ON CONGESTION SYSTEMS WITH NEGATIVE EXPONENTIAL DESIRED SERVICE TIME DISTRIBUTIONS

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1. Introduction

Consider a congestion system whose behaviour when an incoming customer finds all the servers busy is governed by the displacement rule D, and in which

(a) the interarrival times are independent and identically distributed with distribution function $A(t)$ $(A(0+)=0)$,

(b) the desired service times are independent and identically distributed with distribution function $1-\exp(-\lambda t)$ $(t\geqq 0)$,

(c) there is no waiting for service,

(d) the desired service times and the interarrival times are independent.

This paper derives the efficiency of the system, i.e. the ratio of the mean achieved service time under the statistical equilibrium to the mean desired service time for the following D:

(i) to displace that customer who has been at the counter longest,

(ii) to displace that customer whose unexpired desired service time is least,

(iii) to turn away the newly arrived customer if his desired service time is less than the least unexpired desired service time; otherwise to displace the customer with the least unexpired desired service time.

The initial study of such systems when the interarrival time distribution is negative exponential is due to Cox [1]. But his work is restricted to very simple cases of the desired service time, the number of servers and so on. Downton [2] has improved upon the work of Cox [1] by taking general desired service time distribution and general number of servers. The displacement rules considered by Cox [1] are (i) and (ii) and those considered by Downton [2] are (ii) and (iii).

2. Known results

For telephone traffic systems with the interarrival time distribution function $A(t)$ and the service time distribution function $1-\exp(-\lambda t)$ $(t \geqq 0)$, the following results due to Takács [4] are well-known.

Result 1. The limiting probability that just before an arrival all the servers (channels) are busy in a system with m $(<\infty)$ servers is given by

$$B_m=\left\{\sum_{j=0}^{m}\binom{m}{j}(1/C_j)\right\}^{-1} \tag{1}$$

where $C_0=1$ and

$$C_r=\prod_{k=1}^{r}\frac{\phi(k\lambda)}{1-\phi(k\lambda)} \qquad (r=1,2,\cdots) \tag{2}$$

$\phi(\theta)$ being the Laplace-Stieltjes transform of $A(t)$.

Result 2. When there are infinite number of servers, the limiting distribution of the number of busy servers just before an arrival is given by

$$P_k=\sum_{r=k}^{\infty}(-1)^{r-k}\binom{r}{k}C_r \qquad (k=0,1,\cdots)\,. \tag{3}$$

These results are useful in the present study of congestion systems with incomplete service.

3. System with infinite servers

When the number of servers is infinite we do not have congestion in our system. However, we derive here the limiting probability that the rth largest unexpired service time just before an arrival is greater than x $(\geqq 0)$, as it will be useful to determine the efficiencies under the displacement rules (ii) and (iii) when there are finite number of servers.

Defining the probability by $F_r(x)$ and remembering the fact that the rth largest unexpired service time is greater than x if and only if there are at least r unexpired service times greater than x, we have

$$\begin{aligned} F_r(x)&=\sum_{k=r}^{\infty}\sum_{n=k}^{\infty}P_n\binom{n}{k}\exp(-\lambda xk)(1-\exp(-\lambda x))^{n-k} \\ &=\sum_{k=r}^{\infty}\exp(-\lambda kx)(-1)^{k-r}C_k\sum_{s=r}^{k}\binom{k}{s}(-1)^{s-r} \\ &=\sum_{k=r}^{\infty}\exp(-\lambda kx)(-1)^{k-r}\binom{k-1}{r-1}C_k(x\geqq 0)\,. \end{aligned} \tag{4}$$

For Poisson arrivals, taking the interarrival time distribution function as

$$A(t)=1-\exp(-\mu t) \qquad (t\geqq 0)\,, \tag{5}$$

we get

$$F_r(x)=\exp\{-\phi(x)\}\left\{\sum_{s=r}^{\infty}(\phi(x))^s/s\,!\right\} \tag{6}$$

where

$$\phi(x)=b\exp(-\lambda x)\ , \tag{7}$$

b being equal to μ/λ.

4. Displacement of the customer who has been at the counter longest

As mentioned in the introduction the displacement takes place when all the servers are busy. In this section and the sections which follow we assume that the number of servers is m in the system. The displacement rule here is to displace that customer who has been at the counter longest. As noted by Cox [1] this rule is equivalent to the displacement of a customer chosen independently of the unexpired desired service time.

The limiting probability that a lost service time is greater than x can easily be seen to be

$$G_m(x)=B_m\exp(-\lambda x) \qquad (x\geqq 0)\ , \tag{8}$$

where B_m is given by (1).

The efficiency ratio I_m is then given by

$$I_m=1-B_m\ . \tag{9}$$

Special case

Let $A(t)=1-\exp(-\mu t)$ $(t\geqq 0)$. From (8) and (9) we have

$$G_m(x)=b^m(m!\sum_{j=0}^{m}b^j/j!)^{-1}\exp(-\lambda x) \qquad (x\geqq 0) \tag{10}$$

and

$$I_m=\left\{\sum_{j=0}^{m-1}b^j/j\,!\right\}\left\{\sum_{j=0}^{m}b^j/j\,!\right\}^{-1}. \tag{11}$$

The above expression for the efficiency agrees with that given by Cox [1].

5. Displacement of the customer having the least unexpired desired service time

Here the displacement rule is to displace that customer whose unexpired desired service time is least, when all the servers are busy.

When $m=1$ the limiting distribution function of the lost service time and the efficiency ratio are respectively given by $1-G_1(x)$ $(x\geqq 0)$ and I_1. (It should be noted that for the case $m=1$ we can easily obtain the corresponding results even when the desired service time distribution is general.) When $m\geqq 2$ we proceed as follows to derive these quantities:

It can readily be seen that the limiting probability that the rth largest unexpired desired service time just before an arrival here is greater than x $(\geqq 0)$ is given by $F_r(x)$ if $r=1, 2, \cdots, m-1$. Furthermore, we have the limiting probability that the lost service time at an instant of arrival is greater than x $(\geqq 0)$ as

$$F'_m(x)=\int_{y=0}^{\infty} \exp(-\lambda(y+x))F_{m-1}(x+y)\, dA(y)\,, \tag{12}$$

because the lost service time at the present instant is $>x$ if and only if the $(m-1)$st largest unexpired desired service time just before "the first previous arrival" and the desired service time of the first previous customer exceed $x+$ the length of the interval between the present instant and the instant of "the first previous arrival". Hence

$$F'_m(x)=\sum_{k=m-1}^{\infty} \exp\{-\lambda(k+1)x\}(-1)^{k-m+1}\psi((k+1)\lambda)\binom{k-1}{m-2}C_k \quad (x\geqq 0)\,. \tag{13}$$

The efficiency for the present case is given by

$$I'_m=1-\sum_{k=m-1}^{\infty}(-1)^{k-m+1}(k+1)^{-1}\psi((k+1)\lambda)\binom{k-1}{m-2}C_k\,. \tag{14}$$

Special case

Let $A(t)=1-\exp(-\mu t)$ $(t\geqq 0)$, then

$$F'_m(x)=b(1+b)^{-1}\exp(-\lambda x)-b^{-b}\exp(\mu x)\sum_{s=0}^{m-2}\gamma(1+b+s,\, b\exp(-\lambda x))/s\,! \tag{15}$$

where

$$\gamma(u,\, v)=\int_0^{v} \exp(-t)t^{u-1}dt\,. \tag{16}$$

Furthermore

$$I'_m=\frac{m-1}{b}\frac{\gamma(m-1,b)}{(m-2)!}-\frac{\gamma(m-2,b)}{(m-3)!}-b^{-(1+b)}\sum_{s=0}^{m-2}\frac{\gamma(1+b+s,b)}{s!}+\frac{2+b}{1+b}, \tag{17}$$

where $\gamma(m-2, b)/(m-3)!$ is assumed to be unity for $m=2$. These results are due to Downton [2] ((17) for $m=2$ has also been given by Cox [1]).

6. Conditional displacement

In this section we take the displacement rule as the one which discards the minimum of the desired service time of the incoming customer and the least unexpired desired service time just before his arrival.

It readily follows that the limiting probability that the rth ($1\leqq r\leqq m$) largest unexpired desired service time just before an arrival is greater than x ($\geqq 0$) for this case is the same as $F_r(x)$ given earlier.

Using this fact we can write the limiting probability that the lost service time is greater than x ($\geqq 0$) as

$$\begin{aligned} H_m(x)&=F_m(x)\exp(-\lambda x) \\ &=\sum_{k=m}^{\infty}\exp\{-\lambda(k+1)x\}(-1)^{k-m}\binom{k-1}{m-1}C_k, \end{aligned} \tag{18}$$

because the minimum of the least unexpired desired service time and the desired service time of the incoming customer is $>x$ if and only if each is $>x$. Furthermore, the efficiency ratio can be seen to be

$$I''_m=1-\sum_{k=m}^{\infty}(k+1)^{-1}(-1)^{k-m}\binom{k-1}{m-1}C_k. \tag{19}$$

Special case

Let $A(t)=1-\exp(-\mu t)$ ($t\geqq 0$), then

$$H_m(x)=\exp\{-\lambda x-\phi(x)\}\left\{\sum_{s=m}^{\infty}(\phi(x))^s/s!\right\} \tag{20}$$

where $\phi(x)$ is as defined earlier. Furthermore,

$$I''_m=1-\frac{\gamma(m,b)-(\gamma(m+1,b)/b)}{(m-1)!}, \tag{21}$$

where $\gamma(u, b)$ and b are as defined earlier.

It is interesting to note that (21) is valid not only for the present special case but also for the case of general desired service time distribution when the interarrival time distribution is negative exponential (cf. Downton [2], [3]).

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