THE QUEUE GI/M/2 WITH SERVICE RATE DEPENDING ON THE NUMBER OF BUSY SERVERS

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Summary

The time dependent behaviour of the two server queueing system with recurrent input and negative exponential service times is studied here using certain recurrence relations for the underlying queuelength process. The service times have a varying mean depending on the number of busy servers.

1. Introduction

Multiserver queues with recurrent input and negative exponential service times present several difficulties in their investigations. The equilibrium behaviour of such systems from the imbedded Markov chain point of view has been given by Kendall [3]. The time dependent behaviour has been treated only in the special cases when the inter-arrival time and service time distributions have the properties of a negative exponential distribution. Karlin and McGregor [2] and Saaty [4] study the system M/M/s and Arora [1] considers the queue GI/M/2 when the inter-arrival time distribution is 'phase-type', each phase having a negative exponential distribution. Their approaches have been through the classical methods of solving the difference-differential equations for the transition probabilities. When the number of servers increases or when some other complications or generalities are introduced into the problem, clearly we need other methods which can be easily handled. The method presented here seems to be one of them.

In the queue GI/M/2 which we consider the inter-arrival times have an independent arbitrary distribution and the service times are distributed negative exponentially with a varying mean depending on the number of busy servers. This modification of the service mechanism takes care of the practical situation in which, when only one of the servers is busy, the service of the single customer is either slowed due to some reason or accelerated with the help of the idle server (an example of the latter case is that of two repairmen who help each other when one of them is free).

Let $t_0(=0), t_1, t_2, \cdots,$ be the epochs of arrival into the system and the inter-arrival times $u_n=t_n-t_{n-1}$ $(n=1, 2, \cdots)$ be distributed as $\Pr\{u_n \leqq x\}=A(x)$ $(0\leqq x<\infty)$; let

$$(1) \qquad \psi(\theta)=\int_0^\infty \exp(-\theta t)dA(t) \qquad (\mathrm{Re}\ (\theta)\geqq 0)$$

be its Laplace-Stieltjes transform (L.T.) and mean $\alpha=-\psi'(0)<\infty$.

The customers form a single queue in the order of their arrival and the one at the head of the queue goes into service when any one of the servers becomes free. The distribution of the service times is the same for both the servers and therefore the customer arriving when the system is empty may choose any one of them. Let v_n $(n=1, 2, \cdots)$ be the service times of the customers. When both the servers are busy these have the distribution

$$(2) \qquad \Pr\{v_n\leqq x\}=1-\exp(-\lambda x) \qquad (0\leqq x<\infty)$$

and when there is only one customer in the system, v_n will be distributed as

$$(3) \qquad \Pr\{v_n\leqq x\}=1-\exp(-\lambda_2 x) \qquad (0\leqq x<\infty)\,.$$

For the sake of convenience we write

$$(4) \qquad \sigma=\lambda_2/2\lambda\,.$$

Define $Q(t)$ as the number of customers in the system at time t including those who are being served (queuelength) and let $Q(t_n)=Q(t_n-0)$, $(n=0, 1, 2, \cdots)$. In the following sections we shall derive transforms of the several probabilities connected with the queuelength process $Q(t)$.

2. Transitions in a busy period

We define busy period as the interval of time during which there has been at least one customer in the system. In such a period let

$$(5) \quad {}^0P_{ij}^{(n)}(t)=\Pr\{Q(t)=j,\ t_n\leqq t<t_{n+1},\ Q(t_r)>0\ (r=1, 2, \cdots, n-1)|Q(0)=i\} \quad (i, j, n\geqq 0)\,.$$

Consider the case $n=0$; no arrival occurs after $t_0=0$ and in time $(0, t)$, $i+1-j$ $(i+1\geqq j\geqq 0)$ customers get served. Writing down the probabilities of the possible transitions we have

$$
{}^0P_{ij}^{(0)}(t)=\begin{cases}
\exp(-2\lambda t)\dfrac{(2\lambda t)^{i+1-j}}{(i+1-j)!}[1-A(t)] & (i\geqq j-1,\ j\geqq 2) \quad (6)\\[2ex]
[1-A(t)]\displaystyle\int_0^t \exp(-2\lambda\tau)\frac{(2\lambda\tau)^{i-1}}{(i-1)!}2\lambda d\tau \exp[-\lambda_2(t-\tau)] & (i\geqq 1,\ j=1) \quad (7)\\[2ex]
[1-A(t)]\displaystyle\int_0^t \exp(-2\lambda\tau)\frac{(2\lambda\tau)^{i-1}}{(i-1)!}2\lambda d\tau\{1-\exp[-\lambda_2(t-\tau)]\} & (i\geqq 1,\ j=0)\,. \quad (8)
\end{cases}
$$

When $n>0$, let τ $(0<\tau\leqq t)$ be the time at which the first arrival takes place after the one at $t_0=0$. Accounting for the transitions before and after τ, we have

$$
(9)\quad {}^0P_{ij}^{(n)}(t)=\sum_{r=2}^{i+1}\int_0^t \exp(-2\lambda\tau)\frac{(2\lambda\tau)^{i+1-r}}{(i+1-r)!}dA(\tau)\cdot{}^0P_{rj}^{(n-1)}(t-\tau)
+\int_{\tau=0}^{t}\int_{s=0}^{\tau}\exp(-2\lambda s)\frac{(2\lambda s)^{i-1}}{(i-1)!}2\lambda ds\exp[-\lambda_2(\tau-s)]dA(\tau)
\cdot{}^0P_{1j}^{(n-1)}(t-\tau)\qquad (i\geqq 1,\ j\geqq 0)\,.
$$

Define the transforms

$$
(10)\quad \begin{cases}
\Omega_{ij}^{(n)}(\theta)=\displaystyle\int_0^\infty \exp(-\theta t)\,{}^0P_{ij}^{(n)}(t)dt & (\mathrm{Re}\ (\theta)>0)\\[2ex]
\Omega_j^{(n)}(\theta, z)=\displaystyle\sum_{i=1}^\infty z^{i-1}\Omega_{ij}^{(n)}(\theta) & (|z|<1)\\[2ex]
\Omega_j(\theta, z, \omega)=\displaystyle\sum_{n=0}^\infty \omega^n\Omega_j^{(n)}(\theta, z) & (|\omega|\leqq 1)\,.
\end{cases}
$$

Taking transforms of (6)~(8) we get

$$
(11)\quad \Omega_j^{(0)}(\theta, z)=\frac{[1-\phi(\theta+2\lambda-2\lambda z)]z^{j-2}}{\theta+2\lambda-2\lambda z}\qquad (j\geqq 2)
$$

$$
(12)\quad \Omega_1^{(0)}(\theta, z)=2\lambda\int_0^\infty \exp[-(\theta+\lambda_2)t][1-A(t)]dt\int_0^t \exp[-(2\lambda-2\lambda z)\tau+\lambda_2\tau]d\tau
=(1-\sigma-z)^{-1}\left[\frac{1-\phi(\theta+\lambda_2)}{\theta+\lambda_2}-\frac{1-\phi(\theta+2\lambda-2\lambda z)}{\theta+2\lambda-2\lambda z}\right]
$$

and

$$
(13)\quad \Omega_0^{(0)}(\theta, z)=2\lambda\int_0^\infty \exp(-\theta t)[1-A(t)]dt\int_0^t\{\exp[-(2\lambda-2\lambda z)\tau]
-\exp[-(2\lambda-2\lambda z-\lambda_2)\tau-\lambda_2 t]\}d\tau
$$

$$=\frac{1-\psi(\theta)}{\theta(1-z)}-\frac{1-\psi(\theta+\lambda_2)}{(\theta+\lambda_2)(1-\sigma-z)}+\frac{[1-\psi(\theta+2\lambda-2\lambda z)]\sigma}{(1-z)(1-\sigma-z)(\theta+2\lambda-2\lambda z)}\,.$$

Let A and B be the transforms of the two terms respectively of (9). We have

$$A=\int_0^\infty \exp(-2\lambda\tau)\sum_{r=2}^{\infty}\sum_{i=r-1}^{\infty} z^{i-1}\frac{(2\lambda\tau)^{i+1-r}}{(i+1-r)!}dA(\tau) \cdot\int_{t=\tau}^{\infty}\exp(-\theta t)\cdot{}^0P_{rj}^{(n-1)}(t-\tau)\,dt \tag{14}$$

$$=\psi(\theta+2\lambda-2\lambda z)\sum_{r=2}^{\infty} z^{r-2}\Omega_{rj}^{(n-1)}(\theta)$$

$$=z^{-1}\psi(\theta+2\lambda-2\lambda z)[\Omega_j^{(n-1)}(\theta, z)-\Omega_{1j}^{(n-1)}(\theta)]$$

and

$$B=2\lambda\int_0^\infty \exp(-\theta t)dt\int_{\tau=0}^{t}\exp(-\lambda_2\tau)dA(\tau)\,{}^0P_{1j}^{(n-1)}(t-\tau) \cdot\int_{s=0}^{\tau}\exp[-(2\lambda-2\lambda z-\lambda_2)s]ds \tag{15}$$

$$=(1-\sigma-z)^{-1}\Omega_{1j}^{(n-1)}(\theta)\int_0^\infty \exp(-\theta\tau)\{\exp(-\lambda_2\tau) -\exp[-(2\lambda-2\lambda z)\tau]\}\,dA(\tau)$$

$$=(1-\sigma-z)^{-1}[\psi(\theta+\lambda_2)-\psi(\theta+2\lambda-2\lambda z)]\Omega_{1j}^{(n-1)}(\theta)\,.$$

Combining (14) and (15) we get

$$\Omega_j^{(n)}(\theta, z)=z^{-1}\psi(\theta+2\lambda-2\lambda z)\Omega_j^{(n-1)}(\theta, z) -\frac{[(1-\sigma)\psi(\theta+2\lambda-2\lambda z)-z\psi(\theta+\lambda_2)]}{z(1-\sigma-z)}\Omega_{1j}^{(n-1)}(\theta) \quad (j\geqq 0,\ n=1, 2, \cdots)\,. \tag{16}$$

The transform with respect to n is obtained by multiplying (11)~(13) and (16) by appropriate powers of ω and summing over n. This procedure gives after rearrangement of terms

$$\Omega_j(\theta, z, \omega)=[z-\omega\psi(\theta+2\lambda-2\lambda z)]^{-1} \cdot\Bigg[\frac{\omega\{z\psi(\theta+\lambda_2)-(1-\sigma)\psi(\theta+2\lambda-2\lambda z)\}}{1-\sigma-z}\sum_{n=0}^{\infty}\omega^n\Omega_{1j}^{(n)}(\theta) + \begin{cases} \dfrac{[1-\psi(\theta+2\lambda-2\lambda z)]z^{i-1}}{\theta+2\lambda-2\lambda z}\Bigg] & (j\geqq 2) \\[2ex] \dfrac{z}{1-\sigma-z}\left\{\dfrac{1-\psi(\theta+\lambda_2)}{\theta+\lambda_2}-\dfrac{1-\psi(\theta+2\lambda-2\lambda z)}{\theta+2\lambda-2\lambda z}\right\}\Bigg] & (j=1) \end{cases} \tag{17}$$

$$\left. \begin{array}{l} \dfrac{[1-\psi(\theta)]z}{\theta(1-z)} - \dfrac{[1-\psi(\theta+\lambda_2)]z}{(\theta+\lambda_2)(1-\sigma-z)} \\ \quad + \dfrac{[1-\psi(\theta+2\lambda-2\lambda z)]\sigma z}{(1-z)(1-\sigma-z)(\theta+2\lambda-2\lambda z)} \Big] \end{array} \right. \qquad (j=0) .$$

To determine this transform completely we consider the equation

$$z=\omega\psi(\theta+2\lambda-2\lambda z) . \tag{18}$$

Clearly, this has only one root $\gamma=\gamma(\theta, \omega)$ in the unit circle $|z|<1$ if Re $(\theta)\geqq 0$ and $|\omega|\leqq 1$ (For proof, see Takács [5] Lemma 1). The left hand side of (17) is a regular function of z in this domain and therefore the root γ must also satisfy the terms inside the square brackets on the right hand side. Substituting γ in it, equating to zero and simplifying we obtain

$$\sum_{n=0}^{\infty} \omega^n \Omega_{1j}^{(n)}(\theta)= \begin{cases} \dfrac{\gamma^{j-2}(\omega-\gamma)(1-\sigma-\gamma)}{\omega(\theta+2\lambda-2\lambda\gamma)[1-\sigma-\omega\psi(\theta+\lambda_2)]} & (j\geqq 2) \\ [1-\sigma-\omega\psi(\theta+\lambda_2)]^{-1}\left\{\dfrac{1-\psi(\theta+\lambda_2)}{\omega(\theta+2\lambda-2\lambda\gamma)}\right\} & (j=1) \\ (1-\gamma)^{-1}[1-\sigma-\omega\psi(\theta+\lambda_2)]^{-1}\left\{\dfrac{(1-\sigma-\gamma)[1-\psi(\theta)]}{\theta} \right. \\ \quad \left. -\dfrac{(1-\gamma)[1-\psi(\theta+\lambda_2)]}{\theta+\lambda_2}+\dfrac{\sigma(\omega-\lambda)}{\omega(\theta+2\lambda-2\lambda\gamma)}\right\} \\ \qquad (j=0) . \end{cases} \tag{19}$$

These and the transforms (17) determine the transition probabilities in a busy period initiated by $i(>0)$ customers. For the distribution of the busy period we need to study the case $i=0$. Then we have

$$ {}^0P_{0j}^{(0)}(t)= \begin{cases} 0 & (j\geqq 2) \\ [1-A(t)]\exp(-\lambda_2 t) & (j=1) \\ [1-A(t)][1-\exp(-\lambda_2 t)] & (j=0) \end{cases} \tag{20}$$

and

$$ {}^0P_{0j}^{(n)}(t)=\int_0^t \exp(-\lambda_2\tau)dA(\tau)\, {}^0P_{1j}^{(n-1)}(t-\tau) \qquad (j\geqq 0,\ n>0) . \tag{21}$$

Taking transforms as before and using (19) we get

$$
\sum_{n=0}^{\infty} \omega^n \Omega_{0j}^{(n)}(\theta) = \begin{cases}
\dfrac{\gamma^{j-2}(\omega-\gamma)(1-\sigma-\gamma)\psi(\theta+\lambda_2)}{(\theta+2\lambda-2\lambda\gamma)[1-\sigma-\omega\psi(\theta+\lambda_2)]} & (j \geqq 2) \quad (22) \\[2ex]
[1-\sigma-\omega\psi(\theta+\lambda_2)]^{-1} \cdot \left\{ \dfrac{(1-\sigma)[1-\psi(\theta+\lambda_2)]}{\theta+\lambda_2} - \dfrac{(\omega-\gamma)\psi(\theta+\lambda_2)}{\theta+2\lambda-2\lambda\gamma} \right\} & (j=1) \quad (23) \\[2ex]
[1-\sigma-\omega\psi(\theta+\lambda_2)]^{-1} \cdot \left\{ \dfrac{[(1-\gamma)(1-\sigma)-\sigma\omega\psi(\theta+\lambda_2)][1-\psi(\theta)]}{\theta(1-\gamma)} - \dfrac{(1-\sigma)[1-\psi(\theta+\lambda_2)]}{\theta+\lambda_2} + \dfrac{\sigma(\omega-\gamma)\psi(\theta+\lambda_2)}{(1-\gamma)(\theta+2\lambda-2\lambda\gamma)} \right\} & (j=0) . \quad (24)
\end{cases}
$$

The distribution of the busy period can be easily derived from (23). Let T be the length of the busy period and $N(T)$ the number of arrivals in it. Let

$$
g^{(n)}(t)dt = \Pr\{t < T < t+dt ;\ N(T)=n\} ; \tag{25}
$$

we have

$$
g^{(n)}(t)dt = {}^0P_{01}^{(n)}(t)\lambda_2 dt \tag{26}
$$

and therefore

$$
\begin{aligned}
\Pi(\theta, \omega) &= \sum_{n=0}^{\infty} \omega^n \int_0^{\infty} \exp(-\theta t) g^{(n)}(t)dt \\
&= \lambda_2 \sum_{n=0}^{\infty} \omega^n \Omega_{01}^{(n)}(\theta) \\
&= \frac{\lambda_2}{1-\sigma-\omega\psi(\theta+\lambda_2)} \left\{ \frac{(1-\sigma)[1-\psi(\theta+\lambda_2)]}{\theta+\lambda_2} - \frac{(\omega-\gamma)\psi(\theta+\lambda_2)}{\theta+2\lambda-2\lambda\gamma} \right\} .
\end{aligned} \tag{27}
$$

3. The busy cycle

The time interval between the epochs of commencement of two consecutive busy periods is known as a busy cycle. Clearly this consists of a busy period and an idle period. As these two distributions are not independent in this system, we derive the distribution of the busy period as follows. Let

$$
R_i^{(n)}(t) = \Pr\{Q(t_n)=0,\ t_n \leqq t,\ Q(t_r)>0\ (r=1, 2, \cdots, n-1) | Q(0)=i\} \quad (n \geqq 1,\ i \geqq 0) . \tag{28}
$$

Analogous to equations (8) and (9) we get the recurrence relations

$$(29)\qquad dR_i^{(1)}(t)=dA(t)\int_0^t \exp(-2\lambda\tau)\frac{(2\lambda\tau)^{i-1}}{(i-1)!}2\lambda d\tau\{1-\exp[-\lambda_2(t-\tau)]\}$$

$$(30)\qquad dR_i^{(n)}(t)=\sum_{r=2}^{i+1}\int_0^t \exp(-2\lambda\tau)\frac{(2\lambda\tau)^{i+1-r}}{(i+1-r)!}dA(\tau)dR_r^{(n-1)}(t-\tau)$$
$$+\int_{\tau=0}^{t}\int_{s=0}^{\tau}\exp(-2\lambda s)\frac{(2\lambda s)^{i-1}}{(i-1)!}2\lambda ds\exp[-\lambda_2(\tau-s)]$$
$$\cdot dA(\tau)dR_1^{(n-1)}(t-\tau)\qquad (n>1).$$

Taking transforms of these probabilities we get

$$(31)\qquad \Gamma(\theta,z,\omega)=\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\omega^n z^{i-1}\int_0^\infty \exp(-\theta t)dR_i^{(n)}(t)$$
$$(\mathrm{Re}(\theta)>0,\ |z|<1,\ |\omega|\leqq 1)$$
$$=[z-\omega\psi(\theta+2\lambda-2\lambda z)]^{-1}$$
$$\cdot\left\{\frac{[(1-\sigma-z)\psi(\theta)-(1-z)\psi(\theta+\lambda_2)+\sigma\psi(\theta+2\lambda-2\lambda z)]\omega z}{(1-z)(1-\sigma-z)}\right.$$
$$\left.-\frac{[(1-\sigma)\psi(\theta+2\lambda-2\lambda z)-z\psi(\theta+\lambda_2)]\omega}{1-\sigma-z}\sum_{n=1}^{\infty}\omega^n\Gamma_1^{(n)}(\theta)\right\}$$

where

$$(32)\qquad \Gamma_1^{(n)}(\theta)=\int_0^\infty \exp(-\theta t)dR_1^{(n)}(t).$$

A discussion about the root γ of the equation (18) which should satisfy the terms in brackets on the right hand side of (31) leads to the result

$$(33)\qquad \sum_{n=1}^{\infty}\omega^n\Gamma_1^{(n)}(\theta)=\frac{(1-\sigma-\gamma)\omega\psi(\theta)-(1-\gamma)\omega\psi(\theta+\lambda_2)+\sigma\gamma}{(1-\gamma)[1-\sigma-\omega\psi(\theta+\lambda_2)]}.$$

The transform (31) is now completely determined.

For the busy cycle we need to start with the arrival of a customer. We have

$$(34)\qquad dR_0^{(1)}(t)=[1-\exp(-\lambda_2 t)]dA(t)$$

$$(35)\qquad dR_0^{(n)}(t)=\int_0^t \exp(-\lambda_2\tau)dA(\tau)dR_1^{(n-1)}(t-\tau)\qquad (n>1).$$

From these relations we get

$$(36)\qquad \Gamma(\theta,\omega)=\sum_{n=1}^{\infty}\omega^n\int_0^\infty \exp(-\theta t)dR_0^{(n)}(t)$$
$$=\omega\psi(\theta)-\frac{(1-\sigma-\gamma)[1-\omega\psi(\theta)]\omega\psi(\theta+\lambda_2)}{(1-\gamma)[1-\sigma-\omega\psi(\theta+\lambda_2)]}.$$

In our further discussion we shall suppress n and setting $\omega=1$ we shall denote the transforms (17), (22~24), (27), (31) and (36) as $\Omega_j(\theta, z)$, $\Omega_{0j}(\theta)$, $\Pi(\theta)$, $\Gamma(\theta, z)$ and $\Gamma(\theta)$ respectively.

4. General transitions

Suppose $T_1, T_2, \cdots$ are the epochs of commencement of busy periods and T_0, the instant at which the process starts. These epochs T_r ($r=0, 1, 2, \cdots$) form a set of renewal points of the queuelength process for which the busy cycles $Z_r=T_r-T_{r-1}$ ($r=1, 2, \cdots$) are renewal periods. If the process starts with the commencement of a busy period all renewal periods Z_r ($r=1, 2, \cdots$) have the same distribution $R(t)=\sum_{n=1}^{\infty} R_0^{(n)}(t)$ whose L.T. is given by (36). Whereas, if $Q(0)=i>0$, Z_r ($r=2, 3, \cdots$) will have the distribution $R(t)$ and Z_1 will be distributed as $R_i(t)=\sum_{n=1}^{\infty} R_i^{(n)}(t)$, whose L.T. is given by $\Gamma(\theta, z)$ of equation (31) (also see remarks following (36)).

Consider the renewal process $\{S_n\}$, $S_n=Z_1+Z_2+\cdots+Z_n$ ($n=1, 2, \cdots$). Let $N^*(t)$ be the number of renewal points ($T_1, T_2, \cdots$) in this process so that $N^*(t)=\max\{n|S_n\leqq t\}$; also let $U_i(t)=E\{N^*(t)\}$. From renewal theory we have

$$U_0^*(\theta)=\int_0^{\infty} \exp(-\theta t)dU_0(t)=\frac{1}{1-\Gamma(\theta)} \tag{37}$$

when the process starts with a busy period; if at T_0, a busy period is in progress we have

$$U^*(\theta, z)=\sum_{i=1}^{\infty} z^{i-1}\int_0^{\infty} \exp(-\theta t)dU_i(t)=\frac{\Gamma(\theta, z)}{1-\Gamma(\theta)}\,. \tag{38}$$

Substituting for $\Gamma(\theta)$ from (36) (see remarks following this equation) we get

$$U_0^*(\theta)=\frac{(1-\gamma)[1-\sigma-\psi(\theta+\lambda_2)]}{[1-\psi(\theta)][(1-\gamma)(1-\sigma)-\sigma\psi(\theta+\lambda_2)]}\,. \tag{39}$$

Similarly substitution for $\Gamma(\theta, z)$ and $\Gamma(\theta)$ gives the transform (38).

Define the general transition probabilities

$$P_{ij}(t)=\Pr\{Q(t)=j|Q(0)=i\}. \tag{40}$$

To derive these probabilities we proceed as follows: Consider the last renewal point τ in $(0, t)$. Accounting for the possible transitions before and after τ, we can write

$$P_{0j}(t)=\int_0^t dU_0(\tau)\,{}^0P_{0j}(t-\tau) \tag{41}$$

$$P_{ij}(t)={}^0P_{ij}(t)+\int_0^t dU_i(\tau)\,{}^0P_{0j}(t-\tau) \qquad (i\geqq 1)\,. \tag{42}$$

In (42) the first term takes care of the possibility that, when $j>0$, the initial busy period is still in progress at time t. From these relations we have

$$\begin{aligned}\phi_{0j}(\theta)&=\int_0^\infty \exp(-\theta t)P_{0j}(t)dt \qquad (\mathrm{Re}\ (\theta)>0)\\ &=U_0^*(\theta)\Omega_{0j}(\theta)\end{aligned} \tag{43}$$

and

$$\begin{aligned}\phi_j(\theta, z)&=\sum_{i=1}^\infty z^{i-1}\int_0^\infty \exp(-\theta t)P_{ij}(t)dt \qquad (\mathrm{Re}\ (\theta)>0,\ |z|<1)\\ &=\Omega_j(\theta, z)+U^*(\theta, z)\Omega_{0j}(\theta)\,.\end{aligned} \tag{44}$$

In particular, using (39) and (24) we get

$$\phi_{0j}(\theta)=\begin{cases}\dfrac{\gamma^{j-2}(1-\gamma)^2(1-\sigma-\gamma)\phi(\theta+\lambda_2)}{(\theta+2\lambda-2\lambda\gamma)[1-\phi(\theta)][(1-\gamma)(1-\sigma)-\sigma\phi(\theta+\lambda_2)]} & (j\geqq 2) \quad (45)\\[2ex] \dfrac{1-\gamma}{[1-\phi(\theta)][(1-\gamma)(1-\sigma)-\sigma\phi(\theta+\lambda_2)]}\cdot\left\{\dfrac{(1-\sigma)[1-\phi(\theta+\lambda_2)]}{\theta+\lambda_2}-\dfrac{(1-\gamma)\phi(\theta+\lambda_2)}{\theta+2\lambda-2\lambda\gamma}\right\} & (j=1) \quad (46)\\[2ex] \dfrac{1}{\theta}-\dfrac{1-\gamma}{[1-\phi(\theta)][(1-\gamma)(1-\sigma)-\sigma\phi(\theta+\lambda_2)]}\cdot\left\{\dfrac{(1-\sigma)[1-\phi(\theta+\lambda_2)]}{\theta+\lambda_2}-\dfrac{\sigma\phi(\theta+\lambda_2)}{\theta+2\lambda-2\lambda\gamma}\right\} & (j=0)\,. \quad (47)\end{cases}$$

5. Particular cases

The approach adopted in the preceding discussion may also be used with obvious modifications for the system GI/M/1. Results are already available for this queue (see, Takács [5]). Here we shall consider the two server system when the arrivals are in a Poisson process and compare the equilibrium behaviour and the expected queuelengths for the values of $\sigma=1/4$, $1/2$ and 1. It may be noted that the case $\sigma=1/2$ refers to the ordinary system M/M/2 and the case $\sigma=1$, to the system in which the inter-departure times (when busy) have the negative exponential distribution $2\lambda\exp(-2\lambda t)dt$ $(0<t<\infty)$, thus reducing it to the system M/M/1.

Let the arrivals be in a Poisson process of parameter μt so that $\psi(\theta)=\mu(\theta+\mu)^{-1}$. Therefore we have

$$\gamma=\gamma(\theta)=\{\theta+2\lambda+\mu-[(\theta+2\lambda+\mu)^2-8\lambda\mu]^{1/2}\}(4\lambda)^{-1}. \tag{48}$$

As a consequence of the equation (18) we also have the identity

$$\theta+2\lambda-2\lambda\gamma=\mu(1-\gamma)\gamma^{-1}. \tag{49}$$

Using these results we derive the following special cases.

(1) $\sigma=1/4$: We get

$$\Pi(\theta)=\frac{2\gamma(3-4\gamma)}{6\theta+3\lambda-2\mu} \tag{50}$$

and

$$\phi_{0j}(\theta)=\begin{cases}\dfrac{2\gamma^{j-1}(1-\gamma)(3-4\gamma)(\theta+\mu)}{\theta[3(1-\gamma)(2\theta+\lambda+2\mu)-2\mu]} & (j\geqq 1)\\[2ex] \dfrac{1}{\theta}-\dfrac{2(3-4\gamma)(\theta+\mu)}{\theta[3(1-\gamma)(2\theta+\lambda+2\mu)-2\mu]} & (j=0).\end{cases} \tag{51}$$

(2) $\sigma=1/2$: We get

$$\Pi(\theta)=\frac{\lambda(1-2\gamma)}{\theta+\lambda-\mu} \tag{52}$$

and

$$\phi_{0j}(\theta)=\begin{cases}\dfrac{\gamma^{j-1}(1-\gamma)(1-2\gamma)(\theta+\mu)}{\theta[(1-\gamma)(\theta+\lambda+\mu)-\mu]} & (j\geqq 1)\\[2ex] \dfrac{1}{\theta}-\dfrac{(1-2\gamma)(\theta+\mu)}{\theta[(1-\gamma)(\theta+\lambda+\mu)-\mu]} & (j=0).\end{cases} \tag{53}$$

(3) $\sigma=1$: We get

$$\Pi(\theta)=2\lambda\gamma\mu^{-1} \tag{54}$$

and

$$\phi_{0j}(\theta)=\begin{cases}\dfrac{\gamma^{j}(1-\gamma)(\theta+\mu)}{\theta\mu} & (j\geqq 1)\\[2ex] \dfrac{1}{\theta}-\dfrac{\gamma(\theta+\mu)}{\theta\mu} & (j=0).\end{cases} \tag{55}$$

Let P_j be the probability that the queuelength is j as $t\to\infty$. The steady state probabilities can then be obtained from the relation $P_j = \lim_{\theta\to 0} \theta\phi_{0j}(\theta)$ (assuming the existence). Let $E(Q)$ be the mean queuelength and $\rho=\mu/2\lambda$, the traffic intensity of the system. For the special cases given above we have

(56)

σ	P_0	$P_j(j>0)$	$E(Q)$
$\frac{1}{4}$	$\frac{1-\rho}{1+3\rho}$	$\left(\frac{1-\rho}{1+3\rho}\right)4\rho^j$	$\frac{4\rho}{(1-\rho)(1+3\rho)}$
$\frac{1}{2}$	$\frac{1-\rho}{1+\rho}$	$\left(\frac{1-\rho}{1+3\rho}\right)2\rho^j$	$\frac{2\rho}{(1-\rho)(1+\rho)}$
1	$1-\rho$	$(1-\rho)\rho^j$	$\frac{\rho}{1-\rho}$.

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