# COMPOUND PASCAL DISTRIBUTIONS*)

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## 1. Introduction

The usefulness of the compound distributions in describing real data has been properly pointed out by several statisticians in the past (see [2], [3], [4], [6], [7] and also other relevant papers). In this paper the author intends to derive the Pascal-Beta and the Pascal-Gamma distributions, study some of their properties and discuss briefly the problem of estimating their parameters.

## 2. The Pascal-Beta distribution

Let the conditional probability function (p.f.) $f(x/p)$ of the Pascal random variable (r.v.) $X$ be represented by

$$(2.1)\qquad f(x/p)=\begin{cases}\binom{x-1}{k-1}p^k q^{x-k}, & x=k, k+1, \cdots\\ 0, & \text{elsewhere,}\end{cases}$$

where $X$ is the number of experiments to be performed in order to achieve $k$ successful experiments, and $p$ is the probability of achieving success in a single experiment ($q=1-p$). And let the probability density function (p.d.f.) $f(p)$ of the r.v. $p$ be represented by

$$(2.2)\qquad f(p)=\begin{cases}\dfrac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}, & 0<p<1,\ (\alpha, \beta>0)\\ 0, & \text{elsewhere,}\end{cases}$$

where $B(\alpha, \beta)$ is the usual beta function. Then the p.f. of the Pascal-Beta distribution is given by

$$(2.3)\qquad f(x)=\int_0^1 f(x/p)f(p)dp$$

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$$
= \begin{cases} \binom{x-1}{k-1} \dfrac{B(\alpha+k,\, \beta+x-k)}{B(\alpha,\, \beta)}, & x=k,\, k+1, \cdots \\ 0, \quad \text{elsewhere.} & \end{cases}
$$

The above p.f. (2.3) can be rewritten in the form of gamma functions as

$$
(2.4) \qquad f(x)=\mathrm{P}(X=x)=\frac{\Gamma(x)\Gamma(\alpha+k)\Gamma(\beta+x-k)\Gamma(\alpha+\beta)}{\Gamma(k)\Gamma(x-k+1)\Gamma(\alpha+\beta+x)\Gamma(\alpha)\Gamma(\beta)}, \quad x=k,\, k+1, \cdots .
$$

Since we intend to make some statements about the Binomial-Beta distribution, we write its p.f. for the benefit of readers as

$$
(2.5) \quad f(x)=\mathrm{P}(X=x)= \begin{cases} \binom{n}{x} \dfrac{B(\alpha+x,\, n-x+\beta)}{B(\alpha,\, \beta)}, & x=0,\, 1,\, 2, \cdots, n \\ 0, \quad \text{elsewhere,} & \end{cases}
$$

where $n$ is the total number of experiments performed, and $\alpha$ and $\beta$ as defined earlier.

## 3. An approximation to the Pascal-Beta distribution

Utilizing the well-known asymptotic result, namely,

$$
\frac{\Gamma(x+a)}{\Gamma(x)} \text{ is asymptotically equal to } x^{a},
$$

when $x \to \infty$ and $a$ remains fixed, we shall establish a limiting form of the Pascal-Beta distribution in this section. The p.f. (2.3) of the Pascal-Beta distribution is rewritten as

$$
\begin{aligned}
f(x)=\mathrm{P}(X=x) &= \binom{x-1}{k-1} \frac{\Gamma(\alpha+k)\Gamma(\beta+x-k)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+x)\Gamma(\alpha)\Gamma(\beta)} \\
&= \binom{x-1}{k-1} \left(\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\right) \left(\frac{\Gamma(\beta+x-k)}{\Gamma(\beta)}\right) \left(\frac{\Gamma(\alpha+\beta+x)}{\Gamma(\alpha+\beta)}\right)^{-1},
\end{aligned}
$$

which is asymptotically equivalent to

$$
(3.1) \qquad \binom{x-1}{k-1} \left(\frac{\alpha}{\alpha+\beta}\right)^{k} \left(\frac{\beta}{\alpha+\beta}\right)^{x-k},
$$

when $\alpha \to \infty$, $\beta \to \infty$, $k$ and $x-k$ remain fixed. Clearly, the limiting form of the Pascal-Beta distribution is the Pascal distribution with $p = \alpha/(\alpha+\beta)$. Thus for large $\alpha$ and $\beta$ we can approximate the probability of any desired event occurring according to the Pascal-Beta distribution from the p.f. of the Pascal distribution (3.1). One has now the following Limit Theorems:

THEOREM 1. *The Pascal-Beta distribution can be approximated by the Pascal distribution with* $p=\alpha/(\alpha+\beta)$, *provided* $\alpha$ *and* $\beta$ *are large.*

COROLLARY. *The Geometric-Beta distribution can be approximated by the Geometric distribution with* $p=\alpha/(\alpha+\beta)$, *provided* $\alpha$ *and* $\beta$ *are large.*

THEOREM 2. *The Binomial-Beta distribution can also be approximated by the Binomial distribution with* $p=\alpha/(\alpha+\beta)$, *provided* $\alpha$ *and* $\beta$ *are large.*

## 4. Moments of the Pascal-Beta distribution

The $r$th moment of the Pascal-Beta distribution can be computed by using the formula

$$\mathrm{E}x^r=\mathop{\mathrm{E}}_{y}\mathop{\mathrm{E}}_{x}(x^r/y)\,, \tag{4.1}$$

where the first expectation is performed with respect to the conditional probability function of the random variable $X$ and the second expectation is performed with respect to the probability density function of the random variable $Y$.

Now using formula (4.1), we get the first four moments of the Pascal-Beta distribution as

$$\mathrm{E}x=\frac{k(\alpha+\beta-1)}{\alpha-1}\,, \tag{4.2}$$

provided $\alpha>1$;

$$\mathrm{E}x^2=\frac{k(\alpha+\beta-1)}{(\alpha-1)(\alpha-2)}[k(\alpha+\beta-2)+\beta]\,, \tag{4.3}$$

provided $\alpha>2$;

$$\mathrm{E}x^3=\prod_{i=1}^{3}\frac{(k+i-1)(\alpha+\beta-i)}{(\alpha-i)} -3\prod_{i=1}^{2}\frac{(k+i-1)(\alpha+\beta-1)}{(\alpha-i)}+\frac{k(\alpha+\beta-1)}{\alpha-1}\,, \tag{4.4}$$

provided $\alpha>3$;

and

$$\mathrm{E}x^4=\prod_{i=1}^{4}\frac{(k+i-1)(\alpha+\beta-i)}{(\alpha-i)}-6\prod_{i=1}^{3}\frac{(k+i-1)(\alpha+\beta+i)}{(\alpha-i)} \tag{4.5}$$

$$+7\prod_{i=1}^{2}\frac{(k+i-1)(\alpha+\beta-i)}{(\alpha-i)}-\frac{k(\alpha+\beta-1)}{\alpha-1},$$

provided $\alpha>4$. The expressions for the central moments of the Pascal-Beta distribution do not simplify much. Consequently, we shall give below only the expression for its variance. Thus

$$\text{Var}(x)=\frac{k\beta(\alpha+\beta-1)(k+\alpha-1)}{(\alpha-1)^2(\alpha-2)}, \tag{4.6}$$

provided $\alpha>2$.

Using formula (4.1), the mean and the variance of the Binomial-Beta distribution can be shown to be

$$\text{E}x=\frac{n\alpha}{\alpha+\beta} \tag{4.7}$$

and

$$\text{Var}(x)=\frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}. \tag{4.8}$$

## 5. Moment estimators of the parameters

On the basis of the expressions for the mean and the variance of the Pascal-Beta distribution, derived in the previous section, we find the moment estimators for $\alpha$ and $\beta$ to be

$$\alpha^*(\text{moment estimator})=2+\frac{\bar{x}(\bar{x}-k)(1+k)}{s^2k-\bar{x}^2+k\bar{x}}, \tag{5.1}$$

and

$$\beta^*(\text{moment estimator})=(\alpha^*-1)(\bar{x}-k)/k, \tag{5.2}$$

where $\bar{x}$ and $s^2$ are the sample mean and the sample variance of the Pascal-Beta distribution. Since $\alpha^*$ and $\beta^*$ are functions of sample mean and sample variance, they are asymptotically normally distributed with the asymptotic means $\alpha$ and $\beta$ and the asymptotic covariance matrix $\sum(\alpha^*, \beta^*)$ under general regularity conditions. The derivation of the expression for the asymptotic covariance matrix $\sum(\alpha^*, \beta^*)$ is a straightforward task and therefore is left as an exercise. It can be seen that the moment estimators $\alpha^*$ and $\beta^*$ tend to obey the bivariate normal law, provided $\alpha>4$.

Similarly, from the expressions of the mean and the variance of the Binomial-Beta distribution, given in the previous section, we find the moment estimators of $\alpha$ and $\beta$ to be

(5.3) $$\alpha^* = \frac{\bar{x}(n\bar{x} - \bar{x}^2 - s^2)}{ns^2 - n\bar{x} + \bar{x}^2},$$

and

(5.4) $$\beta^* = \alpha^*(n - \bar{x})/\bar{x},$$

where $\bar{x}$ and $s^2$ are the sample mean and the sample variance of the Binomial-Beta distribution. Here $\alpha^*$ and $\beta^*$ tend to obey the bivariate normal law under general regularity conditions and without any restrictions on the parameters $\alpha$ and $\beta$.

## 6. Maximum likelihood estimators of the parameters

Since the maximum likelihood estimators (MLEs) of the parameters are usually asymptotically more efficient than their moment estimators, we shall discuss the problem of obtaining the MLEs of the parameters of the Pascal-Beta distribution in this section. Here the logarithmic likelihood function for the Pascal-Beta distribution is written as

(6.1) $$\begin{aligned}\log L(\alpha, \beta) = {} & \log \Gamma(x) + \log \Gamma(\alpha + k) + \log \Gamma(\beta + x - k) \\ & + \log \Gamma(\alpha + \beta) - \log \Gamma(k) - \log \Gamma(x - k + 1) \\ & - \log \Gamma(\alpha + \beta + x) - \log \Gamma(\alpha) - \log \Gamma(\beta).\end{aligned}$$

This yields the likelihood equations as

(6.2) $$\frac{\partial \log L}{\partial \alpha} = \Psi_\alpha(\alpha + k) + \Psi_\alpha(\alpha + \beta) - \Psi_\alpha(\alpha + \beta + x) - \Psi_\alpha(\alpha) = 0,$$

and

(6.3) $$\frac{\partial \log L}{\partial \beta} = \Psi_\beta(\beta + x - k) + \Psi_\beta(\alpha + \beta) - \Psi_\beta(\alpha + \beta + x) - \Psi_\beta(\beta) = 0,$$

where $\Psi_v(u+v) = \frac{d}{dv} \log \Gamma(u+v)$, which is called the Psi Function (H. T. Davis [1], Vol. I, pp. 275–). From Davis' book it follows that we can write

(6.4) $$\Psi(u) = -C - \sum_{s=0}^{\infty} \left( \frac{1}{u+s} - \frac{1}{s+1} \right),$$

for positive real $u$. In the above expression, $C$ is Euler's constant and is equal to $0.57722\cdots$. Using result (6.4), we can rewrite the likelihood equations (6.2) and (6.3) as

(6.5) $$\sum_{s=0}^{k-1} \frac{1}{\alpha + s} - \sum_{s=0}^{x-1} \frac{1}{\alpha + \beta + s} = 0,$$

and

$$\sum_{s=0}^{x-k-1}\frac{1}{\beta+s}-\sum_{s=0}^{x-1}\frac{1}{\alpha+\beta+s}=0\ . \tag{6.6}$$

We can solve the above two equations by any suitable numerical methods including the Newton-Raphson, the Direct Search [5], the Secant, etc. In this connection we may use the moment estimators of $\alpha$ and $\beta$ as the initial estimators or trial values.

The second-order derivatives of the logarithmic likelihood function are written as

$$\frac{\partial^2 \log L}{\partial\alpha^2}=\Psi^{(1)}_{\alpha^2}(\alpha+k)+\Psi^{(1)}_{\alpha^2}(\alpha+\beta)-\Psi^{(1)}_{\alpha^2}(\alpha+\beta+x)-\Psi^{(1)}_{\alpha^2}(\alpha)\ , \tag{6.7}$$

$$\frac{\partial^2 \log L}{\partial\beta^2}=\Psi^{(1)}_{\beta^2}(\beta+x-k)+\Psi^{(1)}_{\beta^2}(\alpha+\beta)-\Psi^{(1)}_{\beta^2}(\alpha+\beta+x)-\Psi^{(1)}_{\beta^2}(\beta)\ , \tag{6.8}$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\beta}=\Psi^{(1)}_{\alpha,\beta}(\alpha+\beta)-\Psi^{(1)}_{\alpha,\beta}(\alpha+\beta+x)\ , \tag{6.9}$$

where $\Psi^{(1)}_{u}(u+v)$ is the first derivative of the Psi Function with respect to $u$ and is called the trigamma function (Davis [1], Vol. II).

Using the formula for the polygamma functions,

$$\Psi^{(m)}(u)=\frac{d^m\Psi(u)}{du^m}=(-1)^{m+1}m\,!\sum_{s=0}^{\infty}\frac{1}{(u+s)^{m+1}}\ , \tag{6.10}$$

valid for $m>0$ and positive real $u$, we can rewrite expressions (6.7), (6.8) and (6.9) as

$$\frac{\partial^2 \log L}{\partial\alpha^2}=-\left[\sum_{s=0}^{k-1}\frac{1}{(\alpha+s)^2}-\sum_{s=0}^{x-1}\frac{1}{(\alpha+\beta+s)^2}\right], \tag{6.11}$$

$$\frac{\partial^2 \log L}{\partial\beta^2}=-\left[\sum_{s=0}^{x-k-1}\frac{1}{(\beta+s)^2}-\sum_{s=0}^{x-1}\frac{1}{(\alpha+\beta+s)^2}\right], \tag{6.12}$$

and

$$\frac{\partial^2 \log L}{\partial\alpha\partial\beta}=\sum_{s=0}^{x-1}\frac{1}{(\alpha+\beta+s)^2}\ . \tag{6.13}$$

Expressions (6.11), (6.12) and (6.13) may be used to obtain an estimator of the asymptotic covariance matrix of the MLEs of $\alpha$ and $\beta$. Exactly identical approach will enable one to obtain the MLEs of the parameters of the Binomial-Beta distribution and also an estimator of their asymptotic covariance matrix.

## 7. The Pascal-Gamma distribution

Let the conditional p.f. $f(x/\lambda)$ of the Pascal random variable $X$ be represented by

(7.1)
$$f(x/\lambda)=\begin{cases} \binom{x-1}{k-1}\exp(-k\lambda)[1-\exp(-\lambda)]^{x-k}, & x=k, k+1, \cdots, \quad (\lambda>0) \\ 0, & \text{elsewhere}, \end{cases}$$

and let the p.d.f. $f(\lambda)$ of $\lambda$ be given by

$$f(\lambda)=\begin{cases} \frac{\beta^{\alpha}\lambda^{\alpha-1}\exp(-\beta\lambda)}{\Gamma(\alpha)}, & \lambda>0, \quad (\alpha, \beta>0) \\ 0, & \text{elsewhere.} \end{cases} \tag{7.2}$$

Then the p.f. of the Pascal-Gamma distribution is obtained as

$$\begin{aligned} f(x)=\mathrm{P}(X=x)&=\int_0^{\infty} f(x/\lambda)f(\lambda)d\lambda \\ &=\begin{cases} \sum_{j=0}^{x-k}\binom{x-1}{k-1}\binom{x-k}{j}(-1)^j\left(1+\frac{j+k}{\beta}\right)^{-\alpha}, & x=k, k+1, \cdots \\ 0, & \text{elsewhere.} \end{cases} \end{aligned} \tag{7.3}$$

The mean of this Pascal-Gamma distribution is found to be

$$\mathrm{E}x=k\left(1-\frac{1}{\beta}\right)^{-\alpha}, \tag{7.4}$$

provided $\beta>1$, and its variance is found to be

$$\mathrm{Var}(x)=k(k+1)\left(1-\frac{2}{\beta}\right)^{-\alpha}-k\left(1-\frac{1}{\beta}\right)^{-\alpha}-k^2\left(1-\frac{1}{\beta}\right)^{-2\alpha}, \tag{7.5}$$

provided $\beta>2$.

One may obtain several results about the Pascal-Gamma distribution analogous to the Pascal-Beta distribution discussed in this paper. Here we briefly discuss the method of obtaining the moment estimators of the parameters $\alpha$ and $\beta$ of this distribution under three cases and point out a computer approach to their maximum likelihood estimators.

Case I: $\alpha$ known and $\beta$ unknown

When $\alpha$ is known, a moment estimator of $\beta$ based on expression (7.4) is given by

$$\beta^* = \frac{\bar{x}^{1/\alpha}}{\bar{x}^{1/\alpha} - k^{1/\alpha}}, \tag{7.6}$$

where $\bar{x}$ is the sample mean.

Case II: $\beta$ known and $\alpha$ unknown

When $\beta$ is known, a moment estimator of $\alpha$ based on expression (7.4) is given by

$$\alpha^* = \frac{\log \bar{x} - \log k}{\log \beta - \log(\beta - 1)}. \tag{7.7}$$

Case III: $\alpha$ and $\beta$ both are unknown

Here the moment estimators of $\alpha$ and $\beta$, based on expressions (7.4) and (7.5), are the solutions of the equations

$$\bar{x} - k\left(1 - \frac{1}{\beta}\right)^{-\alpha} = 0 \tag{7.8}$$

and

$$s^2 - k(k+1)\left(1 - \frac{2}{\beta}\right)^{-\alpha} + k\left(1 - \frac{1}{\beta}\right)^{-\alpha} + k^2\left(1 - \frac{1}{\beta}\right)^{-2\alpha} = 0, \tag{7.9}$$

where $\bar{x}$ is the sample mean and $s^2$ is the sample variance. These equations can be solved by one of several numerical methods including the Secant, the Direct Search technique [5], etc.

In the present age of electronic computers, one may find it convenient to obtain the maximum likelihood estimators of $\alpha$ and $\beta$ by maximizing the right-hand side of expression (7.3) with respect to $\alpha$ and $\beta$ through the Direct Search technique [5].

## Summary

In this paper, the *Pascal-Beta* and the *Pascal-Gamma* distributions have been derived by compounding the Pascal distribution with the Beta distribution and the Gamma distribution. These two compound distributions include Pascal-Uniform, Geometric-Beta, Geometric-Uniform, Pascal-Exponential, Geometric-Gamma and Geometric-Exponential as special cases. The methods of computing the moments of these compound distributions have been given. The moment estimators of the parameters of the Pascal-Beta distribution have been derived explicitly and the necessary condition has been pointed out for the asymptotic normality of these estimators. The essential expressions have been derived to obtain the maximum likelihood estimators of the parameters of the Pascal-Beta distribution numerically. The moment estimators of the parameters of the Binomial-Beta distribution have been also derived

explicitly. It has been pointed out that under certain conditions the Pascal-Beta distribution can be approximated by the Pascal distribution, the Geometric-Beta distribution can be approximated by the Geometric distribution, and the Binomial-Beta distribution can be approximated by the Binomial distribution. The paper also contains some discussions on the methods of obtaining the moment and maximum likelihood estimators of the parameters of the Pascal-Gamma distribution.

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