty . \tag{3.1}$$

It is required to obtain

$$E(\alpha_1^2) = \pi^{-1}\int \alpha_1^2 \exp\left\{-\frac{1}{2}(\alpha_1^2 + \alpha_2^2)\right\}(\alpha_1^2)^{-1/2}(\alpha_2^2)^{-1/2}\, d\alpha_1^2\, d\alpha_2^2 . \tag{3.2}$$

The range of integration in (3.2) is $\alpha_1^2 < \alpha_2^2$, $0 < \alpha_1^2 < \infty$, $0 < \alpha_2^2 < \infty$.

Transforming α^2 variates to θ variates, we find that

$$E(\alpha_1^2) = E(\theta_1\theta_2) = \pi^{-1}\int \theta_1^{1/2}\theta_2 \exp\left\{-\frac{1}{2}\theta_2(1+\theta_1)\right\} d\theta_2\, d\theta_1 . \tag{3.3}$$

The range of integration in (3.3) is $0 < \theta_2 < \infty$, $0 < \theta_1 < 1$, and integration must be performed with respect to θ_2 first. Integrating with respect to θ_2 we find that

$$\begin{aligned} E(\theta_1\theta_2) &= 4\pi^{-1}\int_0^1 \theta_1^{1/2}(1+\theta_1)^{-2}\, d\theta_1 \\ &= 4\pi^{-1}\left\{[-\theta_1^{1/2}(1+\theta_1)^{-1}]_0^1 + \int_0^1 \frac{d\theta_1}{2\sqrt{\theta_1}\,(1+\theta_1)}\right\} . \end{aligned} \tag{3.4}$$

The integral in (3.4) is evaluated by setting $\theta_1 = t^2$, and we find that

$$E(\alpha_1^2) = E(\theta_1\theta_2) = (\pi - 2)/\pi . \tag{3.5}$$

The result (3.5) agrees with the one given by Govindarajulu ([6], p. 1302, in his notation $\nu_{1,2}^{(2)}$).

Often only the transformation from x's to y's is sufficient in case of exponential populations. Suppose $x_1 < x_2 < \cdots < x_N$ is an ordered sample from the exponential population, say,

$$g(x) = \exp\{-x\}, \qquad 0 < x < \infty . \tag{3.6}$$

Tanis [12] proves that x_1 and $\sum_{i=1}^{N}(x_i - x_1)/N$ are independently distributed. He uses this property to characterize the exponential population. Now transform the x variates to the y variates and observe that the joint density of y's is

$$\begin{aligned} &g(y_1, y_2, \cdots, y_N) \\ &\quad = N! \exp\{-N[y_1 + ((N-1)y_2 + (N-2)y_3 + \cdots + y_N)/N]\} . \end{aligned} \tag{3.7}$$

Obviously $y_1=x_1$, and $\sum_{i=1}^{N}(x_i-x_1)/N=((N-1)y_2+\cdots+y_N)/N$. Thus from (3.7) we see that x_1 and $\sum_{i=1}^{N}(x_i-x_1)/N$ are independent.

It may be noted that this property of independence of x_1 and $\sum_{i=1}^{N}(x_i-x_1)/N$ also holds for the joint density

$$(3.8)\qquad g(x_1, x_2, \cdots, x_N) \\ =(N-1)!N^p(\Gamma(p))^{-1}x_1^{p-1}\exp\{-(x_1+x_2+\cdots+x_N)\} ,$$

where the range of x's in (3.8) is $x_1<x_2<\cdots<x_N$, and $0<x_i<\infty$, $i=1, \cdots, N$.

Further, we observe that the iterated integral of Tanis ([12], p. 271) reduces to a product of independent integrals by the transformation of x's to y's.

The distribution of ratios of linear functions of ordered exponential variates, see, e.g., [4], and [9], may be easily derived by transforming ordered variates x's, from an exponential population, to θ's.

We conclude this section with an illustration of the application of our method to the normal population

$$(3.9)\qquad g(x)=(\sqrt{2\pi}\sigma)^{-1}\exp\left\{-\frac{1}{2\sigma^2}x^2\right\} , \qquad -\infty<x<\infty .$$

Following Cramér ([1], p. 483) we construct the following problem. An ordered sample of size 2, $x_1<x_2$, is available from the normal population (3.9), find the variance of the linear function $z=cx_1+(1-c)x_2$.

Obviously the c.f. $\phi(t)$ of z is

$$(3.10)\qquad \phi(t)=(\pi\sigma^2)^{-1}\int\exp\left\{-\frac{1}{2\sigma^2}(x_1^2+x_2^2)\right\}\exp\{it(cx_1+(1-c)x_2)\}\,dx_1\,dx_2 .$$

The range of integration in (3.10) is $x_1<x_2$, $-\infty<x_1<\infty$, $-\infty<x_2<\infty$. Transforming x_1 and x_2 to v_1 and ξ_1, we find that

$$(3.11)\qquad \phi(t)=(\pi\sigma^2)^{-1}\int_0^\infty\int_{-\infty}^\infty \exp\left\{-\frac{1}{2\sigma^2}(2v_1^2+2\xi_1v_1+\xi_1^2) \right. \\ \left. +it((1-c)\xi_1+v_1)\right\}dv_1\,d\xi_1 .$$

Integrating with respect to v_1, we find that

$$(3.12)\qquad \phi(t)=(\sqrt{\pi}\sigma)^{-1}\int_0^\infty\exp\left\{-\frac{1}{4\sigma^2}\xi_1^2+it\left(\frac{1}{2}-c\right)\xi_1+\frac{(it)^2}{4}\sigma^2\right\}d\xi_1 .$$

The integral (3.12) can be evaluated in terms of Ruben's K functions

[11], which perhaps are inevitable in the distribution theory of linear functions of ordered normal variates. Differentiating (3.12) with respect to it twice and setting $t=0$, we find that

$$\text{variance } (z)=\frac{\sigma^2}{2}+2\left(\frac{1}{2}-c\right)^2\sigma^2-\frac{(\frac{1}{2}-c)^2}{\pi}\cdot 4\sigma^2 \tag{3.13}$$
$$=\frac{\sigma^2}{2}\left[1+4\left(\frac{1}{2}-c\right)^2-\frac{8(\frac{1}{2}-c)^2}{\pi}\right].$$

Here c is a non-negative constant less than or equal to unity.

4. Concluding remarks and acknowledgment

In this paper we have pointed out that a simple transformation used in evaluating Dirichlet's multiple integral may also be used as a tool for deriving the distributions of linear functions of ordered gamma variates. This method is very useful in the theory of ordered statistics from an exponential population, and of sufficiently pedagogical or classroom interest to solve some simple problems in the theory of ordered statistics from normal and gamma populations. We can also use it with other populations as the transformation which we have given does not depend on any particular population. However, even theoretically also the transformation does not give a complete formal solution for any population except for the gamma population. This reason prompted the author to choose the title for the present paper.

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