NOTE ON HIGHER ORDER SPECTRA

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Recently the notion of higher order spectra is gradually attracting attentions of statisticians [2, 4, 5, 7, 8, 9, 10].

Introduction of higher order spectra is quite natural when we try to analyse the non-Gaussian property of a stochastic process or the non-linearity of a system operating under a random input.

For the stationary case, the higher order spectra are understood as giving the correlations between frequency components with frequencies whose sum is equal to zero [6].

Though such an explanation exhausts the essential characteristics of higher order spectra, it is often too general to give a definite meaning to the results obtained by a real data analysis. The results of analyses of ordinary spectra and cross-spectra can be understood completely on the basis of linear transformation theory and they suggest the direction of development of models or theories about the phenomena under observation. In contrast to this, higher order spectra seem to be still in want of a sufficiently general theory which gives an overall understandability to them, and their physical meanings have been understood only where a proper model or theory existed before the observation [4]. Besides the computational and statistical difficulties, this fact may sometimes have led people to somewhat pessimistic attitude towards the higher order spectral analysis [2, p. 1372].

The purpose of this short note is to introduce a new notion of "mixed spectrum" which situates between the moment function and the spectrum and relates the notion of higher order spectra to the well established linear theory of ordinary spectra and cross-spectra. As was mentioned by J. W. Tukey [10, p. 348], the meaningfulness and understandability are the most important characteristics required of the statistics. There is a paper by L. J. Tick [7] in which cross-bispectrum is treated as giving a "frequency response" of a quadratic system, and here we shall also treat the case of cross-bispectrum.

We observe the third order moment $M_{yxx}(\tau_1, \tau_2)=E\{[y(t)-Ey(t)] \cdot x(t+\tau_1)x(t+\tau_2)\}$ of which existence and the independence of t are assumed. We assume the processes to be real. Cross-bispectrum $B_{yx}(f_1, f_2)$ is defined [7] as the Fourier transform of $M_{yxx}(\tau_1, \tau_2)$ and is given by

$$(1)\qquad B_{yxx}(f_1, f_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-i\,2\pi(f_1\tau_1+f_2\tau_2))M_{yxx}(\tau_1, \tau_2)\,d\tau_1 d\tau_2$$

where the integrability of $M_{yxx}(\tau_1, \tau_2)$ is assumed. Obviously, by putting $y(t)=x(t)$ in this definition of $B_{yxx}(f_1, f_2)$ we get the bispectrum of $x(t)$. Now we define a quantity $B_{yxx}(f;\sigma)$ by

$$(2)\qquad B_{yxx}(f;\sigma)=\int_{-\infty}^{\infty} \exp(-i\,2\pi f\tau)M_{yxx}(\sigma-\tau, -\tau)d\tau\ .$$

We tentatively suggest here the term "mixed spectrum" for this type of quantities, as it relates to the spectrum mixed in time and frequency. Taking into account the relation

$$(3)\qquad \begin{aligned} M_{yxx}(\sigma-\tau, -\tau)&=E\{[y(t)-Ey(t)]x(t+\sigma-\tau)x(t-\tau)\} \\ &=E\{[y(t+\tau)-Ey(t+\tau)]x(t+\sigma)x(t)\}\ , \end{aligned}$$

we can see that the mixed spectrum $B_{yxx}(f;\sigma)$ gives the cross-spectral density between $y(t)$ and $x(t+\sigma)x(t)$ under the assumption of stationarity. Thus the meaning of the mixed spectrum for a fixed σ is well understood as an ordinary cross-spectrum for which we have a completely established theory and methods of measurement. The relation between this mixed spectrum and the cross-bispectrum is given by the following:

$$(4)\qquad B_{yxx}(f, f')=\int_{-\infty}^{\infty} \exp(-i\,2\pi f\sigma)B_{yxx}(-f-f';\sigma)d\sigma\ .$$

Extensions of this notion of mixed spectrum to more general cases of higher order by increasing the dimensions of f and σ will be straightforward.

From the point of view of mixed spectrum, it is natural to replace the cross-bispectrum by a quantity $C_{yxx}(f, f')$ which is defined as follows:

$$(5)\qquad C_{yxx}(f, f')=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-i\,2\pi(f\tau+f'\sigma))m_{yxx}(\tau, \sigma)d\tau d\sigma$$

where $m_{yxx}(\tau, \sigma)=E\{y(t+\tau)-Ey(t+\tau)\}x(t+\sigma)x(t)$. Obviously, when $y(t)=x(t)$ we have $C_{yxx}(f, f')=B_{yxx}(f, f')$. It can be seen that $C_{yxx}(f, f')$ corresponds directly to the representation

$$(6)\qquad y(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau, \sigma)x(t-\tau+\sigma)x(t-\tau)d\tau d\sigma$$

of a quadratic transformation of $x(t)$. $C_{yxx}(f, f')$ is the Fourier transform of $B_{yxx}(f;\sigma)$,

(7) $$C_{yxx}(f, f')=\int_{-\infty}^{\infty}\exp(-i2\pi f'\sigma)B_{yxx}(f;\sigma)d\sigma\ ,$$

and from (4) and (7)

(8) $$C_{yxx}(-f-f', f)=B_{yxx}(f, f')\ .$$

When the input $x(t)$ is Gaussian and is with a power spectral density function $p(f)$, it was shown [7] that for a quadratic system

(9) $$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(\tau_1, \tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

we have

(10) $$B_{yxx}(f_1, f_2)=2H(-f_1, -f_2)p(f_1)p(f_2)$$

where $H(f_1, f_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-i2\pi(f\tau_1+f\tau_2))h(\tau_1, \tau_2)d\tau_1 d\tau_2$ and $h(\tau_1, \tau_2)$ is symmetric in τ_1 and τ_2. Taking account of the relation (8), we get, in the present case of Gaussian input,

(11) $$C_{yxx}(f, f')=B_{yxx}(f', -f-f')=2H(-f', f+f')p(f')p(f+f')$$
$$=2G(f, f')p(f')p(f+f')$$

where $G(f, f')=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-i2\pi(f\tau+f'\sigma))g(\tau, \sigma)\,d\tau d\sigma$ and $g(\tau, \sigma)=h(\tau-\sigma, \tau)$, and we can see that the representation of this quadratic system in the form

(12) $$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\tau, \sigma)x(t-\tau+\sigma)x(t-\tau)d\tau d\sigma$$

quite naturally corresponds to the definition of $C_{yxx}(f, f')$.

It seems that, for the analysis of input-output relations of a nonlinear system under stationary random input, the adoption of the present mixed spectrum enables us to treat the numerical data by using the method of analysis of multiple input linear systems [1, 3] and that this naturally leads us to an understanding of the meaning of bispectrum, trispectrum and other higher order spectra of the input process.

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